Dynamic pricing in combinatorial markets

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1 Introduction

A combinatorial market consists of a set of indivisible goods and a set of buyers, where each buyer has a valuation function that represents the buyers' preferences over the subsets of items. From an optimization point of view, the goal is to find an allocation of the items to buyers in such a way that the total sum of the buyers' values is maximized – this sum is called the social welfare. An optimal allocation can be found efficiently in various settings [9, 19, 26, 29], but the problem becomes significantly more difficult if one would like to realize the optimal social welfare through simple mechanisms.

A great amount of work concentrated on finding optimal pricing schemes. Given a price for each item, we define the utility of a buyer for a bundle of items to be the value of the bundle with respect to the buyer's valuation, minus the total price of the items in the bundle. A pair of pricing and allocation is called a Walrasian equilibrium if the market clears (that is, all the items are assigned to buyers) and everyone receives a bundle that maximizes her utility. Given any Walrasian equilibrium, the corresponding price vector is referred to as Walrasian pricing, and the definition implies that the corresponding allocation maximizes social welfare.

Walrasian equilibria were introduced already in the late 1800s [30] for divisible goods. A century later, Kelso and Crawford [22] defined gross substitutes functions and verified the existence of Walrasian prices for such valuations. It is worth mentioning that the class of gross substitutes functions coincides with that of $M^\natural$-concave functions, introduced by Murota and Shioura [24]. The fundamental role of the gross substitutes condition was recognized by Gul and Stacchetti [20] who verified that it is necessary to ensure the existence of a Walrasian equilibrium.

Although Walrasian equilibria have distinguished properties, Cohen-Addad et al. [10] and independently Hsu et al. [21] observed that Walrasian prices are not powerful enough to control the market on their own. To prove this, they gave the following example. The market stays of two buyers, Alice and Bob, and two goods, a and b. Both A and B are unit-demand. For A, a has value $R \gg 1$ and b has value 1, and for B, both items have value 1. We get a Walrasian pricing by setting the price of a to $R-1$ and the price of b to 0. (It is not difficult to verify this pricing is indeed Walrasian.) If B arrives first, he buys b and then A takes a. In this case, social welfare is maximized. But on the other hand, if A arrives first, she has the same utility for a and b, and it can happen she decides to take b. Then B has negative utility for a and does not buy anything. This results in 1 being the social welfare, instead of $R+1$.

The reason behind this is that different bundles of items might have the same utility for the same buyer, and in such cases, ties must be broken by a central coordinator in order to ensure that the optimal social welfare is achieved. Furthermore, this problem cannot be resolved by finding Walrasian prices where ties do not occur as [21] showed that minimal Walrasian prices necessarily induce ties. There exists a Walrasian price vector that is coordinate-wise minimal among all Walrasian price vectors. It is called minimal Walrasian price vector. It is known the minimal Walrasian price is the same as the VCG payments in the unit-demand case. In the previous example, the minimal Walrasian prices are $p(a) = p(b) = 0$. As before, if A is the first buyer, she gets a and B gets b, and the social welfare is again $R+1$. But if B comes first, he may choose a, then A gets b, and the social welfare is only 2. In [21], the authors defined the so-called over-demand for a good. Let g denote a good in the market and $s_g \in \mathbb{Z}$, $s_g \geq 1$ denote its supply, that is, there are s_g copies of g in the market. The demanders for g at price p are those buyers who have a bundle containing g that maximizes their utility. The over-demand for g is the maximum of zero and the difference between the number of demanders and the supply s_g. The authors of [21] showed there exist unit-demand valuations such that at the minimal Walrasian price, some good has over-demand $\Omega(n)$, where n denotes the number of players. That means if a buyer chooses randomly a good in the demand set, roughly $n/2$ buyers want to take g. That means ties can not be broken arbitrarily, some tie-breaking rules must be used. If a buyer's tie-breaking rule can only depend on the buyer's valuation and the prices, the authors also proved there exists a distribution over unit-demand valuations such that for any set of tie-breaking rules, the expected over-demand from n buyers is $\Omega(n)$.

To overcome these difficulties, [10] introduced the notion of dynamic pricing schemes, where prices can be redefined between buyer arrivals. Dynamic pricing schemes were introduced as an alternative to posted-price mechanisms that are capable of maximizing social welfare even without a central tie-breaking coordinator. In this model, the buyers arrive in sequential order, and each buyer selects a bundle of the remaining items that maximizes her utility. The buyers' preferences are known in advance, and the seller is allowed to update the prices between buyer arrivals based upon the remaining set of items, but without knowing the identity of the next buyer. They showed with a simple example that when using static prices, one can reach no better result than $2/3$ of the optimal social welfare. Consider a market with three unit-demand buyers A, B and C and three items a, b, c. A values a and b by 1 and c by 0, B values b and c by 1 and a by 0, and C values c and a by 1 and b by 0. We can assume $1 > p(a) \geq p(b) \geq p(c) \geq 0$. If A arrives first, she gets b. If the second player arriving is C, she buys c. Finally, B arrives, but the only remaining item in the market is a, which has value 0 for B. Therefore the social welfare achieved is only 2, whereas the optimal social welfare is 3.

Cohen-Addad et al. proposed a scheme maximizing social welfare for matching or unit-demand markets, where the valuation of each buyer is determined by the most valuable item in her bundle. In each phase, the algorithm constructs a so-called *relation graph* and performs various computations upon it. Then the prices are updated based on the structural properties of the graph.

The main open problem in [10] asked whether any market with gross substitutes valuations has a dynamic pricing scheme that achieves optimal social welfare.

Berger et al. [4] considered markets beyond unit-demand valuations, and provided a polynomial-time algorithm for finding optimal dynamic prices up to three multi-demand buyers. Their approach is based on a generalization of the relation graph of [10] that they call a *preference graph*, and on a new directed graph termed the *item-equivalence graph*. They showed that there is a strong connection between these two graphs, and provided a pricing scheme based on these observations.

Further results on posted-price mechanisms considered matroid rank valuations [3], relaxations such as combinatorial Walrasian equilibrium [18], and online settings [5–8, 11, 12, 14, 16, 17].

In this work[1], the focus is on multi-demand combinatorial markets. In this setting, each buyer t has a positive integer bound $b(t)$ on the number of desired items, and the value of a set is the sum of the values of the $b(t)$ most valued items in the set. In particular, if we set each $b(t)$ to one then we get back the unit-demand case.

For multi-demand markets, the problem of finding an allocation that maximizes social welfare is equivalent to a maximum weight b-matching problem in a bipartite graph with vertex classes corresponding to the buyers and items, respectively. Note that, unlike in the case of Walrasian equilibrium, clearing the market is not required as a maximum weight b-matching might leave some

[1] A preliminary version of the work appeared on ArXiv [2].

of the items unallocated. The high-level idea of our approach is to consider the dual of this problem and to define an appropriate price vector based on an optimal dual solution with distinguished structural properties.

Based on the primal-dual interpretation of the problem, we give a simpler proof of a result of Cohen-Addad et al. [10] on unit-demand valuations first. Although this can be considered a special case of bi-demand markets, we discuss it separately as an illustration of our techniques.

When the total demand of the buyers exceeds the number of available items, ensuring the optimality of the final allocation becomes more intricate. Therefore, we consider instances satisfying the following property:

$$\text{each buyer } t \in T \text{ receives exactly } b(t) \text{ items in every optimal allocation.} \tag{OPT}$$

While this is a restrictive assumption, it is a reasonable condition that holds for a wide range of applications, and also appeared in [4] and recently in [27]. For example, if the total number of items is not less than the total demand of the buyers and the value of each item is strictly positive for each buyer, then it is not difficult to check that (OPT) is satisfied.

The problem becomes significantly more difficult for larger demands. Berger et al. [4] observed that bundles that are given to a buyer in different optimal allocations satisfy strong structural properties. For markets with up to three multi-demand buyers, they grouped the items into at most eight equivalence classes based on which buyer could get them in an optimal solution, and then analyzed the item-equivalence graph for obtaining optimal dynamic pricing. We show that, when assumption (OPT) is satisfied, these properties follow from the primal-dual interpretation of the problem, and give a new proof of their result for such instances.

The main result of our work is an algorithm for determining optimal dynamic prices in bi-demand markets with an arbitrary number of buyers, that is when the demand $b(t)$ is two for each buyer t. Besides structural observations on the dual solution, the proof relies on uncrossing sets that are problematic in terms of resolving ties. However, in a recent manuscript, Pashkovich and Xie [27] showed that the result of Berger et al. [4] can be generalized from three to four buyers. They further extended the results of the current work on bi-demand valuations to the case when each buyer is ready to buy up to three items.

This work is organized as follows. Basic definitions and notation are given in Section 2, while Section 3 provides structural observations on optimal dynamic prices in multi-demand markets. Unit- and multi-demand markets up to three buyers are discussed in Section 4. Section 5 solves the bi-demand case under the (OPT) condition. In Section 6, we drop the (OPT) condition. In Section 7 and Section 8, the four buyers' and tri-demand case is discussed by following the proof in [27]. In Section 9, there is a short discussion on other social welfare functions.

2 Preliminaries

Basic notation. We denote the sets of *real*, *non-negative real*, *integer*, and *positive integer* numbers by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, and $\mathbb{Z}_{>0}$, respectively. Given a ground set S and subsets $X, Y \subseteq S$, the *difference* of X and Y is denoted by $X - Y$. If Y consists of a single element y, then $X - \{y\}$ and $X \cup \{y\}$ are abbreviated by $X - y$ and $X + y$, respectively. The *symmetric difference* of X and Y is $X \triangle Y := (X - Y) \cup (Y - X)$. For a function $f : S \to \mathbb{R}$, the total sum of its values over a set X is denoted by $f(X) := \sum_{s \in X} f(s)$. The *inner product* of two vectors $x, y \in \mathbb{R}^S$ is $x \cdot y := \sum_{s \in S} x(s) y(s)$. Given a set S, an *ordering* of S is a bijection σ between S and the set of integers $\{1,\dots,|S|\}$. For a set $X \subseteq S$, we denote the restriction of the ordering to $S-X$ by $\sigma|_{S-X}$. Given orderings σ_1 and σ_2 of disjoint sets S_1 and S_2, respectively, we denote by $\sigma = (\sigma_1, \sigma_2)$ the ordering of $S \coloneqq S_1 \cup S_2$ where $\sigma(s) = \sigma_1(s)$ for $s \in S_1$ and $\sigma_2(s) + |S_1|$ for $s \in S_2$.

Let $G = (S, T; E)$ be a bipartite graph with vertex classes S and T and edge set E. We will always denote the *vertex set* of the graph by $V \coloneqq S \cup T$. For a subset $X \subseteq V$, we denote the *set of edges induced by* X by $E[X]$, while $G[X]$ stands for the *graph induced by* X. The graph obtained from G by *deleting* X is denoted by $G - X$. Given a subset $F \subseteq E$, the *set of edges in* F *incident to a vertex* $v \in V$ is denoted by $\delta_F(v)$. Accordingly, the *degree* of v in F is $d_F(v) \coloneqq |\delta_F(v)|$. For a set $Z \subseteq T$, the *set of neighbors of* Z *with respect to* F is denoted by $N_F(Z)$, that is, $N_F(Z) \coloneqq \{s \in S \mid \text{there exists and edge } st \in F \text{ with } t \in Z\}$. The subscript F is dropped from the notation or is changed to G whenever F is the whole edge set.

Market model. A combinatorial market consists of a set S of *indivisible items* and a set T of *buyers*. We consider *multi-demand*[2] markets, where each buyer $t \in T$ has a valuation $v_t \colon S \to \mathbb{R}_+$ over individual items together with an upper bound $b(t)$ on the number of desired items, and the *value* of a set $X \subseteq S$ for buyer t is defined as $v_t(X) \coloneqq \max\{v_t(X') \mid X' \subseteq X, |X'| \le b(t)\}$. *Unit-demand* and *bi-demand* valuations correspond to the special cases when $b(t) = 1$ and $b(t) = 2$ for each $t \in T$, respectively.

Given a *price vector* $p \colon S \to \mathbb{R}_+$, the *utility* of buyer t for X is defined as $u_t(X) \coloneqq v_t(X) - p(X)$. The buyers, whose valuations are known in advance, arrive in an undetermined order, and the next buyer always chooses a subset of at most her desired number of items that maximizes her utility. In contrast to static models, the prices can be updated between buyer-arrivals based on the remaining sets of items and buyers. The goal is to set the prices at each phase in such a way that no matter in what order the buyers arrive, the final allocation maximizes the social welfare. Such a pricing scheme and allocation are called *optimal*. It is worth emphasizing that a buyer may decide either to take or not to take an item which has 0 utility, that is, it might happen that the bundle of items that she chooses is not inclusionwise minimal. This seemingly tiny degree of freedom actually results in difficulties that one has to take care of.

Lemma 1. *We may assume that all items are allocated in every optimal allocation.*

Proof. One can find an optimal allocation that uses an inclusionwise minimum number of items by relying on a weighted b-matching algorithm, see [28]. Setting the price of unused items to a large value ensures that no buyer takes them. Hence every optimal allocation uses the same set of items, meaning that the remaining items play no role in the problem and so can be deleted. □

In particular, when (OPT) is assumed, Lemma 1 implies that the number of items coincides with the total demand of the buyers.

Weighted b-matchings. Let $G = (S, T; E)$ be a bipartite graph and recall that $V \coloneqq S \cup T$. Given an upper bound $b \colon V \to \mathbb{Z}_+$ on the vertices, a subset $M \subseteq E$ is called a *b-matching* if $d_M(v) \le b(v)$ for every $v \in V$. If equality holds for each $v \in V$, then M is called a *b-factor*. Notice that if $b(v) = 1$ for each $v \in V$, then a b-matching or b-factor is simply a matching or perfect matching, respectively. Kőnig's classical theorem [23] gives a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph.

[2] Multi-demand valuations are special cases of weighted matroid rank functions for uniform matroids, see [3].

Theorem 1 (Kőnig)**.** *There exists a perfect matching in a bipartite graph $G = (S, T; E)$ if and only if $|S| = |T|$ and $|N(Y)| \geq |Y|$ for every $Y \subseteq T$.*

Let $w : E \to \mathbb{R}$ be a weight function on the edges. A function $\pi\colon V \to \mathbb{R}$ on the vertex set $V = S \cup T$ is a *weighted covering* of w if $\pi(s) + \pi(t) \geq w(st)$ holds for every edge $st \in E$. An edge st is called *tight with respect to* π if $\pi(s) + \pi(t) = w(st)$. The *total value* of the covering is $\pi \cdot b = \sum_{v\in V} \pi(v) \cdot b(v)$. We refer to a covering of minimum total value as *optimal*. The celebrated result of Egerváry [15] provides a min-max characterization for the maximum weight of a matching or a perfect matching in a bipartite graph.

Theorem 2 (Egerváry)**.** *Let $G = (S, T; E)$ be a graph, $w : W \to \mathbb{R}$ be a weight function. Then the maximum weight of a matching is equal to the minimum total value of a non-negative weighted covering π of w. If G has a perfect matching, then the maximum weight of a perfect matching is equal to the minimum total value of a weighted covering π of w.*

3 Multi-demand markets and maximum weight b-matchings

A combinatorial market with multi-demand valuations can be naturally identified with an edge-weighted complete bipartite graph $G = (S, T; E)$ where S is the set of items, T is the set of buyers, and for every item s and buyer t the weight of edge $st \in E$ is $w(st) := v_t(s)$. We extend the demands to S as well by setting $b(s) = 1$ for every $s \in S$. Then an optimal allocation of the items corresponds to a maximum weight subset $M \subseteq E$ satisfying $d_M(v) \leq b(v)$ for each $v \in S \cup T$.

3.1 Structure of weighted coverings

In general, a b-factor or even a maximum weight b-matching can be found in polynomial time (even in non-bipartite graphs, see e.g. [28]). When b is identically one on S, then the following folklore characterization follows easily from Kőnig's and Egerváry's theorems[3].

Lemma 2. *Let $G = (S, T; E)$ be a bipartite graph, $w : E \to \mathbb{R}_+$ be a weight function, and $b\colon V \to \mathbb{Z}_{>0}$ be an upper bound function satisfying $b(s) = 1$ for $s \in S$.*

(a) G has a b-factor if and only if $|S| = b(T)$ and $|N(X)| \geq b(X)$ for every $X \subseteq T$.

(b) The maximum w-weight of a b-matching is equal to the minimum total value of a non-negative weighted covering π of w.

Proof. Let $G' = (S', T; E')$ denote the graph obtained from G by taking $b(t)$ copies of each vertex $t \in T$ and connecting them to the vertices in $N_G(t)$. It is not difficult to check that G has a b-factor if and only if G' has a perfect matching, thus the first part of the theorem follows by Theorem 1.

To see the second part, for each copy $t' \in T'$ of an original vertex $t \in T$, define the weight of edge st' as $w'(st') := w(st)$. Then the maximum w-weight of a b-matching of G is equal to the maximum w'-weight of a matching of G'. Now take an optimal non-negative weighted covering π' of w' in G'. As the different copies of an original vertex $t \in T$ share the same neighbors in G', each of them receives the same value in any optimal weighted covering of w' - define $\pi(t)$ to be this value. Then π is a non-negative weighted covering of w in G with a total value equal to that of π', hence the theorem follows by Theorem 2. □

[3]The same results follow by strong duality applied to the linear programming formulations of the problems.

Given a weighted covering π, the *subgraph of tight edges* with respect to π is denoted by $G_\pi = (S, T; E_\pi)$. In what follows, we prove some easy structural results on the relation of optimal b-matchings and weighted coverings.

Lemma 3. *Let $G = (S, T; E)$ be a bipartite graph, $w : E \to \mathbb{R}_+$ be a weight function, and $b\colon V \to \mathbb{Z}_{>0}$ be an upper bound function satisfying $b(s) = 1$ for $s \in S$.*

(a) For any optimal non-negative weighted covering π of w, a b-matching $M \subseteq E$ has maximum weight if and only if $M \subseteq E_\pi$ and $d_M(v) = b(v)$ for each v with $\pi(v) > 0$.

(b) For any optimal weighted covering π of w, a b-factor $M \subseteq E$ has maximum weight if and only if $M \subseteq E_\pi$.

Proof. Let M be a maximum weight b-matching and π be an optimal non-negative weighted covering. We have $w(M) = \sum_{st\in M} w(st) \le \sum_{st\in M}(\pi(s) + \pi(t)) \le \sum_{v\in V} \pi(v) \cdot b(v)$, and equality holds throughout if and only if M consists of tight edges and $\pi(v) = 0$ if $d_M(v) < b(v)$.

Now consider the b-factor case. Let M be a maximum weight b-factor and π be an optimal weighted covering. We have $w(M) = \sum_{st\in M} w(st) \le \sum_{st\in M}(\pi(s) + \pi(t)) = \sum_{v\in V} \pi(v) \cdot b(v)$, and the inequality is satisfied with equality if and only if M consists of tight edges. □

Following the notation of [4], we call an edge $st \in E$ *legal* if there exists a maximum weight b-matching containing it, and say that s is *legal for t*. A subset $F \subseteq \delta(t)$ is *feasible* if there exists a maximum weight b-matching M such that $\delta_M(t) = F$; in this case $N_F(t)$ is called *feasible for t*[4]. Notice that a feasible set necessarily consists of legal edges. The essence of the following technical lemma is that there exists an optimal non-negative weighted covering for which G_π consists only of legal edges, thus giving a better structural understanding of optimal dual solutions; for an illustration see Figure 1.

Lemma 4. *The optimal π attaining the minimum in Lemma 2(b) can be chosen such that*

(a) an edge st is tight with respect to π if and only if it is legal, and

(b) $\pi(v) = 0$ for some $v \in V$ if and only if there exists a maximum weight b-matching M with $d_M(v) < b(v)$.

Furthermore, such a π can be determined in polynomial time.

Proof. In both cases, the 'if' part follows by Lemma 3. Let M and π be a maximum weight b-matching and an optimal non-negative weighted covering, respectively. To prove the lemma, we will modify π in two phases.

In the first phase, we ensure (a) to hold. Take an arbitrary ordering $e_1, \dots, e_m$ of the edges, and set $\pi_0 := \pi$ and $w_0 := w$. For $i = 1, \dots, m$, repeat the following steps. Let $\varepsilon_i := \max\{w_{i-1}(M) \mid M \text{ is a } b\text{-matching}\} - \max\{w_{i-1}(M) \mid M \text{ is a } b\text{-matching containing } e_i\}$. Notice that $\varepsilon_i > 0$ exactly if e_i is not legal. Let w_i denote the weight function obtained from w_{i-1} by increasing the weight of e_i by $\varepsilon_i/2$, and let π_i be an optimal non-negative weighted covering of w_i. Due to the definition of ε_i, a b-matching M has maximum weight with respect to w_i if and only if it has maximum weight with respect to w_{i-1}, and in this case $w_i(M) = w_{i-1}(M)$. That is, the sets of maximum weight b-matchings with respect to w and w_m coincide, and the weights of legal edges do not change, therefore π_m is an optimal non-negative weighted covering of w as well.

In the second phase, we ensure (b) to hold. Take an arbitrary ordering $v_1, \dots, v_n$ of the vertices, for $j = 1, \dots, n$, repeat the following steps. Let $\delta_j := \max\{w_{m+j-1}(M) \mid M \text{ is a } b\text{-matching}\} -$

[4]The notion of feasibility is closely related to 'legal allocations' introduced in [4]. However, 'legal subsets' are different from feasible ones, hence we use a different term here to avoid confusion.

$\max\{w_{m+j-1}(M) \mid M$ is a b-matching, $d_M(v_j) \le b(v_j) - 1\}$. Then $\delta_j > 0$ if and only if the degree of v_j is $b(v_j)$ in every maximum weight b-matching. Let w_{m+j} denote the weight function obtained from w_{m+j-1} by decreasing the weight of the edges incident to v_j by $\delta_j/(b(v_j)+1)$ and let π_{m+j} be an optimal non-negative weighted covering of w_{m+j}. Due to the definition of δ_j, a b-matching M has maximum weight with respect to w_{m+j-1} if and only if it has maximum weight with respect to w_{m+j}, and in this case $w_{m+j}(M) = w_{m+j-1}(M) - \delta_j \cdot b(v_j)$. That is, the sets of maximum weight b-matchings with respect to w and w_{m+n} coincide. Let π' denote the weighted covering of w obtained by increasing the value of $\pi_{m+n}(v_\ell)$ by $\delta_\ell/(b(v_\ell)+1)$ for $\ell = 1, \dots, n$. As the total value of π' is greater than that of π_{m+n} by exactly $\max\{w(M) \mid M$ is a b-matching$\} - \max\{w_{m+n}(M) \mid M$ is a b-matching$\}$, π' is an optimal non-negative weighted covering of w.

As $\varepsilon_i > 0$ whenever e_i is not legal and $\delta_j > 0$ whenever there is no a maximum weight b-matching M with $d_M(v_j) < b(v_j)$, π' satisfies both (a) and (b) as required. □

Remark 3. If the market satisfies property (OPT), the lemma implies that there exists an optimal non-negative weighted covering that is positive for every buyer and every item.

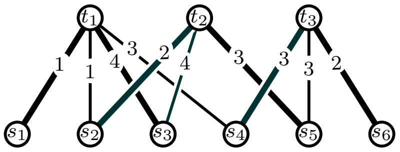
Feasible sets play a key role in the design of optimal dynamic pricing schemes. After the current buyer leaves, the associated bipartite graph is updated by deleting the vertices corresponding to the buyer and her bundle of items, and the prices are recomputed for the remaining items. It follows from the definitions that the scheme is optimal if and only if the prices are always set in such a way that any bundle of items maximizing the utility of an agent t forms a feasible set for t.

3.2 Adequate orderings

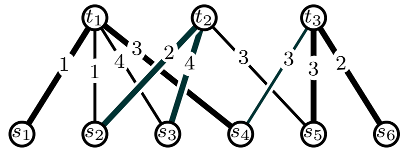
The high-level idea of our approach is as follows. First, we take an optimal non-negative weighted covering π provided by Lemma 4. If we define the price of an item $s \in S$ to be $\pi(s)$, then for any $t \in T$ we have $u_t(s) = v_t(s) - \pi(s) = w(st) - \pi(s) \le \pi(t)$ and, by Lemma 4(a), equality holds if and only if s is feasible for t. This means that each buyer prefers choosing items that are legal for her. For unit-demand valuations, such a solution immediately yields an optimal dynamic pricing scheme as explained in Section 4.1. However, when the demands are greater than one, a collection of legal items might not form a feasible set, see an example in Figure 1. In order to control the choices of the buyers, we slightly perturb the item prices by choosing an ordering $\sigma\colon S \to \{1, \dots, |S|\}$ and set the price of item s to be $\pi(s) + \delta \cdot \sigma(s)$ for some sufficiently small $\delta > 0$. Here the value of $\sigma(s)$ will be set in such a way that any bundle of items maximizing the utility of a buyer will form a feasible set for her, as needed.

Given a bipartite graph $G = (S, T; E)$ and upper bounds $b\colon V \to \mathbb{Z}_{>0}$ with $b(s) = 1$ for $s \in S$, we call an ordering $\sigma\colon S \to \{1, \dots, |S|\}$ *adequate for* G if it satisfies the following condition: for any $t \in T$, there exists a b-factor in G that matches t to its first $b(t)$ neighbors according to the ordering σ. For ease of notation, we introduce the *slack* of π to denote $\Delta(\pi) := \min\big\{\min\{\pi(t)+\pi(s)-w(st) \mid st \in E, st \text{ is not tight}\},\ \min\{\pi(v) \mid v \in V, \pi(v) > 0\}\big\}$, where the minimum over an empty set is defined to be $+\infty$. Using this terminology, the above idea is formalized in the following lemma.

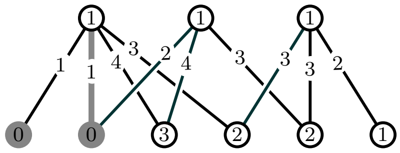
Lemma 5. *Assume that* (OPT) *is satisfied. Let $G = (S, T; E)$ be the edge-weighted bipartite graph associated with the market, π be a weighted covering provided by Lemma 4, and σ be an adequate ordering for G_π. For $\delta := \Delta(\pi)/(|S| + 1)$, setting the prices to $p(s) := \pi(s) + \delta \cdot \sigma(s)$ results in optimal dynamic prices.*



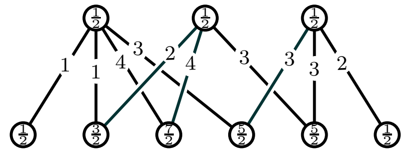
(a) Maximum weight b-matching $M_1 = \{t_1s_1, t_1s_3, t_2s_2, t_2s_5, t_3s_4, t_3s_6\}$.



(b) Maximum weight b-matching $M_2 = \{t_1s_1, t_1s_4, t_2s_2, t_2s_3, t_3s_5, t_3s_6\}$



(c) An optimal non-negative weighted covering π. Notice that s_1t_1 is tight but not legal, and $\pi(s_1) = \pi(s_2) = 0$ although $d_M(s_1) = d_M(s_2) = 1$ for every maximum weight b-matching.



(d) An optimal non-negative weighted covering satisfying the conditions of Lemma 4.

Figure 1: A bipartite graph corresponding to a market with three buyers having demand two and six items. The numbers denote the weights of the edges; all the remaining edges have weight 0. There are two maximum weight b-matchings M_1 (Figure 1a) and M_2 (Figure 1b). Notice that both s_3t_1 and s_4t_1 are legal, but they do not form a feasible set.

Proof. By (OPT), every optimal solution is a b-factor. Observe that for any $s \in S$ and $t \in T$, we have

$$\begin{aligned} u_t(s) &= v_t(s) - p(s) \\ &= w(st) - (\pi(s) + \delta \cdot \sigma(s)) \\ &\leq \pi(t) - \delta \cdot \sigma(s). \end{aligned}$$

Here equality holds if and only if st is tight with respect to π, in which case $u_t(s) = \pi(t) - \delta \cdot \sigma(s) > \pi(t) - \Delta(\pi) \cdot |S|/(|S|+1) > 0$ by the choice of δ and by Lemma 4(b). Furthermore, if st is tight and $s't$ is a non-tight edge of G, then $u_t(s') \leq \pi(t) - \Delta(\pi) \leq \pi(t) - \delta(|S|+1) < u_t(s)$ by the choice of δ. Concluding the above, we get that no matter which buyer arrives next, she strictly prefers legal items over non-legal ones, and legal items have strictly positive utility values for her. That is, she chooses the first $b(t)$ of its neighbors in G_π according to the ordering σ. As σ is adequate for G_π, the statement follows by Lemma 3(b). $\square$

It is worth emphasizing that the application of Lemma 5 provides optimal dynamic prices for a single round; the prices should be updated before the arrival of each buyer accordingly.

For a $\pi\colon V \to \mathbb{R}$ and $\sigma\colon S \to \{1, \dots, |S|\}$, the *combination* of π and σ is an ordering $\sigma'\colon S \to \{1, \dots, |S|\}$ that is obtained by pre-ordering the elements of S according to their π values in a non-decreasing order, and then items having the same π value are ordered according to σ. We denote the combination of π and σ by $\pi \circ \sigma$. The following technical lemma will be useful in the inductive proof.

Lemma 6. *Let $G = (S, T; E)$ be an edge-weighted bipartite graph with all edges having weight one, and $b\colon V \to \mathbb{Z}_{>0}$ be an upper bound function satisfying $b(s) = 1$ for $s \in S$ such that G admits a b-factor. Furthermore, let π be a weighted covering provided by Lemma 4, and σ be an adequate ordering for $G_\pi = (S, T; E_\pi)$. Then $\pi \circ \sigma$ is an adequate ordering for G.*

Proof. Let $\sigma' := \pi \circ \sigma$ denote the combination of π and σ. We claim that for any $t \in T$, the first $b(t)$ neighbors of t in G_π according to σ coincides with the first $b(t)$ neighbors of t in G according to σ'. Indeed, this follows from the fact that the edge-weights are identically 1, hence the value of $\pi(s)$ is exactly $1 - \pi(t)$ if $st \in E_\pi$ and strictly less if $st \in E \setminus E_\pi$. That is, in the ordering σ', the edges in E_π precede the edges in $E \setminus E_\pi$. As G admits a b-factor by assumption, t has at least $b(t)$ neighbors in G_π, and the lemma follows. □

4 Unit- and multi-demand markets

4.1 Unit-demand markets

Based on the primal-dual interpretation of the problem, first we give a simpler proof of a result of Cohen-Addad et al. [10] on unit-demand valuations as an illustration of our approach.

Theorem 4 (Cohen-Addad et al.)**.** *Every unit-demand market admits an optimal dynamic pricing that can be computed in polynomial time.*

Proof. Consider the bipartite graph associated with the market, take an optimal cover π provided by Lemma 4, and set the price of item s to be $\pi(s)$. For a pair of buyer $t \in T$ and $s \in S$, we have

$$\begin{aligned} u_t(s) &= v_t(s) - p(s) \\ &= w(st) - p(s) \\ &\leq (\pi(s) + \pi(t)) - \pi(s) \\ &= \pi(t). \end{aligned}$$

By Lemma 4(a), strict equality holds if and only if st is legal. We claim that no matter which buyer arrives next, she either chooses an item that is legal (and so forms a feasible set for her), or she takes none of the items and the empty set is feasible for her.

To see this, assume first that $\pi(t) > 0$. By Lemma 4(b), there exists at least one item legal for t, and those items are exactly the ones maximizing her utility. Now assume that $\pi(t) = 0$. By Lemma 4(b), the empty set is feasible for t. Furthermore, for any item $s \in S$, the utility $u_t(s)$ is negative unless s is legal for t, in which case $u_t(s) = 0$. Notice that a buyer may decide to take or not to take any item with zero utility value. However, she gets a feasible set in both cases by the above, thus concluding the proof. □

4.2 Multi-demand markets up to three buyers

The aim of the section is to settle the existence of optimal dynamic prices in multi-demand markets with a bounded number of buyers, under the assumption (OPT).

Theorem 5 (Berger et al.)**.** *Every multi-demand market with property* (OPT) *and at most three buyers admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. By Lemma 5, it suffices to show the existence of an adequate ordering for G_π, where π is an optimal non-negative weighted covering provided by Lemma 4. For a single buyer, the statement is meaningless. For two buyers t_1 and t_2, $|S| = b(t_1) + b(t_2)$ by assumption (OPT). Let σ be an ordering that starts with items in $N_{G_\pi}(t_1) \triangle N_{G_\pi}(t_2)$ and then puts the items in $N_{G_\pi}(t_1) \cap N_{G_\pi}(t_2)$

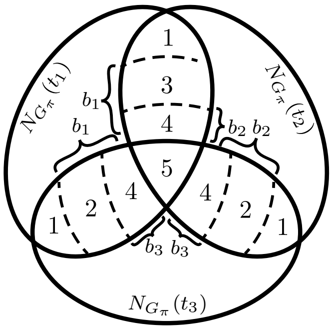


Figure 2: Definition of the labeling Θ for three buyers. Notice that some parts might be empty, e.g. if $|X_{12}| \leq b_2$, then there are no items with label 1 and 3 in the intersection of $N_{G_\pi}(t_1)$ and $N_{G_\pi}(t_2)$.

at the end of the ordering. Then, after the deletion of the first $b(t_i)$ neighbors of t_i according to σ, the remaining $b(t_{3-i})$ items are in $N_{G_\pi}(t_{3-i})$, hence σ is adequate.

Now we turn to the case of three buyers. Let t_1, t_2 and t_3 denote the buyers, and let b_i, v_i, and u_i denote the demand, valuation, and utility function corresponding to buyer t_i, respectively. Without loss of generality, we may assume that $b_1 \geq b_2 \geq b_3$. The proof is based on the observation that a set is feasible if and only if its deletion leaves 'enough' items for the remaining buyers, formalized as follows.

Claim 1. *A set $F \subseteq N_{G_\pi}(t_i)$ is feasible for t_i if and only if $|F| = b_i$ and $|N_{G_\pi}(t_j) - F| \geq b_j$ for $j \neq i$.*

Proof. The conditions are clearly necessary. To prove sufficiency, we show that the constraints of Lemma 2(a) are fulfilled after deleting t_i and F from G_π, that is, $|S - F| = b(T) - b_i$ and $|N_{G_\pi}(Y) - F| \geq b(Y)$ for $Y \subseteq T - t_i$. By (OPT) and the assumption that every item is legal for at least two buyers, $|S - F| = b(T) - b_i$ holds for $Y = T - t_i$. Furthermore, one-element subsets have enough neighbors by assumption, and the claim follows. ☐

For $I \subseteq \{1,2,3\}$, let $X_I \subseteq S$ denote the set of items that are legal exactly for buyers with indices in I, that is, $X_I \coloneqq \big(\bigcap_{i\in I} N_{G_\pi}(t_i)\big) - \big(\bigcup_{i\notin I} N_{G_\pi}(t_i)\big)$. We may assume that $X_1 = X_2 = X_3 = \emptyset$. Indeed, given an adequate ordering for $G_\pi - (X_1 \cup X_2 \cup X_3)$ where the demands of t_i is changed to $b_i - |X_i|$ for $i \in \{1,2,3\}$, putting the items in $X_1 \cup X_2 \cup X_3$ at the beginning of the ordering results in an adequate solution for the original instance.

By assumption, $|X_{12}| + |X_{13}| + |X_{23}| + |X_{123}| = b_1 + b_2 + b_3$. Furthermore, $|X_{ij}| \leq b_i + b_j$ holds for $i \neq j$, as otherwise in any allocation there exists an item that is legal only for t_i and t_j but is not allocated to any of them, contradicting (OPT). We first define a labeling $\Theta\colon S \to \{1,2,3,4,5\}$ so that for each buyer i and set X_{ij}, the number of items in X_{ij} with label at most $4 - i$ is $\max\{0, |X_{ij}| - b_j\}$. We will make sure that each buyer i selects all items with label at most $4 - i$ that are legal for her, which will be the key to satisfy the constraints of Claim 1, see Figure 2.

All the items in X_{123} are labeled by 5. If $|X_{12}| \leq b_2$, then all the items in X_{12} are labeled by 4. If $b_1 \geq |X_{12}| > b_2$, then b_2 items are labeled by 4 and the remaining $|X_{12}| - b_2$ items are labeled by 3 in X_{12}. If $|X_{12}| > b_1$, b_2 items are labeled by 4, $b_1 - b_2$ items are labeled by 3, and the remaining $|X_{12}| - b_1$ items are labeled by 1 in X_{12}. We proceed with X_{13} analogously. If $|X_{13}| \leq b_3$, then all the items in X_{13} are labeled by 4. If $b_1 \geq |X_{13}| > b_3$, then b_3 items are labeled by 4 and the remaining $|X_{13}| - b_3$ items are labeled by 2 in X_{13}. If $|X_{13}| > b_1$, b_3 items are labeled by 4, $b_1 - b_3$

items are labeled by 2, and the remaining $|X_{13}| - b_1$ items are labeled by 1 in X_{13}. Similarly, if $|X_{23}| \leq b_3$, then all the items in X_{23} are labeled by 4. If $b_2 \geq |X_{23}| > b_3$, then b_3 items are labeled by 4 and the remaining $|X_{23}| - b_3$ items are labeled by 2 in X_{23}. If $|X_{23}| > b_2$, then b_3 items are labeled by 4, $b_2 - b_3$ items are labeled by 2, and the remaining $|X_{23}| - b_2$ items are labeled by 1 in X_{23}.

Now let σ be any ordering of the items satisfying the following condition: if the label of item s_1 is strictly less than that of item s_2, then s_1 precedes s_2 in the ordering, that is, $\Theta(s_1) < \Theta(s_2)$ implies $\sigma(s_1) < \sigma(s_2)$. We claim that σ is adequate for G_π. To see this, it suffices to verify that the set F of the first $b(t_i)$ neighbors of t_i according to σ fulfills the requirements of Claim 1 for $i = 1, 2, 3$. Let $\{i, j, k\} = \{1, 2, 3\}$. First we show that F contains all the items $s \in X_{ij} \cup X_{ik}$ with $\Theta(s) \leq 4 - i$.

Claim 2. *We have* $|\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4 - i\}| \leq b_i$.

Proof. Suppose to the contrary that this does not hold. Then $b_i < \max\{0, |X_{ij}| - b_j\} + \max\{0, |X_{ik}| - b_k\}$ by the definition of the labeling. Since $|X_{ij}| \leq b_i + b_j$ and $|X_{ik}| \leq b_i + b_k$, we have $\max\{0, |X_{ij}| - b_j\} \leq b_i$ and $\max\{0, |X_{ik}| - b_k\} \leq b_i$. Therefore, if $b_i < \max\{0, |X_{ij}| - b_j\} + \max\{0, |X_{ik}| - b_k\}$, then both maximums must be positive on the right hand side. However, this leads to $b_i + b_j + b_k < |X_{ij}| + |X_{ik}|$, contradicting $b_i + b_j + b_k = |X_{ij}| + |X_{ik}| + |X_{jk}| + |X_{ijk}|$. ☐

By Claim 2, F contains all the items $s \in X_{ij} \cup X_{ik}$ with $\Theta(s) \leq 4 - i$, we have $|X_{ij} - F| \leq b_j$ and $|X_{ik} - F| \leq b_k$. Thus we get

$$
\begin{aligned}
|N_{G_\pi}(t_j) - F| &= |X_{ij} - F| + |X_{jk}| + |X_{ijk} - F| \\
&= |S| - |X_{ik} - F| - |F| \\
&\geq (b_i + b_j + b_k) - b_k - b_i \\
&= b_j.
\end{aligned}
$$

An analogous computation shows that $|N_{G_\pi}(t_k) - F| \geq b_k$. That is, F is indeed a feasible set for t_i, concluding the proof of the theorem. ☐

5 Bi-demand markets

This section is devoted to the proof of the main result of this work, the existence of optimal dynamic prices in bi-demand markets. The algorithm aims at identifying subsets of buyers whose neighboring set in G_π is 'small', meaning that other buyers should take no or at most one item from it. If no such set exists, then an adequate ordering is easy to find. Otherwise, by examining the structure of problematic sets, the problem is reduced to smaller instances.

Theorem 6. *Every bi-demand market with property* (OPT) *admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. Let $G = (S, T; E)$ and w be the bipartite graph and weight function associated with the market. Take an optimal non-negative weighted covering π of w provided by Lemma 4, and consider the subgraph $G_\pi = (S, T; E_\pi)$ of tight edges. For simplicity, we call a subset $M \subseteq E_\pi$ a $(1, 2)$*-factor* if $d_M(s) = 1$ for every $s \in S$ and $d_M(t) = 2$ for every $t \in T$. By (OPT) and Lemmas 1 and 3, there is a one-to-one correspondence between optimal allocations and $(1, 2)$-factors of G_π. Therefore, by Lemma 5, it suffices to show the existence of an adequate ordering σ for G_π.

We prove by induction on $|T|$. The statement clearly holds when $|T| = 1$, hence we assume that $|T| \geq 2$. As there exists a $(1,2)$-factor in G_π, we have $|N_{G_\pi}(Y)| \geq 2|Y|$ for every $Y \subseteq T$ by Lemma 2(a). We distinguish three cases.

Case 1. $|N_{G_\pi}(Y)| \geq 2|Y| + 2$ for every $\emptyset \neq Y \subsetneq T$.

For any $t \in T$ and $s_1, s_2 \in N_{G_\pi}(t)$, the graph $G_\pi - \{s_1, s_2, t\}$ still satisfies the conditions of Lemma 2(a), hence $\{s_1, s_2\}$ is feasible for t. Therefore, σ can be chosen arbitrarily.

Case 2. $|N_{G_\pi}(Y)| \geq 2|Y| + 1$ for $\emptyset \neq Y \subsetneq T$ and there exists Y for which equality holds.

We call a set $Y \subseteq T$ *dangerous* if $|N_{G_\pi}(Y)| = 2|Y|+1$. By Lemma 2(a), a pair $\{s_1, s_2\} \subseteq N_{G_\pi}(t)$ is not feasible for buyer t if and only if there exists a dangerous set $Y \subseteq T - t$ with $s_1, s_2 \in N_{G_\pi}(Y)$. In such case, we say that Y *belongs to buyer* t. Notice that the same dangerous set might belong to several buyers.

Claim 3. *Assume that Y_1 and Y_2 are dangerous sets with $Y_1 \cup Y_2 \subsetneq T$.*

(a) If $Y_1 \cap Y_2 = \emptyset$ and $N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2) \neq \emptyset$, then $|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| = 1$ and $Y_1 \cup Y_2$ is dangerous.

(b) If $Y_1 \cap Y_2 \neq \emptyset$, then both $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ are dangerous.

Proof. Observe that

$$\begin{aligned}(2|Y_1| + 1) + (2|Y_2| + 1) &= |N_{G_\pi}(Y_1)| + |N_{G_\pi}(Y_2)| \\ &= |N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| + |N_{G_\pi}(Y_1) \cup N_{G_\pi}(Y_2)| \\ &= |N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| + |N_{G_\pi}(Y_1 \cup Y_2)|.\end{aligned}$$

Assume first that $Y_1 \cap Y_2 = \emptyset$. Then $|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| \leq 1$ as otherwise $|N_{G_\pi}(Y_1 \cup Y_2)| \leq 2(|Y_1| + |Y_2|) = 2|Y_1 \cup Y_2|$, contradicting the assumption of Case 2. If $|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| = 1$, then $|N_{G_\pi}(Y_1 \cup Y_2)| = 2|Y_1 \cup Y_2| + 1$ and so $Y_1 \cup Y_2$ is dangerous.

Now consider the case when $Y_1 \cap Y_2 \neq \emptyset$. Then

$$\begin{aligned}|N_{G_\pi}(Y_1) \cap N_{G_\pi}(Y_2)| + |N_{G_\pi}(Y_1 \cup Y_2)| &\geq |N_{G_\pi}(Y_1 \cap Y_2)| + |N_{G_\pi}(Y_1 \cup Y_2)| \\ &\geq (2|Y_1 \cap Y_2| + 1) + (2|Y_1 \cup Y_2| + 1) \\ &= (2|Y_1| + 1) + (2|Y_2| + 1).\end{aligned}$$

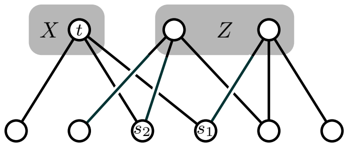
Therefore, we have equality throughout, implying that both $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ are dangerous. □

Let $Z \subseteq T$ be an inclusionwise maximal dangerous set.

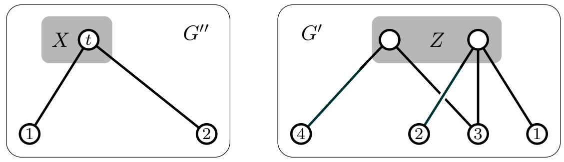
Subcase 2.1. There is no dangerous set disjoint from Z.

First we show that if a pair $s_1, s_2 \in N_{G_\pi}(t)$ is not feasible for a buyer $t \in T - Z$, then $s_1, s_2 \in N_{G_\pi}(Z)$. Indeed, if $\{s_1, s_2\}$ is not feasible for t, then there is a dangerous set X belonging to t with $s_1, s_2 \in N_{G_\pi}(X)$. Since $t \notin X \cup Z$ and $Z \cap X \neq \emptyset$ by the assumption of the subcase, Claim 3(b) applies implying that $X \cup Z$ is dangerous as well. The maximal choice of Z implies $X \cup Z = Z$, hence Z belongs to t and $s_1, s_2 \in N_{G_\pi}(Z)$.

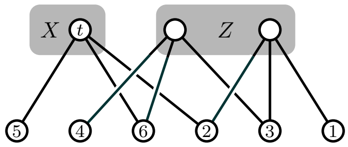
Now take an arbitrary buyer $t_0 \in T - Z$ who shares a neighbor with Z, and let $s_0 \in N_{G_\pi}(t_0) \cap N_{G_\pi}(Z)$. Let σ' be an arbitrary ordering of the items in $S - N_{G_\pi}(Z)$. Furthermore, let $G'' := G_\pi[Z \cup N_{G_\pi}(Z)] - s_0$. For any $(1,2)$-factor of G_π containing s_0t_0, its restriction to G'' is a $(1,2)$-factor as well. In G'', some of the edges might not be contained in any of the $(1,2)$-factors. Still, by induction and Lemma 6, there exists an adequate ordering σ'' of the items in G''. Finally, let



(a) The graph of tight edges corresponding to the instance in Figure 1, where Z is an inclusionwise maximal dangerous set, and X is an inclusionwise minimal dangerous set disjoint from Z.



(b) The graphs $G' = G_\pi - (X \cup (N_{G_\pi}(X) - s_1))$ and $G'' = G_\pi - (Z \cup (N_{G_\pi}(Z) - s_1))$, together with an adequate ordering σ' and an arbitrary ordering σ'', respectively.



(c) Construction of the ordering $\sigma = (\sigma', \sigma''|_{N_{G_\pi}(X)-s_2}, \sigma''')$, where σ''' is the trivial ordering of the one element set $\{s_2\}$.

Figure 3: An illustration of the inductive step in Subcase 2.2.2.

σ''' denote the trivial ordering of the single element set $\{s_0\}$. We set $\sigma \coloneqq (\sigma', \sigma'', \sigma''')$. Then any buyer $t \in T - Z$ will choose at most one item from $N_{G_\pi}(Z)$, hence the adequateness of σ follows from that of σ'' and the assumption of the subcase.

Subcase 2.2. There exists a dangerous set disjoint from Z.

Let X be an inclusionwise minimal dangerous set disjoint from Z.

Subcase 2.2.1. For any $t \in X$ and for any $s_1, s_2 \in N_{G_\pi}(t)$, the set $\{s_1, s_2\}$ is feasible.

Take an arbitrary buyer $t_0 \in T - X$ who shares a neighbor with X and let $s_0 \in N_{G_\pi}(t_0) \cap N_{G_\pi}(X)$. Let G' denote the graph obtained by deleting $X \cup (N_{G_\pi}(X) - s_0)$. For any $(1,2)$-factor of G_π containing s_0t_0, its restriction to G' is a $(1,2)$-factor as well. In G', some of the edges might not be contained in any of the $(1,2)$-factors. Still, by induction and Lemma 6, there exists an adequate ordering σ' of the items in G'. Let σ'' be an arbitrary ordering of the items in $N_{G_\pi}(X) - s_0$, and define $\sigma \coloneqq (\sigma', \sigma'')$. Then t_0 chooses at most one item from $N_{G_\pi}(X)$ (namely s_0), since she has at least one neighbor outside of $N_{G_\pi}(X)$ and those items have smaller indices in the ordering. Thus the adequateness of σ follows from that of σ' and from the assumption that any pair $s_1, s_2 \in N_{G_\pi}(t)$ form a feasible set for $t \in X$.

Subcase 2.2.2. There exists $t_0 \in X$ and $s_1, s_2 \in N_{G_\pi}(t)$ such that $\{s_1, s_2\}$ is not feasible.

The following claim is the key observation of the proof.

Claim 4. *$X \cup Z = T$ and $N_{G_\pi}(X) \cap N_{G_\pi}(Z) = \{s_1, s_2\}$.*

Proof. Let $Y \subseteq T - t_0$ be a dangerous set with $s_1, s_2 \in N_{G_\pi}(t_0)$. As $t_0 \in T - (Z \cup Y)$ and Z is inclusionwise maximal, either $Y \subseteq Z$ or $Y \cap Z = \emptyset$ by Claim 3(b). In the latter case, X and Y are dangerous sets with $X \cup Y \subsetneq T$. Furthermore, $|N_{G_\pi}(X) \cap N_{G_\pi}(Y)| \geq 2$ since s_1 and s_2 are contained in both. Hence, by Claim 3(a), $X \cap Y \neq \emptyset$. But then $X \cap Y$ is dangerous by Claim 3(b), contradicting the minimality of X. Therefore, we have $Y \subseteq Z$. By Claim 3(a), $X \cup Z = T$. As $|N_{G_\pi}(X)| = 2|X| + 1$, $|N_{G_\pi}(Z) = 2|Z| + 1$, and $|S| = 2|T| = 2|T| + 2|Z|$, the claim follows. $\square$

Let G' and G'' denote the graphs obtained by deleting $X \cup (N_{G_\pi}(X) - s_1)$ and $Z \cup (N_{G_\pi}(Z) - s_1)$, respectively, see Figure 3. For any $(1,2)$-factor of G_π containing $t_0 s_2$, its restriction to G' is a $(1,2)$-factor as well. In G', some of the edges might not be contained in any of the $(1,2)$-factors. Still, by induction and Lemma 6, there exists an adequate ordering σ' of the items in G'. Let σ'' be an arbitrary ordering of the items in $N_{G_\pi}(X) - s_2$. Finally, let σ''' denote the trivial ordering of the single element set $\{s_2\}$. Let $\sigma := (\sigma', \sigma''|_{N_{G_\pi}(X)-s_1}, \sigma''')$. We claim that σ is adequate. Indeed, if a buyer $t \in Z$ arrives first, then she chooses two items from $N_{G_\pi}(Z) - s_2$ according to σ'. As σ' is adequate for G' and $G'' - s_1 + s_2$ has a $(1,2)$-factor, the remaining graph has a $(1,2)$-factor as well. If a buyer $t \in X$ arrives first, then she chooses two items from $N_{G_\pi}(X) - s_2$ that form a feasible set, since the only pair that might not be feasible for her is $\{s_1, s_2\}$ by Claim 4.

Case 3. $|N_{G_\pi}(T')| = 2|T'|$ for some $\emptyset \neq T' \subsetneq T$.

We claim that there exists a set T' satisfying the assumption if and only if G_π is not connected. Indeed, if G_π is not connected, then necessarily the number of items is exactly twice the number of buyers in every component as the graph is supposed to have a $(1,2)$-factor. To see the other direction, let $S' := N_{G_\pi}(T')$, $T'' := T - T'$, $S'' := S - S'$, and consider the subgraphs $G' := G_\pi[T' \cup S']$ and $G'' := G_\pi[T'' \cup S'']$. As every tight edge is legal and all the vertices in S' are matched to vertices in T' in any optimum b-matching, G_π contains no edges between T'' and S'. Therefore, G_π is not connected, and it is the union of G' and G''. By induction, there exist adequate orderings σ' and σ'' of S' and S'', respectively. Then the ordering $\sigma := (\sigma', \sigma'')$ is adequate with respect to π.

By Lemma 4, π can be determined in polynomial time, hence the graph of tight edges is available. The algorithm for determining an adequate ordering for G_π is presented as Algorithm 1. To see that all steps can be performed in polynomial time, it suffices to show how to decide whether a pair $\{s_1, s_2\}$ of items forms a feasible set for a buyer t, and how to find an inclusionwise maximal or minimal dangerous set, if exists, efficiently. Checking the feasibility of $\{s_1, s_2\}$ for t reduces to finding a $(1,2)$-factor in $G_\pi - \{s_1, s_2, t\}$. Dangerous sets can be found as follows: take two copies of each vertex $t \in T$, and connect them to the vertices in $N_{G_\pi}(t)$. Furthermore, add a dummy vertex w_0 to the graph and connect it to every vertex in S. Let $G' = (S', T'; E')$ denote the graph thus obtained. For a set $Y \subseteq T$, let $Y' \subseteq T'$ consist of the copies of the vertices in Y plus the vertex w_0. It is not difficult to check that $Y \subseteq T$ is an inclusionwise minimal or maximal dangerous set of G_π if and only if Y' is an inclusionwise minimal or maximal subset of T' with $|N_{G'}(Y')| = |Y'|$. Hence Y can be determined, for example, by relying on Kőnig's alternating path algorithm [23]. When an inclusionwise minimal dangerous set X is needed that is disjoint from Z, then the same approach can be applied for the graph $G_\pi - Z$. □

Remark 7. Theorem 6 settles the existence of optimal dynamic prices when the demand of each buyer is exactly two. However, the proof can be straightforwardly extended to the case when the demand of each buyer is at most two.

Algorithm 1 Determining an adequate ordering for bi-demand markets with property (OPT).

Input: Graph G_π of tight edges, upper bounds $b(t) = 2$ for $t \in T$ and $b(s) = 1$ for $s \in S$.
Output: Adequate ordering σ of the items.

if $|N_{G_\pi}(Y)| \geq 2|Y| + 2$ for every $\emptyset \neq Y \subsetneq T$ **then**
  Let σ be an arbitrary ordering of S.
else if $|N_{G_\pi}(Y)| \geq 2|Y| + 1$ for every $\emptyset \neq Y \subsetneq T$, and there exists Y for which equality holds **then**
  Determine an inclusionwise maximal dangerous set Z.
  if there exists no dangerous set disjoint from Z **then**
    Take an item $s_0 \in N_{G_\pi}(Z)$ that has a neighbor $t_0 \in T - Z$.
    Let σ' be an arbitrary ordering of $S - N_{G_\pi}(Z)$.
    Determine an adequate ordering σ'' for $G'' := G_\pi[Z \cup (N_{G_\pi}(Z) - s_0)]$.
    Let σ''' be the trivial ordering of the single item s_0.
    Set $\sigma := (\sigma', \sigma'', \sigma''')$.
  else
    Determine an inclusionwise minimal dangerous set X disjoint from Z.
    if $\{s_1, s_2\}$ is feasible for any $t \in X$ and $s_1, s_2 \in N_{G_\pi}(t)$ **then**
      Take item $s_0 \in N_{G_\pi}(X)$ that has a neighbor $t_0 \in T - X$.
      Determine an adequate ordering σ' for $G' := G_\pi - (X \cup (N_{G_\pi}(X) - s_0))$.
      Let σ'' be an arbitrary ordering of $N_{G_\pi}(X) - s_0$.
      Set $\sigma := (\sigma', \sigma'')$.
    else (Observation: $X \cup Z = T$ and $N_{G_\pi}(X) \cap N_{G_\pi}(Z) = \{s_1, s_2\}$.)
      Determine an adequate ordering σ' for $G' := G_\pi - (X \cup (N_{G_\pi}(X) - s_1))$.
      Let σ'' be an arbitrary ordering of the items in $G'' := G_\pi - (Z \cup (N_{G_\pi}(Z) - s_1))$.
      Let σ''' be the trivial ordering of the single item s_2.
      Set $\sigma := (\sigma', \sigma''|_{N_{G_\pi}(X) - s_1}, \sigma''')$.
else (Observation: the graph G_π is not connected.)
  Let $\emptyset \neq T' \subsetneq T$ be a set with $|N_{G_\pi}(T')| = 2|T'|$.
  Determine an adequate ordering σ' for $G' := G_\pi[T' \cup N_{G_\pi}(T')]$.
  Determine an adequate ordering σ'' for in $G'' := G_\pi - (T' \cup N_{G_\pi}(T'))$.
  Set $\sigma := (\sigma', \sigma'')$.
return σ

6 Markets where property (OPT) is not satisfied

Our goal is to give optimal dynamic pricing schemes for multi-demand markets with at most three buyers and for bi-demand markets, without assuming (OPT). In both cases, the proof is based on the following idea: We add small valued dummy items to the market so that (OPT) is satisfied, then we determine optimal dynamic prices for the modified market, and show that the same prices are optimal for the original instance as well.

Formally, consider a market for which (OPT) does not hold, that is, the number of items is less than the total demand of the buyers. Let $G = (S, T; E)$ be the bipartite graph associated with the market, and take a minimum weighted covering π provided by Lemma 4. For ease of discussion, let us denote the set of buyers who might receive fewer items than their demand in an optimal solution by

$$\hat{T} := \{t \in T \mid d_M(t) < b(t) \text{ for some maximum weight } b\text{-matching } M\}.$$

We call the items in S *real*. Now extend the graph by adding a set $\hat{S}$ of $b(T) - |S|$ *dummy items*; we refer to edges going between these items and buyers as *dummy edges*. We define the value of a dummy item (and so the weight of the corresponding dummy edge) to be 2ε for each buyer, where $\varepsilon := 1/4 \cdot \Delta(\pi)$. By Lemma 4(b) and the assumption that every item is used in every optimal allocation, ε is strictly positive, hence the modified instance satisfies (OPT). Let $G^+ = (S^+, T, E^+)$ and w^+ denote the graph and weight function thus obtained, respectively. It is not difficult to check that the maximum weight b-factors of G^+ are exactly those that can be obtained from a maximum weight b-matching of G by adding $|\hat{S}|$ dummy edges.

Lemma 7. *There exists a minimum weighted covering π^+ of w^+ such that*

(a) $\pi^+(t) = \varepsilon$ for each $t \in \hat{T}$,

(b) $\pi^+(t) > \varepsilon$ for each $t \in T - \hat{T}$, and

(c) if $t \in \hat{T}$, $s \in S$ and st is not legal, then $w(st) - \pi^+(s) < 0$.

Furthermore, such a π^+ can be determined in polynomial time.

Proof. Let π^+ be an extension of π by setting $\pi^+(\hat{s}) := 2\varepsilon$ for $\hat{s} \in \hat{S}$. It is not difficult to check that π^+ is a weighted covering of w^+. Furthermore, as the total value of π^+ equals the total value of π plus $2\varepsilon|\hat{S}|$ which is exactly the difference between the maximum weight of a b-factor in G^+ and the maximum weight of a b-matching in G, π^+ is a minimum weighted covering.

Now increase $\pi^+(t)$ by ε for $t \in T$ and decrease $\pi^+(s)$ by ε for $s \in S^+$. As (OPT) holds for the modified instance, the total value of π^+ does not change, hence it remains a minimum weighted covering. By Lemma 4(b), $\pi(t) = 0$ for $t \in \hat{T}$ and $\pi(t) > 0$ otherwise. Furthermore, by the assumption that every item is used in every optimal allocation, $\pi(s) > 0$ for $s \in S$. These together show that π^+ satisfies (a) and (b).

By Lemma 4(a), for every $st \in E$ such that $t \in \hat{T}$, $s \in S$, and st is not legal, we have $w(st) - \pi(s) < \pi(t) = 0$, therefore $w(st) - \pi^+(s) < 0$ by the choice of ϵ. This proves the last part of the claim. ☐

6.1 Multi-demand markets up to three buyers

Theorem 8 (Berger et al.). *Every multi-demand market with at most three buyers admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. For a single buyer, the statement is meaningless.

For two buyers t_1 and t_2, if the dummy items are in $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$, labelling items in $N_{G^+_{\pi^+}}(t_1) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$ and $N_{G^+_{\pi^+}}(t_2) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$ by 1 and items in $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ by 2 results in optimal allocations, because for $i = 1, 2$, buyer t_i has positive utility for all real items in $N_{G^+_{\pi^+}}(t_i)$, negative utility for items not in $N_{G^+_{\pi^+}}(t_i)$, and she prefers items in $N_{G^+_{\pi^+}}(t_i) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$. If the dummy items are in, say, $N_{G^+_{\pi^+}}(t_1) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$, we chose $\max\{0, b_2 - |N_{G^+_{\pi^+}}(t_2) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))|\}$ items from $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ and increase their prices by ε. This way, t_2 gets all items which are legal only for her and she gets $\max\{0, b_2 - |N_{G^+_{\pi^+}}(t_2) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))|\}$ items from $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ as her utility is still positive for them. Buyer t_1 takes real items in $N_{G^+_{\pi^+}}(t_1) - (N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2))$ and the items in $N_{G^+_{\pi^+}}(t_1) \cap N_{G^+_{\pi^+}}(t_2)$ whose prices remained unchanged.

Now we turn to the case of three buyers. Add dummy items to the instance as described before, and let π^+ be a minimum weighted covering provided by Lemma 7. Let t_1, t_2 and t_3 denote the buyers, and let b_i, v_i, and u_i denote the demand, valuation, and utility function corresponding to buyer t_i, respectively. For $I \subseteq \{1,2,3\}$, let $X_I \subseteq S^+$ denote the set of items that are legal exactly for buyers with indices in I, that is, $X_I := \big(\bigcap_{i\in I} N_{G^+_{\pi^+}}(t_i)\big) - \big(\bigcup_{i\notin I} N_{G^+_{\pi^+}}(t_i)\big)$. Without loss of generality, we may assume that $b_1 \geq b_2 \geq b_3$. However, unlike before, we cannot assume $X_1 = X_2 = X_3 = \emptyset$ due to the presence of dummy items.

Similarly to the case when property (OPT) holds, we define a labeling $\Theta : S^+ \to \{1,2,3,4,5\}$ such that any b_i items with the smallest labels in $N_{G^+_{\pi^+}}(t_i)$ form a feasible set for t_i. That is, for an appropriately small $\delta > 0$, setting the prices to $\pi^+(s) + \delta \cdot \Theta(s)$ for each item s where $\delta := \Delta(\pi^+)/(|S|+1)$, results in optimal dynamic prices for the modified instance. Unfortunately, when the prices are restricted to the set of original items, optimality might not be met due to the absence of dummy items. This is because a buyer might replace the missing dummy items with real items that she did not take before, which results in a suboptimal solution. To resolve this, as in the bi-demand case, we further increase the prices by ε to ensure that buyers have negative utility from items they should not choose. Notice that Observation 9 holds again.

We have seen, when the market satisfy the property (OPT) and $X_1 = X_2 = X_3 = \emptyset$, it is enough to ensure $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\} \subseteq F$ for $\{i,j,k\} = \{1,2,3\}$, where F be a set of b_i items with the largest utility for t_i. Now, if there are $\hat{s}$ dummy items with label $\Theta(\hat{s}) \leq 4-i$, t_i simply skips them, so we also have to ensure she does not take too much real items with label greater than $4-i$. If some X_i ($i \in \{1,2,3\}$) is not empty, but it contains only real items, if we label them by 0, t_i always buys them, therefore we can reduce the problem to the case when X_i is empty and the demand of t_i is $b_i - |X_i|$. If some X_i contains dummy items, the reduction will be more difficult. The following claim shows how the conditions for the feasible sets change when $X_1 = X_2 = X_3 = \emptyset$:

Claim 5. *Assume $X_1 = X_2 = X_3 = \emptyset$. Let $i \in \{1,2,3\}$ and let F be the following set: if t_i has at positive utility for at least b_i items in $N_{G^+_{\pi^+}}(t_i)$, F is the set of the first b_i items with the largest utility. If t_i has at positive utility for less than b_i items in $N_{G^+_{\pi^+}}(t_i)$, F contains all of them. If*

(a) F contains all real items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$,

(b) The difference $b_i - |F|$ is at least the number of dummy items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$,

(c) The difference $b_i - |F|$ is at most the number of dummy items in $N_{G^+_{\pi^+}}(t_i)$,

then F is feasible for t_i.

Proof. Let F be a set of items in $N_{G^+_{\pi^+}}(t_i)$ as stated above. We extend F with dummy items the following way: If there are dummy items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$, we add them to F. If the cardinality of the set we got this way is strictly less than b_i, we further extend it by adding dummy items from $N_{G^+_{\pi^+}}(t_i)$ with label at least 4 until the resulting sets cardinality becomes b_i. By 5(b) and 5(c), this can be achieved. Let F' denote the resulting set. Then $|F'| = b_i$ and F' contains all real and dummy items in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \leq 4-i\}$. We have $|X_{ij} - F'| \leq b_j$ and

$|X_{ik} - F'| \le b_k$. Thus we get

$$\begin{aligned} |N_{G^+_{\pi+}}(t_j) - F'| &= |X_{ij} - F'| + |X_{jk}| + |X_{ijk} - F'| \\ &= |S| - |X_{ik} - F'| - |F'| \\ &\ge (b_i + b_j + b_k) - b_k - b_i \\ &= b_j. \end{aligned}$$

An analogous computation shows that $|N_{G^+_{\pi+}}(t_k) - F'| \ge b_k$. Since we get F' by adding only dummy items to F, this proves the feasibility of F. □

We will apply a similar labeling procedure as when property OPT holds, then increase some prices by ε. As dummy items are completely equivalent, either none or all of them are legal for each buyer. We divide the proof into three cases based on whether dummy items are legal only for two or all three of the buyers.

Case 1. The dummy items are in X_i for some $i \in \{1, 2, 3\}$.

By Lemma 7, each buyer has positive utility from her legal real items and negative utility from her non-legal items.

If the number of dummy items is b_i, then all real items are non-legal for t_i, and her utility from real items is negative. Therefore we can apply the labeling procedure for the other two buyers. Otherwise, let b'_i denote the difference between b_i and the number of dummy items, that is, $b'_i := b_i - |\hat{S}|$. If there are real items in X_i, we label them by 0 and decrease b'_i by the number of real items in X_i. Regardless if there are real items in X_i or not, $b'_i = b_i - |X_i|$. We delete the dummy items from the graph and the real items from X_i, if there is any, then apply the labeling procedure for three buyers, but with b'_i in place of b_i. Now $|\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \le 4-i\}| \le b'_i$. We select $\max(0, b'_i - |\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \le 4-i\}|)$ items from $X_{ijk} \cup \{s \in X_{ij} \cup X_{ik} \mid \Theta(s) > 4-i\}$, starting with the items with lower labels, and leave their prices unchanged, while the prices of all other items in $X_{ijk} \cup \{s \in X_{ij} \cup X_{ik} \mid \Theta(s) > 4-i\}$ are increased by ε. This way we achieve that t_i has non-negative utility from exactly b'_i items. Despite the price increasing, t_j $(j \ne i)$ has positive utility for all items in $N_{G^+_{\pi+}}(t_j)$ and since we start the price increasing with the items with lower labels, the order of items in $N_{G^+_{\pi+}}(t_j)$ does not change, and the conditions of Claim 5 hold.

From now on, we can assume $X_1 = X_2 = X_3 = \emptyset$. Otherwise, we label the items in X_i by 0, delete them from the graph, and replace b_i by $b_i - |X_i|$.

Case 2. The dummy items are in X_{13}.

We apply a similar labeling procedure that we used when the market satisfies property (OPT). The items in X_{123} get label 5. As before, items in X_{ij} are labeled by 4, θ or 1, where $\theta = 3$ if $\{i, j\} = \{1, 2\}$, otherwise $\theta = 2$. However, dummy items are preferred to get higher labels. That is, we label as many dummy items by 4 as possible, and if the number of dummy items is more than the number of items to be labeled by 4, we proceed with labeling dummy items by 2, and then by 1 if necessary. The proof of Theorem **??** shows that t_i takes every item in $\{s \in X_{ij} \cup X_{ik} \mid \Theta(s) \le 4 - i\}$.

We distinguish three subcases:

Subcase 1. $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \le 2\}| > b_3$ and there is no item in X_{13} with label 1.

We do not change the prices in X_{23}. If $b_1 \le |X_{13}| + |\{s \in X_{12} \mid \Theta(s) \le 3\}|$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε, otherwise we select $b_1 - (|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \le 3\})$

items from $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$, starting with the ones in X_{12}. We leave the prices of the selected items unchanged, but we increase the prices of items in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ which were not selected by ε. This way, t_3 takes all items in X_{23} with label 1 (remember, there are no items in X_{13} with label 1) as they are real items. t_2 takes all items in X_{23} with label 1 and 2, since their prices were not increased. If $b_1 \leq |X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}|$, t_1 takes all items from $\{s \in X_{12} \mid \Theta(s) \leq 3\}$ as these are real items, and t_1 gets all real or dummy items in $\{s \in X_{12} \mid \Theta(s) \leq 2\}$, since we increased the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$. If $b_1 > |X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}|$, t_1 gets X_{13} and all items in $X_{12} \cup X_{123}$ with unchanged prices. In both cases, t_1 takes all real items in X_{12} with label 1 and 3, and she also takes all real items in X_{13} with label 2. Moreover, the difference between b_1 and the real items she takes is at least the number of dummy items in X_{13} with label 1 and 2. The way we increased some prices, we ensured conditions 5(b), 5(c) are fulfilled.

Subcase 2. $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| > b_3$ and exists an item in X_{13} with label 1.

As in the previous case, we need to ensure 5(b), 5(c) hold. In this case, $|\{s \in X_{13} \mid \Theta(s) > 1\}| = b_1$ and $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| > b_3$, which implies $|X_{12}| + |X_{123}| + |\{s \in X_{23} \mid \Theta(s) = 4\}| = b_2$, therefore there is no item in X_{12} with label 1 or 3. We increase the prices in $X_{12} \cup X_{123}$ by ε. We select $b_3 - |\{s \in X_{13} \mid \Theta(s) = 1\}| - |\{s \in X_{23} \mid \Theta(s) = 1\}|$ items from $\{s \in X_{23} \mid \Theta(s) = 2\}$ (the assumption $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| > b_3$ shows this can be done), and leave their prices unchanged, but we increase the prices of the remaining items in $\{s \in X_{23} \mid \Theta(s) = 2\}$ by ε, and we also increase the prices in $\{s \in X_{23} \mid \Theta(s) = 4\}$ by ε. This way, t_1 only takes items from X_{13}, which is enough, since there are no items in X_{12} with label 1 or 3. t_2 takes all items in X_{23} with label 1 and 2, and t_3 takes all real items in $X_{13} \cup X_{23}$ with label 1.

Subcase 3. $|\{s \in X_{13} \mid \Theta(s) = 1\}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| \leq b_3$.

In the case when property (OPT) holds, t_1 could choose freely from $\{s \in X_{12} \cup X_{13} \mid \Theta(s) = 4\}$ when $|\{s \in X_{12} \cup X_{13} \mid \Theta(s) \leq 3\}| < b_1$, now we will force her to buy as many items from $\{s \in X_{13} \mid \Theta(s) = 4\}$ as possible. We do this in the following way: if $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \geq b_1$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| < b_1$, we choose $b_1 - (|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}|)$ items from $\{s \in X_{12} \mid \Theta(s) = 4\}$, and if the items in $\{s \in X_{12} \mid \Theta(s) = 4\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ which were not chosen by ε. We do the same with t_3. If we have to choose items from X_{123}, we start with the items which are chosen because of t_1, if there are any. If there is no chosen item because of t_1 or we have to choose more, we choose from the items with increased price, but we decrease their price by ε. First, we check the case when the first buyer is t_1, and assume we increased the price of all items in X_{123}. If $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \geq b_1$, then t_1 has negative utility for the items not in $X_{13} \cup \{s \in X_{12} \mid \Theta(s) \leq 3\}$. Since $|\{s \in X_{13} \cup X_{12} \mid \Theta(s) \leq 3\}| \leq b_1$, t_1 takes all real items in $\{s \in X_{13} \mid \Theta(s) \leq 3\}$ and she also takes real items in $\{s \in X_{13} \mid \Theta(s) \leq 2\}$. By the price increasing, 5(b) and 5(c) also hold. If $|X_{13}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| < b_1$, t_1 gets all real items in $X_{13} \cup \{s \in X_{12} \mid \Theta(s) \leq 3\}$ and the items in $\{s \in X_{12} \mid \Theta(s) = 4\}$ whose price were not changed. One can verify that 5(b) and 5(c) hold again. If there are items in X_{123} with unchanged prices, and the number of them is $b_1 - |X_{12}| - |X_{13}|$, t_1 gets all real items in X_{13}, X_{12} and the items in X_{123} whose price were not changed. 5(b) and 5(c) hold again. The remaining case is when exists at least one item in X_{123} whose price was not changed and the number of items in X_{123} with unchanged prices is greater than $b_1 - |X_{12}| - |X_{13}|$. That means we left their prices unchanged because of t_3, that is $|X_{13}| + |X_{23}| < b_3$. That also means there is no item in X_{13} with label 1 or 2. It is not difficult to see that t_1 takes all items in $\{s \in X_{12} \mid \Theta(s) \leq 3\}$. Secondly, if the first buyer is t_3, the reasoning goes the same way as with t_1: if we increased the price of all items in X_{123} and $|X_{13}| + |\{s \in X_{23} \mid \Theta(s) = 1\}| \geq b_3$, then t_3 has negative utility for the items not in $X_{13} \cup \{s \in X_{23} \mid \Theta(s) = 1\}$. Since $|\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}| \leq b_3$, t_3 takes all real items in $\{s \in X_{13} \mid \Theta(s) = 1\}$ and she also gets $\{s \in X_{23} \mid \Theta(s) = 1\}$. If $|X_{13}| + |\{s \in X_{23} \mid \Theta(s) = 1\}| < b_3$, t_3 buys all real items in $X_{13} \cup \{s \in X_{23} \mid \Theta(s) = 1\}$ and the items in $\{s \in X_{23} \mid \Theta(s) > 1\}$ whose price were not changed. 5(b) and 5(c) holds again. If there are items in X_{123} with unchanged prices, and the number of them is $b_3 - |X_{13}| - |X_{23}|$, t_3 buys all real items in X_{13}, X_{23} and the items in X_{123} whose price were not changed. When exists at least one item in X_{123} whose price was not changed and the number of items in X_{123} with unchanged prices is greater than $b_3 - |X_{13}| - |X_{23}|$, $|X_{12}| + |X_{13}| < b_1$ holds. That means there is no item in X_{13} with label 1. It is not difficult to see that t_3 gets all items in $\{s \in X_{23} \mid \Theta(s) = 1\}$. It is not difficult to check 5(b) and 5(c) holds. Finally, if the first buyer is t_2, she gets all items in $\{s \in X_{23} \mid \Theta(s) \leq 2\} \cup \{s \in X_{12} \mid \Theta(s) = 1\}$ as we only increased prices in $\{s \in X_{12} \cup X_{23} \cup X_{123} \mid \Theta(s) \geq 4\}$.

Case 3. The dummy items are in X_{12}.

The initial labeling procedure is the same as in Case 2, then we increase some of the prices. First, we want to ensure 5(b), 5(c) holds if t_1 is the first buyer. We do this the following way: if $|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| \geq b_1$, we increase the prices in $\{s \in X_{13} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| < b_1$, we choose $b_1 - (|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}|)$ items from $\{s \in X_{13} \mid \Theta(s) = 4\}$, and if the items in $\{s \in X_{13} \mid \Theta(s) = 4\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{13} \mid \Theta(s) = 4\} \cup X_{123}$ which were not chosen by ε. We proceed similarly with t_2 instead of t_1 to ensure 5(b), 5(c) holds if she is the first buyer in the market. If $|X_{12}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| \geq b_2$, we increase the prices in $\{s \in X_{23} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{12}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}| < b_2$, we choose $b_2 - (|X_{12}| + |\{s \in X_{23} \mid \Theta(s) \leq 2\}|)$ items from $\{s \in X_{23} \mid \Theta(s) = 4\}$, and if the items in $\{s \in X_{23} \mid \Theta(s) = 4\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{23} \mid \Theta(s) = 4\} \cup X_{123}$ which were not chosen by ε. If we have to choose items from X_{123}, we start with the items which are chosen because of t_1, if there is any. If there is no chosen item because of t_1 or we have to choose more, we choose from the items with increased price, but we decrease their price by ε.

Now let us assume t_1 is the first buyer. We also assume first that we increased the price of all items in X_{123}. t_1 gets all items from $\{s \in X_{13} \mid \Theta(s) \leq 2\}$ and she also takes the real items in $\{s \in X_{12} \mid \Theta(s) \leq 3\}$. If $|X_{12}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| < b_1$, she buys all real items in X_{12}, $\{s \in X_{13} \mid \Theta(s) \leq 2\}$ and the items in $\{s \in X_{13} \mid \Theta(s) = 4\}$ whose price were not changed. If there are items in X_{123} with unchanged prices, and the number of them is $b_1 - |X_{12}| - |X_{13}|$, t_1 gets all items in X_{12}, X_{13} and the items in X_{123} whose price were not changed. The remaining case is when exists at least one item in X_{123} whose price was not changed and the number of items in X_{123} with unchanged prices is greater than $b_1 - |X_{12}| - |X_{13}|$. That means we left their prices unchanged because of t_2, that is $|X_{12}| + |X_{23}| < b_2$. That also means there is no item in X_{12} with label 1 or 3. It is not difficult to see that t_1 buys all items in $\{s \in X_{13} \mid \Theta(s) \leq 2\}$, as their prices are unchanged. If t_2 is the first buyer, the reasoning goes similarly. The only thing which is different from the previous case is when there exists at least one item in X_{123} with unchanged price, but the number of these items is greater than $b_2 - |X_{12}| - |X_{23}|$. Now, it means $b_1 > |X_{12}| + |X_{13}|$, which does not mean there are no items in X_{12} with label 1 or 3, it only means there are no items in X_{12} with label 1, but that is enough as t_2 has to buy all real items in X_{12} with label 1, if there is any, but she can leave real items in X_{12} with label 3 or 4. If the first buyer is t_3, she takes all items in $\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}$.

Case 4. The dummy items are in X_{23}.

We start with the same labeling procedure as in Case 2 and Case 3. We distinguish five subcases:

Subcase 1. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| > b_2$ and there is at least one item in X_{23} with label 2.

The assumption that there is at least one item in X_{23} with label 2 shows $|\{s \in X_{23} \mid \Theta(s) = 4\}| = b_3$, thus with $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| > b_2$, it implies $|X_{13}| < b_1$, therefore there is no item in X_{13} with label 1. We increase the prices in $X_{13} \cup X_{123}$ by ε. If $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) = 1\}| \geq b_2$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 3\} \cup \{s \in X_{12} \mid \Theta(s) = 4\}$ by ε. If $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) = 1\}| < b_2$, we select some items from $\{s \in X_{12} \mid \Theta(s) = 3\}$ such way that the number of the selected items are $b_2 - |\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) = 1\}|$, and leave their prices unchanged, but we increase the prices of the unselected items in X_{12} with label 3 and the prices of the label 4 items in X_{12} by ε. This way, if t_1 comes first, she gets $\{s \in X_{12} \mid \Theta(s) \leq 3\}$ and $\{s \in X_{13} \mid \Theta(s) = 2\}$ (remember, there are no items in X_{13} with label 1). If t_2 comes first, she takes $\{s \in X_{12} \mid \Theta(s) = 1\}$ and the real items in $\{s \in X_{23} \mid \Theta(s) \leq 2\}$. If t_3 comes first, she gets the real items in $\{s \in X_{23} \mid \Theta(s) = 1\}$. The price increasing shows 5(b) and 5(c) holds.

Subcase 2. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| > b_2$ and there are no items in X_{23} with label 2.

We leave all prices unchanged. If t_1 comes first, she takes $\{s \in X_{13} \mid \Theta(s) \leq 2\}$ and $\{s \in X_{12} \mid \Theta(s) \leq 3\}$. If t_2 or t_3 comes first, they get all real items in $\{s \in X_{12} \mid \Theta(s) = 1\}$ and $\{s \in X_{13} \mid \Theta(s) = 1\}$, respectively. As the dummy items are in $\{x \in X_{23} \mid \Theta(s) = 4\}$, 5(b), 5(c) holds automatically.

Subcase 3. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$, $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| > b_3$ and exists an item in X_{23} with label 1.

The assumption that there is at least one item in X_{23} with label 1 shows $|\{s \in X_{23} \mid \Theta(s) > 1\}| = b_2$, thus with $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| > b_3$, it implies $|X_{13}| < b_1$, therefore there is no item in X_{13} with label 1. We increase the prices in $X_{12} \cup X_{123}$ by ε. We also increase the prices in $\{s \in X_{13} \mid \Theta(s) > 1\}$.

If t_1 comes first, she takes $\{s \in X_{12} \mid \Theta(s) = 3\}$ (there are no items in X_{12} with label 1) and $\{s \in X_{13} \mid \Theta(s) \leq 2\}$, as in $N_{G_\pi}(t_1)$, we only left the prices unchanged in $\{s \in X_{13} \mid \Theta(s) = 1\}$, which means the order of items in $N_{G_\pi}(t_1)$ remained unchanged. If t_2 comes first, she gets the real items in $\{s \in X_{23} \mid \Theta(s) \leq 2\}$. If the first buyer is t_3, she takes $\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}$. Therefore all three conditions of Claim 5 hold again.

Subcase 4. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$, $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| > b_3$ and there is no item in X_{23} with label 1.

If $|X_{23}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \geq b_2$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε. If $|X_{23}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| < b_2$, we select $b_2 - (|X_{23}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}|)$ items from $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$, starting with the items in $\{s \in X_{12} \mid \Theta(s) = 4\}$, and we increase the prices of the unselected items in $\{s \in X_{12} \mid \Theta(s) = 4\} \cup X_{123}$ by ε.

When the first buyer is t_1, she takes $\{s \in X_{12} \mid \Theta(s) \leq 3\} \cup \{s \in X_{13} \mid \Theta(s) \leq 2\}$, as we only increased prices of items with label 4 or 5. If the first buyer is t_2, she gets $\{s \in X_{23} \mid \Theta(s) = 2\}$ (remember, there is no item in X_{23} with label 1) and she also buys $\{s \in X_{12} \mid \Theta(s) \leq 3\}$ as these are real items and $|\{s \in X_{23} \mid \Theta(s) = 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$. If t_3 is the first buyer, she gets $\{s \in X_{13} \mid \Theta(s) = 1\}$. Observe that the conditions of Claim 5 hold.

Subcase 5. $|\{s \in X_{23} \mid \Theta(s) \leq 2\}| + |\{s \in X_{12} \mid \Theta(s) \leq 3\}| \leq b_2$ and $|\{s \in X_{23} \mid \Theta(s) = 1\}| + |\{s \in X_{13} \mid \Theta(s) \leq 2\}| \leq b_3$.

To ensure 5(b) and 5(c). holds, we increase some prices the following way: If $|X_{23}|+|\{s \in X_{12} \mid \Theta(s) = 1\}| \geq b_2$, we increase the prices in $\{s \in X_{12} \mid \Theta(s) > 1\} \cup X_{123}$ by ε. If $|X_{23}| + |\{s \in X_{12} \mid \Theta(s) = 1\}| < b_2$, we choose $b_2 - (|X_{23}| + |\{s \in X_{12} \mid \Theta(s) = 1\}|)$ items from $\{s \in X_{12} \mid \Theta(s) > 1\}$, and if the items in $\{s \in X_{12} \mid \Theta(s) > 1\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{12} \mid \Theta(s) > 1\} \cup X_{123}$ which were not chosen by ε. We do the same with b_3 instead of b_2: If $|X_{23}| + |\{s \in X_{13} \mid \Theta(s) = 1\}| \geq b_3$, we increase the prices in $\{s \in X_{13} \mid \Theta(s) > 1\} \cup X_{123}$ by ε. If $|X_{23}| + |\{s \in X_{13} \mid \Theta(s) = 1\}| > b_3$, we choose $b_3 - (|X_{23}| + |\{s \in X_{13} \mid \Theta(s) = 1\}|)$ items from $\{s \in X_{13} \mid \Theta(s) > 1\}$, and if the items in $\{s \in X_{13} \mid \Theta(s) > 1\}$ are not enough, we further choose from X_{123}. We increase the prices of the others in $\{s \in X_{13} \mid \Theta(s) > 1\} \cup X_{123}$ which were not chosen by ε. If we have to choose items from X_{123}, we start with the items which are already chosen, if there is any. If there is no chosen item or we have to choose more, we choose from the items with increased price, but we decrease their price by ε.

If t_1 comes first, she gets $\{s \in X_{12} \mid \Theta(s) \leq 3\}$ and $\{s \in X_{13} \mid \Theta(s) \leq 2\}$, as in $N_{G^+_{\pi^+}}(t_1)$, we only increased the prices of items with label 4 or 5. It is easy to check for t_2 and t_3 that they take $\{s \in X_{12} \cup X_{23} \mid \Theta(s) \leq 2\}$ and $\{s \in X_{13} \cup X_{23} \mid \Theta(s) = 1\}$, respectively.

Case 5. The dummy items are in X_{123}.

By Lemma 7, a buyer has positive utility from her legal real items and negative utility from her non-legal items. We apply the same labeling procedure that we used in the proof of Theorem **??**, that is, when the market satisfies property (OPT). Thus dummy items are now labeled by 5.

As before, t_i gets all items in $N_{G^+_{\pi^+}}(t_i)$ with label no greater than $4 - i$, since these are real items. That implies 5(a) and 5(b). As $|N_{G^+_{\pi^+}}(t_i)| \geq b_i$ and t_i has positive utility for all real items in $N_{G^+_{\pi^+}}(t_i)$, 5(c) automatically holds. $\square$

6.2 Bi-demand markets

For convenience, let S denote S^+. In the proof of Theorem 6, we showed the existence of an adequate ordering σ. In Lemma 5, we saw that, for $\delta := \Delta(\pi)/(|S| + 1)$, setting the prices to $p(s) := \pi(s) + \delta \cdot \sigma(s)$ results in optimal dynamic pricing if (OPT) holds. However, when S contains dummy items besides the real ones, the pricing defined this way might not result in an optimal allocation. This is because when a buyer chooses items from her neighbors according to σ, the dummy items are not there in real life, therefore the buyer might skip dummy items in its neighborhood in $G^+_{\pi^+}$. As a consequence, she might take two items that are not allowed to her (that is, she takes two items from $N_{G^+_{\pi^+}}(Y)$ where $|N_{G^+_{\pi^+}}(Y)| \leq 2|Y| + 1$ for some $\emptyset \neq Y \subsetneq T$) or she might take an item which is not feasible for her (that is, the item is not her neighbor in $G^+_{\pi^+}$). However, if we start with the minimum weighted covering π^+ described in Lemma 7, property 7(c) shows that if a buyer skips her dummy neighbors in the ordering, she does not take real items which are not legal for her as she has negative utility for them. That is, it is enough to ensure that if a buyer t has dummy neighbors, then she does not take two items from $N_{G^+_{\pi+}}(Y)$ for every Y set with $|N_{G^+_{\pi+}}(Y)| \leq 2|Y|+1$, $t \notin Y$ when she skips dummy items in the ordering. Recall that a set Y of buyers is dangerous if $|N_{G^+_{\pi+}}(Y)| = 2|Y|+1$ and tight if $|N_{G^+_{\pi+}}(Y)| = 2|Y|$. That means, we have to pay attention to dangerous and tight sets when pricing the items in the market.

The idea of the proof of Theorem 6 is the following: we set the prices to $p(s) := \pi(s) + \delta \cdot \sigma(s)$, where σ is an adequate ordering which is determined the same way as previously with the property (OPT). Then we increase some of the prices by ε to ensure that if a buyer has dummy items as neighbors, she will not take two items from $N_{G^+_{\pi+}}(Y)$ if Y is a dangerous or tight set. We will use the following observations.

Observation 9.

(a) In $G^+_{\pi+}$, the neighborhoods of dummy items are the same. As a result, for every $Y \subseteq T$, all dummy items are in $N_{G^+_{\pi+}}(Y)$ or all of them are in $S - N_{G^+_{\pi+}}(Y)$,

(b) A buyer $t \in \hat{T}$ has negative utility for a real item s if st is not legal, therefore buyers in $\hat{T}$ only take items that are feasible for them. Also, buyers in $T - \hat{T}$ take only feasible items as they have at least $b(t)$ neighbors in $G^+_{\pi+}$,

(c) By the choice of ε, if we increase $p(s)$ by ε for some item s, the utility of $t \in \hat{T}$ for s becomes negative,

(d) By the choice of ε, if $t \in T - \hat{T}$, st is an edge in $G^+_{\pi+}$, and we increase $p(s)$ by ε, the utility of t for s remains positive and still higher than for any s' where $s't$ is not legal.

Now we are ready to prove Theorem 6 without assuming (OPT).

Theorem 10. *Every bi-demand market admits an optimal dynamic pricing scheme, and such prices can be computed in polynomial time.*

Proof. As before, we prove by induction on $|T|$, and the proof goes very similarly to the proof with the (OPT) assumption. The statement holds when $|T| = 1$, therefore $|T| \geq 2$ can be assumed.

Case 1. $|N_{G^+_{\pi+}}(Y)| \geq 2|Y|+2$ for every $\emptyset \neq Y \subsetneq T$.

For any $t \in T$ and $s_1, s_2 \in N_{G^+_{\pi+}}(t)$, the graph $G_\pi - \{s_1, s_2, t\}$ still satisfies the conditions of Lemma 2(a), hence $\{s_1, s_2\}$ is feasible for t. Therefore σ can be chosen arbitrarily, since the current pricing ensures buyers will not buy items which are not optimal for them (Observation 9(b)).

Case 2. $|N_{G^+_{\pi+}}(Y)| \geq 2|Y|+1$ for $\emptyset \neq Y \subsetneq T$ and there exists Y dangerous set, that is $|N_{G^+_{\pi+}}(Y)| = 2|Y|+1$.

Let Z be an inclusionwise maximal dangerous set.

Subcase 2.1. There is no dangerous set disjoint from Z.

We have already shown in Section 5 that if a pair $s_1, s_2 \in N_{G^+_{\pi+}}(t)$ is not feasible for a buyer $t \in T - Z$, then $s_1, s_2 \in N_{G^+_{\pi+}}(Z)$.

First we consider the case when $|S - N_{G^+_{\pi+}}(Z)| \geq 2$. If $t \in T - Z$ and t has only one neighbor in $S - N_{G^+_{\pi+}}(Z)$, then for $T' = Z + t_0 \neq T$ we get $|N_{G^+_{\pi+}}(T')| = 2|T'|$. This case will be discussed later on (see Case 3). From now on, we assume that each $t \in T - Z$ has at least two neighbors in $N_{G^+_{\pi+}}(Z)$. Similarly as in the proof when (OPT) holds, let $t_0 \in T - Z$ be an arbitrary buyer who shares a neighbor with Z, and let $s_0 \in N_{G^+_{\pi+}}(t) \cap N_{G^+_{\pi+}}(Z)$. If it is possible, we choose t_0 and s_0 in such a way that s_0 is a dummy item. Let σ' be an arbitrary ordering of the items in

$S - N_{G^+_{\pi^+}}(Z)$, σ'' be an adequate ordering of the items in G'' where G'' is obtained by deleting the items in $S - (N_{G^+_{\pi^+}}(Z) - s_0)$ and the buyers in $T - Z$, and σ''' be the trivial ordering of the single element set $\{s_0\}$. We consider the ordering $\sigma = (\sigma', \sigma'', \sigma''')$ of items in S.

Subcase 2.1.1. All dummy items are in $N_{G^+_{\pi^+}}(Z)$.

If s_0 is dummy, any buyer from Z will choose items from $N_{G^+_{\pi^+}}(Z)$ (see Observation 9(b)), but she will not take the dummy s_0. As σ'' was an adequate ordering of the items in G'', the remaining graph still admits a $(1,2)$-factor. Buyers from $T - Z$ will take two real items from $S - N_{G^+_{\pi^+}}(Z)$ as they have at least two neighbors in $S - N_{G^+_{\pi^+}}(Z)$.

If s_0 is not dummy, we increase its price by ε. This way, buyers in Z who have dummy neighbors have negative utility for s_0, therefore such buyers will not take s_0 even after the deletion of the dummy items. If a buyer in Z has no dummy neighbors, she has at least two cheaper neighbors in $N_{G^+_{\pi^+}}(Z)$ than s_0, which means that she will not take s_0 either. Again, a buyer from $T - Z$ will take two items from $S - N_{G^+_{\pi^+}}(Z)$.

Subcase 2.1.2. All dummy items are in $S - N_{G^+_{\pi^+}}(Z)$.

We increase all prices in $N_{G^+_{\pi^+}}(Z)$ by ε. This way, if a buyer in $T - Z$ has less than two real neighbors in $S - N_{G^+_{\pi^+}}(Z)$, she will not take items from $N_{G^+_{\pi^+}}(Z)$ by Observation 9(c). For a buyer in Z, the order of neighbors in G_π remains unchanged and she still prefers items in $N_{G^+_{\pi^+}}(Z)$ than items in $S - N_{G^+_{\pi^+}}(Z)$ by Observation 9(d).

We finished the discussion of the case when there is no dangerous set disjoint from Z and $|S - N_{G^+_{\pi^+}}(Z)| \geq 2$. Now let us assume that $|S - N_{G^+_{\pi^+}}(Z)| = 1$, and let y_0 denote the single element in $S - N_{G^+_{\pi^+}}(Z)$. As $|S| = 2|T|$, there is only one buyer in $T - Z$ (namely t_0). If t_0 has only two neighbors in $N_{G^+_{\pi^+}}(Z)$, $X = \{t_0\}$ is a dangerous set disjoint from Z, contradicting the assumption of Subcase 2.1. Hence t_0 has at least three neighbors in $N_{G^+_{\pi^+}}(Z)$. As before, s_0 denotes a neighbor of t_0 in $N_{G^+_{\pi^+}}(Z)$. We define $\sigma = (\sigma', \sigma'', \sigma''')$ the same way as when $|S - N_{G^+_{\pi^+}}(Z)| \geq 2$. First, we discuss the case when y_0 is a dummy item. Notice that y_0 is the only dummy item by Observation 9(a). Let y_1 denote the earliest neighbor of t_0 in $N_{G^+_{\pi^+}}(Z)$ according to σ. Let $k \in \{1, \ldots, |S|\}$ denote the place of y_1 in the ordering. Then the price of y_1 is $\varepsilon + \delta \cdot k$ and t_0 has $\varepsilon - \delta \cdot k$ utility for y_1. We increase the price of every item in $N_{G^+_{\pi^+}}(Z)$ by $\varepsilon - \delta \cdot \frac{2k+1}{2}$. As a result, t_0 has positive utility only for y_0 and y_1, while for buyers in Z, the utilities for their neighbors in $G^+_{\pi^+}$ remain positive and the order of items remains unchanged. If t_0 is the first buyer, she takes y_0 and y_1, and any buyer in Z takes items according to σ. If y_0 is a real item, we do not change the prices. This way, t_0 takes at most one item from $N_{G^+_{\pi^+}}(Z)$. A buyer from Z does not take y_0, since y_0 is feasible only for t_0, and she does not take s_0 which is at the end of the ordering.

Subcase 2.2. There exists a dangerous set disjoint from Z.

Let X be an inclusionwise minimal dangerous set disjoint from Z.

Subcase 2.2.1. For any $t \in X$ and for any $s_1, s_2 \in N_{G^+_{\pi^+}}(t)$, the set $\{s_1, s_2\}$ is feasible.

We define an adequate ordering $\sigma := (\sigma', \sigma'')$ the same way as before with the (OPT) assumption. If the dummy items are in $N_{G^+_{\pi^+}}(X)$, a buyer from $T - X$ who has no dummy neighbors chooses at most one item from $N_{G^+_{\pi^+}}(X)$ (namely s_0), and a buyer from $T - X$ who has dummy neighbors also chooses at most one item from $N_{G^+_{\pi^+}}(X)$, which is s_0 only if s_0 is real. If s_0 is dummy and the buyer has real neighbors in $N_{G^+_{\pi^+}}(X)$, then she chooses one of them, but if she has only dummy neighbors in $N_{G^+_{\pi^+}}(X)$, her utility is negative from the real items in $N_{G^+_{\pi^+}}(X)$ by Observation 9(b), therefore she does not take anything from $N_{G^+_{\pi^+}}(X)$. A buyer from X takes items from $N_{G^+_{\pi+}}(X)$, since if she has at least two real neighbors, she chooses two of them, but if she has at most one real neighbor, she does not take anything from $S - N_{G^+_{\pi+}}(X)$ as her utility is negative for them by Observation 9(b). If the dummy items are in $S - N_{G^+_{\pi+}}(X)$, we increase the prices in $N_{G^+_{\pi+}}(X) - \{s_0\}$ by ε. This way, a buyer from $T - X$ takes at most one item from $N_{G^+_{\pi+}}(X)$ (which is s_0), since if she has dummy neighbors, her utility is negative from $N_{G^+_{\pi+}}(X) - \{s_0\}$ by Observation 9(c), otherwise she has at least two cheaper real neighbors in $S-(N_{G^+_{\pi+}}(X)-\{s_0\})$. A buyer from X takes items from $N_{G^+_{\pi+}}(X)$ which does not cause a problem as X is an inclusionwise minimal dangerous set.

Subcase 2.2.2. There exists $t \in X$ and $s_1, s_2 \in N_{G^+_{\pi+}}(t)$ such that $\{s_1, s_2\}$ is not feasible.

In this case, we have already shown in the proof with assumption (OPT) that $X \cup Z = T$ and $N_{G^+_{\pi+}}(X) \cap N_{G^+_{\pi+}}(Z) = \{s_1, s_2\}$. We have defined an adequate ordering $\sigma := (\sigma', \sigma''|_{N_{G^+_{\pi+}}(X)-s_1}, \sigma''')$, where σ' and σ'' are adequate for the corresponding smaller graphs and σ''' is the trivial ordering of s_2. First, we assume s_1 and s_2 are both dummy. By Observation 9(a), that means there are no other dummy items. Any buyer in X takes items only from $N_{G^+_{\pi+}}(X) - \{s_1, s_2\}$, since if she has at least two real neighbors, she takes two of them, but if she has at most one real neighbor, she does not choose items which are not feasible for her (that is which are in $N_{G^+_{\pi+}}(Z) - \{s_1, s_2\}$) as her utility is negative for them by Observation 9(b). The reasoning is the same for buyers in Z. Now assume that s_1 is real and s_2 is dummy. We switch the roles of s_1 and s_2 if s_2 is real and s_1 is dummy. Now any buyer in X chooses items from $N_{G^+_{\pi+}}(X)-\{s_2\}$ and any buyer in Z chooses items from $N_{G^+_{\pi+}}(Z) - \{s_2\}$, since those buyers, who are in X and have dummy neighbor, have negative utility for $N_{G^+_{\pi+}}(Z) - \{s_1, s_2\}$ and those buyers, who are in Z and have dummy neighbor, have negative utility for $N_{G^+_{\pi+}}(X) - \{s_1, s_2\}$. Thirdly, if s_1 and s_2 are real items, we increase the price of s_2 by ε. This way, if all dummy items are in $N_{G^+_{\pi+}}(Z)$, any buyer from Z takes at most one item from $N_{G^+_{\pi+}}(Z) \cap N_{G^+_{\pi+}}(X)$ (namely s_1), since buyers with dummy neighbors have negative utility for s_2 and buyers with only real neighbors have at least two cheaper neighbors in $N_{G^+_{\pi+}}(X) - \{s_2\}$. Buyers in X do not take s_2 either as they have cheaper neighbors in $N_{G^+_{\pi+}}(X) - \{s_2\}$. When all dummy items are in $N_{G^+_{\pi+}}(X)$, the proof goes the same way.

Case 3. $|N_{G^+_{\pi+}}(T')| = 2|T'|$ for some $\emptyset \neq T' \subsetneq T$.

As we showed in the case when (OPT) holds, there exists T' satisfying the assumption if and only if $G^+_{\pi+}$ is not connected. Suppose $G^+_{\pi+}$ has k components, and determine an adequate ordering for the components of $G^+_{\pi+}$, separately. Then $\sigma = (\sigma', \sigma'', \ldots \sigma^{(k)})$ is an adequate ordering for the whole graph since if a buyer has no dummy neighbors, she has at least two real neighbors in her own component, and if a buyer has dummy neighbors, she has negative utility for all items which are not in her own component as Observation 9(b) shows. □

7 Multi-demand markets up to four buyers

As we have already seen, if property (OPT) does not hold, it causes many technical difficulties. Therefore, we only consider the case where (OPT) is satisfied. In [27], Pashkovich and Xie proved the following theorem:

Theorem 11 (Pashkovich, Xie)**.** *Every multi-demand market with four players admits an optimal dynamic pricing scheme when property* (OPT) *holds, and such prices can be computed in polynomial time.*

The proof given by Pashkovich and Xie is discussed in this Section. They also observed it is enough to show the existence of a σ adequate ordering. The theorem can be proved by induction on $|S|$. As before, the main goal is to give an adequate ordering of the items in the market, that is to give an order which ensures if an arbitrary agent t enters the market and chooses $b(t)$ items from $N_{G_\pi}(t)$ according to the ordering, the remaining items can be allocated to the other agents in a way where every $t' \in T$ gets $b(t')$ items that are in $N_{G_\pi}(t')$. It can be assumed every demand is positive and there is at least one buyer whose demand is at least 2. If G_π is not connected, one can construct adequate orderings in each component separately. So one can assume G_π is connected, which also means for any buyer t, $N_{G_\pi}(t)$ is strictly greater than $b(t)$. One can also assume there is no item in S that is feasible only to one player, since if s is only feasible for t, we can remove it and decrease $b(t)$ by one. Then, if σ' is an adequate ordering in the new problem and σ_0 denotes the trivial ordering on s, $\sigma := (\sigma_0, \sigma')$ is an adequate ordering in the original problem. That means, the statement is true if $|S| \leq 4$.

Lemma 8. *If there exists an $s \in S$ with $s \in N_{G_\pi}(t_i)$ for all $i = 1...4$, then there exists a σ adequate ordering.*

Proof. Assuming there is an $s \in S$ with $s \in N_{G_\pi}(t_i)$ for all $i = 1...4$. Let $t \in T$ be an arbitrary buyer with $b(t) \geq 2$. Now consider the smaller problem we get by increasing $b(t)$ by 1 and removing s from S. Using the inductive hypothesis, there is a σ' adequate ordering on the items in $S - \{s\}$. Let σ'' be the trivial ordering on s and $\sigma := (\sigma', \sigma'')$. One can show σ is an adequate ordering in the original problem instance. If a buyer $t' \neq t$ enters the shop first, she gets $b(t')$ items according to σ and does not take the item s. As σ' was an adequate ordering in the smaller instance, we can allocate $b(t'')$ and $b(t''')$ feasible items from the remaining items in $S - \{s\}$ to the other two players t'' and t''' who are not t, and $b(t) - 1$ items to t. But since s is optimal to all players, we can allocate s to t. If the first player arriving to the market is t, she takes $b(t) - 1$ items from $S - \{s\}$ according to σ'' and an extra item. If the extra item is s, we can complete it to an optimal allocation as σ'' was a proper ordering in the smaller case, and if the extra item is not s, we can replace it by s in any optimal allocation since s is feasible for all players. □

The key step in the proof is introducing the so-called *removable sets* of items. The existence of these sets with the inductive hypothesis guarantees the existence of optimal dynamic pricing. To define the removable sets, we need to fix an arbitrary $O = \{O_1, O_2, O_3, O_4\}$ optimal allocation of elements in S.

Suppose there is an $s_c \in S$ item, which is allocated to t_c in O and the following holds: if $t \neq t_c$ and $s_c \in N_{G_\pi}(t)$, then there is an $s_t \in O_t$ item with $s_t \in N_{G_\pi}(t_c)$. Let T_r be the set of buyers for whom s_c is feasible, i.e., $T_r := \{t \in T : s_c \in N_{G_\pi}(t_c)\}$ and $S_r := \{s_c\} \cup \{s_t : t \in T_r - \{t_c\}\}$. Then S_r is called *removable set of type I* and s_c is the *central item*.

Lemma 9. *Assume that an S_r removable set of type I exists with central item s_c. Consider the problem instance we get by removing S_r and decreasing the demands of players in T_r by 1. By induction, there is a σ'' adequate ordering in the smaller graph. Let σ' denote the trivial ordering on s_c and σ''' an arbitrary ordering of $S_r - \{s_c\}$. Then $\sigma = (\sigma', \sigma'', \sigma''')$ is an adequate ordering in the original problem.*

Proof. **Case 1.** The first player t arriving is in $T - T_r$.

Since $t \notin T_r$, $s_c \notin N_{G_\pi}(t)$, and since $b(t)$ was not decreased in the smaller example, she takes exactly those items she would take in the smaller problem instance. The fact every item in S_r can be allocated to distinct buyers in T_r and the adequateness of σ'' shows that after t leaves the shop, the remaining items can be allocated optimally to $T - \{t\}$.

Case 2. The first player t arriving is in $T_r - \{t_c\}$.

Since $s_c \in N_{G_\pi}(t)$ and s_c is the first item according to σ, t takes s_c, then $b(t) - 1$ items from $S - S_r$, and does not take items from $S_r - \{s_c\}$. The item in $S_r - \{s_c\}$ corresponding to t is in $N_{G_\pi}(t_c)$, which implies the remaining items can be allocated to $T - \{t\}$ optimally.

Case 3. The first player arriving is t_c.

If t_c comes first, she takes s_c and other $b(t_c) - 1$ items from $S - S_r$ according to σ''. It is easy to see there is an optimal allocation where t_c gets exactly these $b(t_c)$ items. □

From now on, we make a directed graph from G_π in the following way: if an edge connects a buyer t and an item in O_t, then it is directed towards T, and all the other edges which connect t and items in $N_{G_\pi}(t) - O_t$ are directed towards S. Assume there exists $S_r \subseteq S$ for which $|O_t \cap S_r| = 1$ $\forall t \in T$ and consider the subgraph G of G_π induced by T and S_r. If there is a directed path in G from x to y for every $x, y \in S$, then S_r is a *removable set of type II.*

Lemma 10. *Assume that an S_r removable set of type II exists. Consider the smaller problem instance we get by removing S_r and decreasing the demands of all players by 1. By induction, there is a σ' adequate ordering on $S - S_r$ in the smaller graph. Let σ'' denote an arbitrary ordering of S_r. Then $\sigma = (\sigma', \sigma'')$ is an adequate ordering in the original problem.*

Proof. Let t denote the first buyer entering the shop. t gets $b(t) - 1$ items according to σ' and an other item x.

Case 1. $x \in S_r \cap O_t$.

The items left after t leaves can be allocated optimally to the remaining buyers: buyer t' gets $b(t') - 1$ items from $(S - S_r) \cap N_{G_\pi}(t')$ (this can be done as σ' was adequate) and the only item in $S_r \cap O_{t'}$.

Case 2. $x \in S_r \cap O_{\hat{t}}$ for some $\hat{t} \neq t$.

Let y denote the unique item in $S_r \cap O_t$ and P denote an $x \to y$ directed path in G. If t' is a buyer who is not in P, she can get $b(t') - 1$ element from $S - S_r$ and the unique element in $S_r \cap O_{t'}$. Otherwise, if t' is in P, then she gets the item in S_r which follows t' in the directed path P.

Case 3. $x \in S - S_r$.

Let t_0 denote the agent for whom $x \in O_{t_0}$ and x_0 denote the single item in $S_r \cap O_{t_0}$. Let P be an $x_0 \to y$ directed path in G. As in the previous case, if $t' \neq t_0$ is not in P, $b(t')$ items can be allocated to her optimally. If $t' \neq t_0$ is in P, t' gets $b(t') - 1$ items from $S - S_r$ and the item in S_r which follows t' in P. t_0 gets $b(t_0) - 2$ items from $S - S_r$, x_0 and the item which follows t_0 in P. □

We have seen that if S has a removable subset of type I or II, we can give an appropriate ordering using the inductive hypothesis. The next step in the proof of Theorem 11 is to prove either there is a removable set of type I or type II, or there is an item that is feasible for all four agents which also implies the existence of optimal dynamic pricing by Lemma 8. Let $\{t_1, t_2, t_3, t_4\}$ denote the set of buyers. We will need the following lemma:

Lemma 11. *There is a directed cycle in G_π.*

Proof. By the assumption, every item is feasible for at least two players and every player has strictly more neighbors than her demand, every indegree and outdegree is at least 1 in G_π, which implies the existence of a directed cycle. □

If C is a directed cycle in G_π, for any player t, $|C \cap O_t| \leq 1$. In [27], such cycle is called *uniquely assigned cycle*. Let C be a directed cycle. As G_π is bipartite, $|C| = 4, 6$ or 8.

Case 1. The length of C is 8.

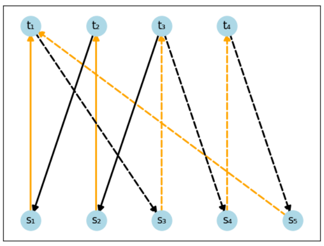
Then C contains all four players, therefore C is a removable set of type II.

Case 2. The length of C is 6.

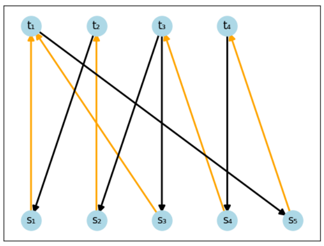
Without loss of generality, we can suppose $C = s_3 \to t_3 \to s_2 \to t_2 \to s_1 \to t_1 \to s_3$ and $s_i \in O_i$ $i = 1...4$. We choose an s_4 item from O_4 arbitrarily. As every item is feasible for at least two buyers, we can assume $s_4 \in O_3$. (The proof goes analogously if $s_4 \in O_2$ or $s_4 \in O_1$.) If $\{s_1, s_2, s_3\} \cap N_{G_\pi}(t_4) \neq \emptyset$, it is easy to check $\{s_1, s_2, s_3, s_4\}$ is a removable set of type II. So it can be assumed $\{s_1, s_2, s_3\} \cap N_{G_\pi}(t_4) = \emptyset$. Since every buyer has more feasible items than her demand, $N_{G_\pi}(t_4) - O_4 \neq \emptyset$. Let s_5 denote an item in $N_{G_\pi}(t_4) - O_4$. Then s_5 is in O_1, O_2 or O_3.

Subcase 2.1. $s_5 \in O_1$.

Consider the following directed cycle: $C' = s_3 \to t_3 \to s_4 \to t_4 \to s_5 \to t_1 \to s_3$. We get a different optimal allocation by exchanging the items along C' (and change the orientation of edges in C'). It is not difficult to check that with the new orientation, $\{s_1, s_3, s_4, s_5\}$ is a removable set of type II, see Figure 4.



(a) A graph with a directed cycle $C = s_3 \to t_3 \to s_2 \to t_2 \to s_1 \to t_1 \to s_3$ of length 6. $s_4 \in O_3$, $\{s_1, s_2, s_3\} \cap N_{G_\pi}(t_4) = \emptyset$ and $s_5 \in O_1$.



(b) After reallocating according to C', $\{s_1, s_3, s_4, s_5\}$ is a removable set of type II.

Figure 4: Subgraph of G_π in Subcase 2.1. The black edges denote the item is feasible for the buyer, and the orange edges denote the item is allocated to the buyer. In Figure 4a, dashed edges are in C'.

Subcase 2.2. $s_5 \in O_2$.

$C' = s_1 \to t_1 \to s_3 \to t_3 \to s_4 \to t_4 \to s_5 \to t_2 \to s_1$ is a directed cycle of length 8, see Figure 5.

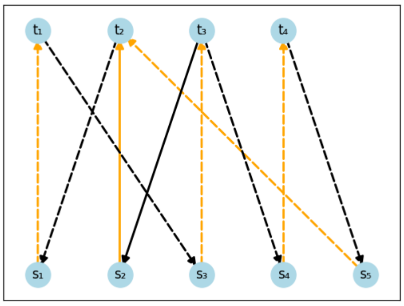
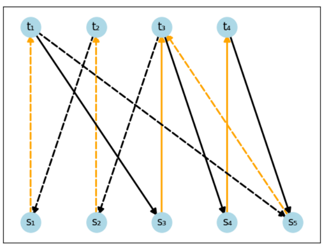


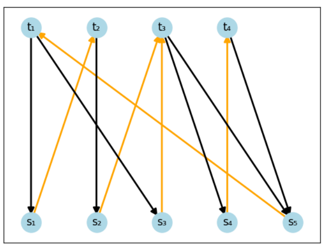
Figure 5: A graph with a directed cycle $C = s_3 \to t_3 \to s_2 \to t_2 \to s_1 \to t_1 \to s_3$ of length 6. $s_4 \in O_3$, $\{s_1, s_2, s_3\} \cap N_{G_\pi}(t_4) = \emptyset$ and $s_5 \in O_2$, as in Subcase 2.2. The directed cycle of length 8 is indicated by dashed lines.

Subcase 2.3. $s_5 \in O_3$.

If s_4 and s_5 are feasible only for t_3 and t_4, then $\{s_4, s_5\}$ is a removable set of type I (both s_4 and s_5 can be the central item). We can assume s_5 is feasible for t_1 or t_2 (the proof goes similarly if s_4 is feasible for t_1 or t_2 by changing the allocation O along the cycle $C' = t_3 \to s_4 \to t_4 \to s_5 \to t_3$). If $s_5 \in N_{G_\pi}(t_1)$, see Figure 6, we get an other optimal allocation by exchanging items along $C'' = t_1 \to s_5 \to t_3 \to s_2 \to t_2 \to s_1 \to t_1$ and changing the orientation of edges in C''. One can check $\{s_1, s_2, s_4, s_5\}$ is a removable set of type II. If $s_5 \in N_{G_\pi}(t_2)$, $s_5 \notin N_{G_\pi}(t_1)$, otherwise s_5 would be an item feasible for all players. Now $\{s_2, s_4, s_5\}$ is a removable set of type I with central item s_5, see Figure 7.



(a) A graph with a directed cycle $C = s_3 \to t_3 \to s_2 \to t_2 \to s_1 \to t_1 \to s_3$ of length 6. $s_4 \in O_3$, $\{s_1, s_2, s_3\} \cap N_{G_\pi}(t_4) = \emptyset$. $s_5 \in O_3$ and $s_5 \in N_{G_\pi}(t_1)$, as in Subcase 2.3. C'' is indicated by dashed lines.



(b) After reallocation, $\{s_1, s_2, s_4, s_5\}$ is a removable set of type II.

Figure 6: Subgraphs of G_π in Subcase 2.3.

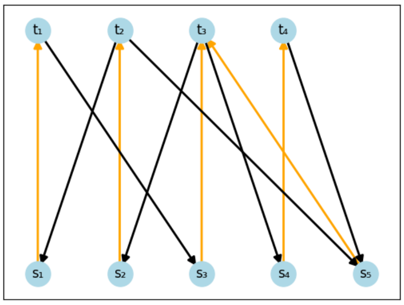


Figure 7: Subgraph of G_π in Subcase 2.3. Directed cycle $C = s_3 \to t_3 \to s_2 \to t_2 \to s_1 \to t_1 \to s_3$ of length 6. $s_4 \in O_3$, $\{s_1, s_2, s_3\} \cap N_{G_\pi}(t_4) = \emptyset$. $s_5 \in O_3$ and $s_5 \in N_{G_\pi}(t_2)$, $s_5 \notin N_{G_\pi}(t_1)$. $\{s_2, s_4, s_5\}$ is a removable set of type I with central item s_5.

Case 3. The length of C is 4.

We can assume C is of the form $s_1 \to t_1 \to s_2 \to t_2 \to s_1$. If s_1 and s_2 are only feasible for t_1 and t_2, there is a removable set of type I. Therefore one can assume s_1 or s_2 is feasible for t_3 or t_4. By exchanging items along C, it can be assumed $s_2 \in N_{G_\pi}(t_3)$. Let s_3 be an arbitrary element of O_3. s_3 is feasible for at least one player besides t_3. If $s_3 \in N_{G_\pi}(t_1)$, $C' := t_3 \to s_2 \to t_2 \to s_1 \to t_1 \to s_3 \to t_3$ is a directed cycle of length 6, see Figure 8. If $s_3 \in N_{G_\pi}(t_2)$ then $s_2 \notin N_{G_\pi}(t_4)$, since s_2 is not feasible for all players. That means $\{s_1, s_2, s_3\}$ is a removable set of type II with s_2 being the central item (Figure 9). If $s_3 \in N_{G_\pi}(t_4)$, let s_4 be an item in O_4. s_4 is feasible for at least one other player. If $s_4 \in N_{G_\pi}(t_1)$ or $s_4 \in N_{G_\pi}(t_2)$, $\{s_1, s_2, s_3, s_4\}$ is a removable set of type II (Figure 10). If s_4 is only feasible for t_3, $\{s_3, s_4\}$ is a removable set of type I with s_3 being the central item (Figure 11).

We proved there exists a removable set of type I or type II, or there is an item that is feasible for all players. By using the inductive hypothesis, we can find a σ adequate ordering, which completes the proof of Theorem 11.

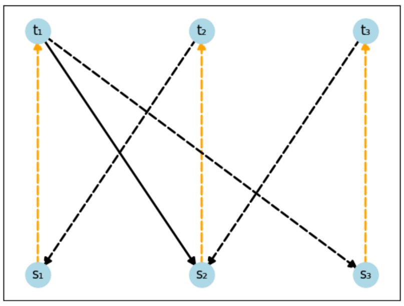


Figure 8: Subgraph of G_π in Case 3. Directed cycle $C = s_1 \to t_1 \to s_2 \to t_2 \to s_1$ of length 4. $s_2 \in N_{G_\pi}(t_3)$, $s_3 \in N_{G_\pi}(t_1)$, $C' := t_3 \to s_2 \to t_2 \to s_1 \to t_1 \to s_3 \to t_3$ is a directed cycle of length 6.

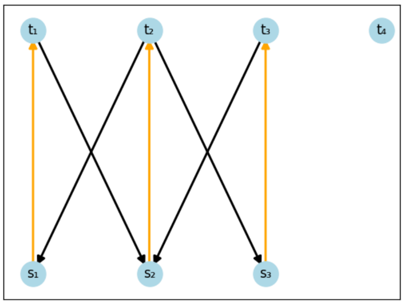


Figure 9: Subgraph of G_π in Case 3. Directed cycle $C = s_1 \to t_1 \to s_2 \to t_2 \to s_1$ of length 4. $s_2 \in N_{G_\pi}(t_3)$, $s_3 \in N_{G_\pi}(t_2)$, $s_2 \notin N_{G_\pi}(t_4)$. $\{s_1, s_2, s_3\}$ is a removable set of type II with s_2 being the central item.

(a) Subgraph of G_π in Case 3. Directed cycle $C = s_1 \to t_1 \to s_2 \to t_2 \to s_1$ of length 4. $s_2 \in N_{G_\pi}(t_3)$, $s_3 \in N_{G_\pi}(t_2)$, $s_3 \in N_{G_\pi}(t_4)$, $s_4 \in N_{G_\pi}(t_1)$, $\{s_1, s_2, s_3, s_4\}$ is a removable set of type II.

(b) Subgraph of G_π in Case 3. Directed cycle $C = s_1 \to t_1 \to s_2 \to t_2 \to s_1$ of length 4. $s_2 \in N_{G_\pi}(t_3)$, $s_3 \in N_{G_\pi}(t_2)$, $s_3 \in N_{G_\pi}(t_4)$, $s_4 \in N_{G_\pi}(t_2)$, $\{s_1, s_2, s_3, s_4\}$ is a removable set of type II.

Figure 10: Subgraphs of G_π in Case 3.

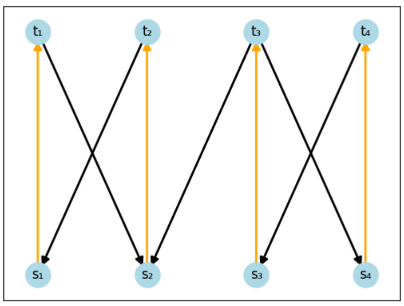


Figure 11: Subgraph of G_π in Case 3. Directed cycle $C = s_1 \to t_1 \to s_2 \to t_2 \to s_1$ of length 4. $s_2 \in N_{G_\pi}(t_3)$, $s_3 \in N_{G_\pi}(t_2)$, $s_3 \in N_{G_\pi}(t_4)$, s_4 is only feasible for t_3, $\{s_3, s_4\}$ is a removable set of type I with s_3 being the central item.

8 Tri-demand markets

In Theorem 6, we saw the existence of optimal dynamic pricing when all demands are at most 2. However, a stronger statement holds. In [27], the authors proved the following theorem:

Theorem 12 (Pashkovich, Xie)**.** *Every multi-demand market admits an optimal dynamic pricing scheme when the demands are at most 3 and property* (OPT) *holds. Such prices can be computed in polynomial time.*

Similarly to the bi-demand case, we can define dangerous sets. Remember, we called a set $Y \subseteq T$ dangerous if $|N_{G_\pi}(Y)| = 2|Y| + 1$. The reason behind this definition was that we do not want a two-demand buyer outside Y to buy two items from $N_{G_\pi}(Y)$. Now as we have three-demand agents also, we want to forbid a three-demand buyer outside Y' to take three items from $|N_{G_\pi}(Y')|$ if $|N_{G_\pi}(Y')| = \sum_{i \in Y'} b(t) + 2$. For this reason, we need to define *two-dangerous sets* and *three-dangerous sets* separately. $Y \subseteq T$ is *two-dangerous* set if $|N_{G_\pi}(Y)| = \sum_{i \in Y} +1$, $|N_{G_\pi}(Y) \cap N_{G_\pi}(T-Y)| \geq 2$ and if $s \in N_{G_\pi}(Y) \cap N_{G_\pi}(T-Y)$, there is an optimal allocation where s is allocated to a player not in Y. If Y is a two-dangerous set, Y is called *maximal two-dangerous set* if for any Y' two-dangerous set with $N_{G_\pi}(Y) \subseteq N_{G_\pi}(Y')$, we have $Y = Y'$. $W \subseteq T$ is called *three-dangerous set* if $|N_{G_\pi}(W)| = \sum_{i \in W} b(i) + 2$, $|N_{G_\pi}(T-W) \cap N_{G_\pi}(W)| \geq 3$ and for any pair $s_1, s_2 \in N_{G_\pi}(T-W) \cap N_{G_\pi}(W)$, there is an optimal allocation where s_1 and s_2 are allocated to agents not in W. Similarly, if W is a three-dangerous set, we say W is *maximal three-dangerous set* if for any W' three-dangerous set with $N_{G_\pi}(W) \subseteq N_{G_\pi}(W')$, we have $W = W'$.

Consider an assignment where some items are already assigned to buyers, but some items are not. When the players arrive in an arbitrary sequence, in each step, we want to ensure the assignment we reached until that point can be extended to an optimal assignment, that is, the remaining items can be allocated to buyers such that a buyer t gets exactly $b(t)$ items and the items are in $N_{G_\pi}(t)$. An assignment is called *extendable* if it can be extended to an optimal assignment. Similarly as in the bi-demand case, it is easy to prove if a pair of items $\{s_1, s_2\}$ is assigned to buyers in a way that s_i $(i = 1, 2)$ is assigned to player for which s_i is optimal and the assignment is not extendable, there exists a Y two-dangerous set in the market. If there are three items $\{s_1, s_2, s_3\}$ such that there is a feasible assignment of $\{s_1, s_2, s_3\}$ which is not extendable but any assignment of two items is extendable, then there exists a W three-dangerous set.

We observed in Lemma 5, when proving the existence of optimal dynamic pricing, it is enough to give an adequate ordering of items in S according to the graph G_π which contains only the feasible edges. In the proof of Theorem 6, we used the inductive hypothesis which states we can give an adequate ordering in smaller problem instances with two-demand buyers. To show the existence of adequate ordering in multi-demand markets with buyers whose demands are at most 3, we need a bit stronger statement in the inductive hypothesis. The authors of [27] proved the following: for every item $s \in S$, there is an adequate ordering which starts with s.

Theorem 13. *For every $s \in S$, there exists an adequate ordering on S which starts with s.*

Proof. Following the proof of [27], one can prove by induction on $|S| + |T|$. The statement is trivial if $|T| = 1$. If $|T| = 2$, let $s \in S$ be an arbitrary item. We get a σ adequate ordering starting with s if we put s on the first place, then in arbitrary order, we continue with items that are optimal for only one of the players. At the end of the ordering, there are the items that are optimal for both players (except s). The case when $|S| = 2$ is also proved since if $|S| = 2$, then $|T| \leq 2$.

In the proof of Theorem 6, when we use the induction hypothesis on smaller induced subgraphs of G_π, Lemma 6 tells us we do not have to check whether every edge of the subgraph is still feasible in the smaller market. But as we want to prove a stronger statement now, if there is an edge st in an induced subgraph G which is not feasible in the smaller problem instance, there is no adequate ordering in G which starts with s. One can use induction only on smaller induced subgraph G which satisfies the following property:

an edge st is in G only if s is feasible for t in the smaller problem instance defined by G. (P)

We need a few lemmas to prove the statement in Theorem 13.

Lemma 12. *If there are two items s_1, s_2 such that there is a feasible assignment of $\{s_1, s_2\}$ which is not extendable, then there is dynamic pricing fixed at s for any item s.*

Proof. Let Z be a maximal two-dangerous set.

Case 1. $s \in N_{G_\pi}(T-Z) \cap N_{G_\pi}(Z)$.

Let G' denote the graph of the following problem instance: we introduce a new unit-demand player whose feasible items are the items in $N_{G_\pi}(T-Z) \cap N_{G_\pi}(Z)$. It satisfies property (P). By induction, there is a σ' adequate ordering in G' starting with s. Let G'' denote the smaller problem we get by deleting Z and $N_{G_\pi}(Z) - \{s\}$. By induction, there is a σ'' adequate ordering in G'' starting with s. Let σ be the following ordering: $\sigma = (\sigma'', \sigma'|_{N_{G_\pi}(Z)-s})$. Then σ is an adequate ordering in the original problem and starts with s.

Case 2. $s \in N_{G_\pi}(Z) - N_{G_\pi}(T-Z)$.

As in Case 1, let G' denote the graph we get by introducing a new unit-demand player whose feasible items are the items in $N_{G_\pi}(T-Z) \cap N_{G_\pi}(Z)$ and let σ' denote an adequate ordering in G' starting with s. Let s' be the item which is in $N_{G_\pi}(T-Z) \cap N_{G_\pi}(Z)$ and has the smallest label according to σ'. Now let G'' denote the subgraph induced by buyers $T-Z$ and items $N_{G_\pi}(T-Z) \cup \{s'\}$. By induction, there is a proper σ'' ordering starting with s'. One can define the following sets of items:

$$S_1 = \{\hat{s} \in N_{G_\pi}(Z) - \{s\} : \sigma'(\hat{s}) < \sigma'(s')\},$$
$$S_2 = \{\hat{s} \in N_{G_\pi}(Z) - \{s\} : \sigma'(\hat{s}) > \sigma'(s')\}.$$

Let σ_s and $\sigma_{s'}$ denote the trivial ordering of s and s'. If we define σ as

$$\sigma = (\sigma_s, \sigma'|_{S_1}, \sigma_{s'}, \sigma''|_{S-N_{G_\pi}(Z)}, \sigma'|_{S_2}),$$

we get an adequate ordering with s being the first element, since $N_{G_\pi}(T-Z) \cap S_1 = \emptyset$.

Case 3. $s \in N_{G_\pi}(T-Z)$.

We can choose an arbitrary s' element of $N_{G_\pi}(T-Z) \cap N_{G_\pi}(Z)$, define G' as in the previous cases, and define G'' with s'. In G', σ' denotes the proper ordering fixed at s' and in G'', σ'' denotes the proper ordering fixed at s. As before, σ_s and $\sigma_{s'}$ denote the trivial ordering of s and s'. Let

$$\sigma := (\sigma_s, \sigma''|_{(T-N_{G_\pi}(Z))-s+s'}, \sigma'|_{N_{G_\pi}(Z)-s'})$$

be an ordering in the original problem. It is easy to check that σ is adequate. ☐

Lemma 13. *If $t \in T$ is a unit-demand buyer, then for every $s \in S$, there is a dynamic pricing fixed at s.*

Proof. If $N_{G_\pi}(x) = \{s\}$, the statement holds since s is not feasible for other players, which means if we remove s and t from the market, the resulting problem instance admits an adequate ordering. By putting s on the first place, we get a dynamic pricing for the original problem. If there is an item $s' \neq s$ in $N_{G_\pi}(t)$, let G' denote the graph we get by removing s' and t. If there is a buyer t'' and $s'' \in N_{G_\pi}(t'')$ such that if we allocate s' to t and s'' to t'', the other items can not be allocated optimally, by using the previous lemma, there is a dynamic pricing fixed at s. Otherwise, property (P) holds and there exists a dynamic pricing fixed at s in G'. By putting s' to the end of the ordering, we get an adequate ordering for the original problem instance. ☐

From now on, we can assume there is no unit-demand player.

Lemma 14. *If for an item $s_1 \in S$ there are two other items s_2 and s_3 such that a feasible assignment of $\{s_1, s_2, s_3\}$ is not extendable, but every feasible assignment of two items is extendable, there is a dynamic pricing fixed at s_1.*

Proof. It is not difficult to see if a feasible assignment of $\{s_1, s_2, s_3\}$ is not extendable, there is a $Z \subset T$ three-dangerous set. We choose Z to be maximal in the following sense: if Z' is another three-dangerous set and $N_{G_\pi}(Z) \subseteq N_{G_\pi}(Z')$, then $Z = Z'$. The maximality of Z implies $s_1 \in N_{G_\pi}(Z)$. Let G' denote the following smaller instance: we remove $T - Z$ and add a new two-demand buyer for whom the feasible items are exactly those items which are in $N_{G_\pi}(T - Z) \cap N_{G_\pi}(Z)$. We also delete $S - N_{G_\pi}(Z)$. In G', the number of items is the same as the sum of the buyers' demands. It is not difficult to see the market we get this way satisfies property (P). By the inductive hypothesis, there is a dynamic pricing fixed at s_1 in G', or equivalently, there is an adequate ordering σ' starting with s_1. Let v_1 and v_2 denote the items in $N_{G_\pi}(T - Z) \cap N_{G_\pi}(Z)$ with the first and second lowest price. Let G'' denote the following smaller problem instance: we remove Z and $N_{G_\pi}(Z) - \{v_1, v_2\}$. Remember, every feasible assignment of two items is extendable. Therefore, G'' satisfying property (P) can be checked easily. By the inductive hypothesis, there is an adequate ordering σ'' in G'' starting with v_1. Let $S_1, ...S_5$ denote the following sets of items:

$$
\begin{aligned}
S_1 &= \{s \in N_{G_\pi}(Z) : \sigma'(s) \leq \sigma'(v_1)\}, \\
S_2 &= \{s \in N_{G_\pi}(Z) : \sigma'(v_1) < \sigma'(s) \leq \sigma'(v_2)\}, \\
S_3 &= \{s \in N_{G_\pi}(Z) : \sigma'(s) > \sigma'(v_2)\}, \\
S_4 &= \{s \in N_{G_\pi}(T - Z) : \sigma''(s) < \sigma''(v_2)\}, \\
S_5 &= \{s \in N_{G_\pi}(T - Z) : \sigma''(s) > \sigma''(v_2)\}.
\end{aligned}
$$

One can get an adequate ordering starting with s_1 by merging σ' and σ'' appropriately. That is, σ is the following:

$$\sigma = (\sigma'|_{S_1}, \sigma''|_{S_4 - v_1}, \sigma'|_{S_2}, \sigma''|_{S_5}, \sigma'|_{S_3}).$$

Since σ' is starting with s_1, $s_1 \in S_1$ and σ gives a dynamic pricing fixed at s_1. If a buyer from Z comes first, she chooses items according to σ' and a buyer from $T - Z$ chooses items according to σ'' and she does not take any item from $N_{G_\pi}(Z) - \{v_1, v_2\}$ as $N_{G_\pi}(T - Z) \cap (S_1 \cup S_2) = \{v_1, v_2\}$. Therefore the optimality of the dynamic pricing follows. ☐

Lemma 15. *If there are no unit-demand players, every feasible assignment of two items is extendable and every feasible assignment of three items is extendable as soon as it contains item s, then there is a dynamic pricing fixed at s.*

Proof. Suppose the conditions hold and let t denote a player with $s \in N_{G_\pi}(t)$. Let G' be the smaller problem we get by deleting s and decreasing $b(t)$ by one. G' satisfies property (P) since if we take an arbitrary item s' and assign it to a buyer t' with $s' \in N_{G_\pi}(t')$ and assign s to t, it is extendable in G, which means it is extendable in G'. By induction, there is a σ' adequate ordering in G'. Let σ_0 denote the trivial ordering on s and $\sigma := (\sigma_0, \sigma')$. If the first buyer is $\hat{t}$ and $s \notin \hat{t}$, then $\hat{t}$ takes items according to σ' and the allocation is extendable. If $s \in \hat{t}$, then one can also allocate the remaining items optimally since feasible assignments of two items are extendable and feasible assignments of three items containing s are extendable. □

Now we are ready to continue the proof of Theorem 13. If there is a unit-demand buyer in the market, Lemma 13 shows us how to find an adequate ordering starting with s. If there is no unit-demand buyer, but there is a feasible assignment of two items which is not extendable, one can find an adequate ordering by Lemma 12. If there is no unit-demand buyer and every feasible assignment of two items is extendable, but there is a feasible assignment of three items containing s which is not extendable, Lemma 14 ensures there is a dynamic pricing fixed at s. If all feasible assignments of size three and containing s are extendable, by Lemma 15, one can find a dynamic pricing fixed at s. □

By Lemma 5, providing an adequate ordering on the set of items implies the existence of optimal dynamic pricing. Therefore, Theorem 13 implies Theorem 12.

9 Other social welfare functions

Until now, when we talked about social welfare, we considered the problem of finding an allocation $\{X_1, ...X_n\}$ of items such that $\sum_{i=1}^n v_i(X_i)$ is maximized, where the v_i valuation functions were multi-demand valuations and buyers had b_i upper bounds on the number of items they are willing to buy. Here, the value of $\sum_{i=1}^n v_i(X_i)$ is usually called as *utilitarian social welfare* and the allocation $\{X_1, ...X_n\}$ is *efficient* since it maximizes the utilitarian social welfare. However, in many cases one can consider other valuation functions and leave the $|X_i| \leq b_i$ restrictions. Naturally arises the question whether the efficiency is the only factor we want to take into account. For example, if an allocation that maximizes utilitarian social welfare allocates all items to one buyer and the empty set to other buyers, that allocation does not seem to be fair. Therefore it suggests not only efficiency should be taken care of but also *fairness*. In the *Max-min welfare problem*, also known as *Santa Claus problem*, the objective is to maximize $\min_i(v_i(X_i))$, which is an NP-hard problem. When the valuation functions are additive, the problem is similar to a job scheduling problem, namely it is related to the makespan minimization on unrelated parallel machines, however it is still an open question to find a constant-factor approximation algorithm for additive valuations. [1]

In [13], Garg, Husic and Végh gave a detailed discussion of a different social welfare concept, the *Nash social welfare*. They call the Nash social welfare a balanced trade-off between utilitarian social welfare and the max-min fairness. They also summarized the best existing approximation results under different classes of valuation functions. In the Nash social welfare problem, the task is to allocate the set of items such a way that

$$(\prod_{i=1}^{n} v_i(X_i)^{w_i})^{1/\sum_{i=1}^n w_i}$$

is maximized, where $w_i > 0$ are weights associated to buyers. When all weights are equal, i.e., $w_i = 1$, we call it the *symmetric* case, otherwise we talk about *asymmetric* Nash social welfare. It turned out the problem is NP-hard even in the symmetric case with two equivalent players and additive valuations since the partition problem reduces to the Nash social welfare problem [25]. Furthermore, for additive valuations, there are no algorithms with a better approximation ratio than 1.069 unless $P = NP$ and for submodular valuations, with a better approximation ratio than 1.5819 [13]. However, for the symmetric case with additive valuations, constant-factor approximation algorithms exist and can be extended to slight generalizations of additive functions. In [13], Garg, Husic and Végh gave a polynomial-time $256e^{3/e}$-approximation algorithm for the symmetric case with Rado valuations. When we talk about Rado valuations, an agent t is associated with a matroid $\mathcal{M}_t = (V_t, I_t)$ and a bipartite graph with node set $S \cup V_t$, where S denotes the set of items in the market. There is a c_t non-negative weight function on the edges of the bipartite graph. For t, the value of $S' \subseteq S$ is the maximum cost of a matching between S' and points in V_t such that the points of V_t covered by the matching must form an independent set in matroid $\mathcal{M}_t$. Rado valuation functions include for example the additive case and the weighted matroid rank valuations. The authors also gave a polynomial-time $256\gamma^3$-approximation algorithm for the asymmetric case with Rado valuations, where $\gamma \geq 2$ is a constant satisfying the following: all w_i weights are in the interval $[1, \gamma - 1]$. Under additive valuations, a stronger 16γ approximation ratio holds. Under subadditive valuations, no better than $O(n)$ approximation algorithm exists both for the symmetric and asymmetric case [13].

10 Conclusions and open problems

We considered the existence of optimal dynamic prices for multi-demand valuations. By relying on the structural properties of an optimal dual solution, we gave polynomial-time algorithms for determining such prices in unit-demand markets and in multi-demand markets with up to three buyers, thus giving new interpretations of the results of Cohen-Addad et al. and Berger et al. We also proved that any bi-demand market satisfying the same technical assumption has a dynamic pricing scheme that achieves optimal social welfare. As we saw, there are a bit stronger results: the tri-demand case and multi-demand markets with up to four buyers are solved by Pashkovich and Xie. An open problem is to decide the existence of optimal dynamic prices in multi-demand markets in general.

We also saw finding an optimal allocation for maximizing the objective of the min-max social welfare or the Nash social welfare is already a hard task. One interesting question would be whether we can realize allocations from the approximation algorithms mentioned previously through static or dynamic pricing, or whether using dynamic pricing instead of static pricing results in a better solution, as it does in the multi-demand combinatorial markets.

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Dynamic pricing in combinatorial markets - Kombinatorikus piacok dinamikus árazása

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2022.05.27.

Szögi Evelin

a hallgató aláírása