NYILATKOZAT

Név: Fekete Fanni

ELTE Természettudományi Kar, szak: Alkalmazott Matematikus Msc.

NEPTUN azonosító: X5DULY

Szakdolgozat címe:
Pricing of Residential Mortgage Backed Securities (EN)
Lakossági jelzáloghitellel fedezett kötvények árazása

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2022. május 31.

Fekete Fanni

a hallgató aláírása

Eötvös Loránd University

Faculty of Science

Dept. of Applied Analysis and
Computational Mathematics

Pricing of Residential Mortgage Backed Securities

Supervisor:
Molnár-Sáska Gábor
Professor

Author:
Fekete Fanni
MSc in Applied Mathematics

Budapest, 2022

Acknowledgement

I would like to thank my supervisor, Prof. Gábor Molnár-Sáska for the guidance, support and the interesting thesis topic, as well as for introducing me to financial mathematics. I also would like to thank my colleagues, dr. Gábor Horváth and dr. Florin Ioniță, during my internship for all of the discussions of this thesis, the helpful notes, explanations and guidance. In addition, I would like to thank my friends and family for the unconditional support and motivation they provided.

Contents

Chapter 1

Introduction

This thesis presents the pricing of Mortgage Backed Securities (MBS) using Monte Carlo simulation of interest rates and prepayment processes. In the second chapter we introduce the financial terminology related to fixed-income securities. In Chapter 3 we summarize the concept of securitization and present different types of securities. In the next chapter we look into the valuation methodology of fixed income securities with different types of principal and interest repayment structures. In Chapter 5 we introduce MBS bonds and the impact that conditional prepayment, and default rates have on their prices, followed in Chapter 6 by introducing a Monte Carlo method to simulate interest rates and prepayment rates which are used to value residential MBS bonds. Finally in conclude in Chapter 8.

Chapter 2

Fixed-Income Securities: Defining Elements

Today there are two ways for an entity to raise capital: issuing equity or fixed-income securities. Fixed income is a commonly used method for governments and companies to borrow money from investors. This chapter is based on the 2023 CFA Program Curriculum Level I book [1] and Brealey's (et. al) Corporate finance book [2].

Fixed–income securities differ from equities in several ways:

- a fixed–income investor has no ownership rights in the business.
- the money borrowed, or **principal**, is repaid at maturity. In addition, interest on the borrowed money is paid periodically.
- fixed–income investors receive payments before the shareowners, therefore the risk is lower.

The most common type of fixed-income securities are **bonds**. Brealey et al. (2022) define bonds in the following way [2]:

Definition 1 (Bond)**.** *A **bond** is a fixed-income instrument that represents a loan made by an investor to a borrower (typically corporate or governmental).*

A bond could be thought of as an I.O.U. ("I Owe You"; a document that recognizes the existence of debt) between the lender and borrower that includes the details of the loan and its payments. Bonds are used by companies, municipalities, states, and sovereign governments to finance projects and operations. Owners of bonds are **debtholders**, or **creditors**, of the **issuer**.

Bond details include the end date when the principal of the loan is due to be paid to the bond owner and usually include the terms for variable or fixed interest payments made by the borrower.

2.1 Bond characteristics

At first we should take a look on the basic features of a bond.

Definition 2 (Issuer)**.** *The entity that wants to borrow money, therefore issues bonds, is called the* ***issuer****. Its responsibilities include paying interest and principal to investors.*

Governments need money in order to fund projects, build schools, hospitals, roads or other infrastructure. Furthermore, if corporations and companies want to grow their business, buy properties, hire employees, buy new machines, they may need money to do so.

An average bank usually cannot provide enough capital in loans for the above reasons, therefore the government's and company's best choice is in issuing bonds. This allows many individual investors to lend a portion of the capital needed and to support these projects.

Definition 3 (Credit risk)**.** *The possibility that the issuer fails to make interest and principal payments as they come due is called* ***credit risk****.*

All bonds are exposed to this risk. Credit rating agencies (CRAs or ratings services) examine the issuers creditworthiness and assign them ratings based on credit risk. Borrowers, who are considered low credit risk are charged lower interest rates. [2]

Definition 4 (Maturity)**.** *The date on which the last payment is made for a bond is called the* ***maturity date****, in other words it is the date on which the bond issuer will pay the bondholder the face value of the bond.*

Once a bond is issued, the time remaining until maturity is called the **tenor of the bond**.

Definition 5 (Face value)**.** *The **face value**, or in other words par value, maturity value or redemption value, is the amount of money the issuer promises to pay the bondholder at the time of maturity. It also serves as a reference amount when calculating interest payments.*

The price someone pays for the bond may change over the life of the bond, but the face value remains the same. Meaning, if the face value was a certain amount of money at the time of issuing, the bondholder will gain that amount of money when the bond matures.

Definition 6 (Market price)**.** *The **market price** of a bond is the price an investor has to pay to buy that bond in the open market.*

Some may think, if a bond's face value is for example 1000 USD, than its market price would be the same, but that is not always the case. A bond's market price depends on a number of factors. We are going to look into those factors in the upcoming chapters.

We can categorize bonds based on their market price compared to their face value in the following way:

- if the market price is bigger than the face value, then the bond is trading at premium;
- if the market price is less than the face value, then the bond is trading at discount;
- if the market price is equal to the face value, then the bond is trading at par.

Definition 7 (Issue price)**.** *The **issue price** is the price at which the bond issuer originally sells the bonds.*

Definition 8 (Coupon rate)**.** *The **coupon rate** is the rate of interest the bond issuer agrees to pay on the face value of the bond, expressed as a percentage. This can be a fixed–, or a floating rate.*

Definition 9 (Coupon date)**.** ***Coupon dates*** *are the dates on which the bond issuer will make interest payments.The payments can be made in any interval: annual, quarterly, monthly, but the standard is semiannual payments.*

Based on the frequency of coupon payments, we can classify bonds into the following:

Plain vanilla bonds It is the most basic type of bond with periodic fixed interest payments during the bonds' life and the principal payed at maturity.

Floating–rate bonds Unlike in the case of vanilla bonds, the interest rate is not fixed. The coupon payments are based on a floating rate of interest like Libor (London Interbank Offered Rate) at the start of the period. Some bonds specify the coupon rate as a reference rate (such as Libor), plus a spread. The coupon rate and the interest paid in every period changes as the reference rate does.

Zero–coupon bond/pure discount bonds These bonds have only one payment at maturity, therefore they do not make coupon payment over a bonds' life. Also they are typically sold at a discount (less than the face value) at issuance.

2.2 Yield measures

There are two widely used yield measures to describe a bond: the **current yield** and the **yield to maturity**. In the following section we are going to discuss these and the differences between them.

Current yield or running yield

Definition 10 (Current yield)**.** *The* ***current yield*** *is the annual coupon divided by the bond's price and expressed as a percentage.*

For example we have a 3-year annual coupon bond with the face value of 100 USD, issued at 95 USD. The coupon rate is 10%, which means the coupon payments are 10 USD every year. In this case the current yield is $\frac{10}{95} = 10,5\%$.

Yield to maturity

Definition 11 (Yield to maturity)**.** *The **yield to maturity (YTM)** is a way to consider a bond's price. Yield to maturity is considered a long-term bond yield but is expressed as an annual rate: is the internal rate of return on a bond's expected cash flows. In other words, it is the expected annual rate of return an investor will earn if the bond is held to maturity. It is also called as **yield to redemption** or **redemption yield**.*

The YTM calculation is considered very useful when evaluating the attractiveness of one bond relative to other bonds of different coupons and maturity in the market. The formula for YTM involves solving for the interest rate in the following equation:

$$YTM = \sqrt[n]{\frac{Face\ value}{Present\ value}} - 1, \tag{2.1}$$

where n represents the number of years until maturity.

2.3 Principal Repayment Structures

There are a few categories of bond that have unique repayment structure, since not every bond make periodic interest payments and a single lump-sum principal payment at maturity. In this section we look at different ways in which principal and interest payments can be made.

2.3.1 Bullet bond

This type of bond makes periodic interest payments and the full amount of principal at maturity.

For example let us consider a 5-year, 1000 USD bullet bond with 6% coupon paid annually. For such a bond the cash flows can be modeled as seen in Table 2.1, on every periodic payment, the interest rates or coupons are $1000 \cdot \frac{6}{100} = 60$, and at maturity we add the last coupon payment to the principal, resulting in 1060 USD.[1]

[1] All the examples are from the CFA Curriculum Level 1 books [1].

Year	Cash Flow	Interest Payment	Principal Repayment
1	60	60	0
2	60	60	0
3	60	60	0
4	60	60	0
5	1060	60	1000

Table 2.1: The repayment structure of a bullet bond

2.3.2 Amortized bond

These bonds make interest and principal payments over the bonds' life. We can talk about fully amortized bonds and partially amortized bonds.

Fully Amortized Bond

Over a fully amortized bonds' life the principle is paid little by little in fixed equal payments, therefore it is repaid in full when maturity is due. These payments that the issuer makes periodically consist of interest and a part of principal.

Consider as an example a 5-year, 1000 USD bond with 6% coupon paid anually and with 6% market interest rate.

Year	Cash Flow	Interest Payment	Principal Repayment
	a = b + c	b	c
1	237.4	60	177.4
2	237.4	49.36	188.04
3	237.4	38.07	199.32
4	237.4	26.11	211.28
5	237.4	13.44	223.96

Table 2.2: The repayment structure of a fully amortized bond

Based on the formula of a bond's present value 4.1, we can calculate the periodic payments in the following way: $PV = -1000$, $r = 0.06$, $FV = 0$.

$$1000 = \frac{PMT}{1.06^1} + \frac{PMT}{1.06^2} + \frac{PMT}{1.06^3} + \frac{PMT}{1.06^4} + \frac{PMT + 0}{1.06^5},$$

and from that we get:

$$PMT = 237.4.$$

The periodic payments consist of interest and principal payments, and we know that the coupon payment (or the interest paid) is 6% anually. We can calculate this coupon by multiplying the outstanding principal amount by $\frac{6}{100}$, then we get the first coupon payment, that is 60. By using the $PMT = interest + principal$ concept, we can get the first principal payment as 177.4. By using these steps, we can derive the interest payments and principal payments, as Table 2.2 presents.

Contrary to bullet bonds, fully amortized bond carry lower credit risk, because in this case, the borrower continuously paid parts of the money amount back, and does not need to pay when maturity comes due.

2.3.3 Partially Amortized Bond

As opposed to the fully amortized bond, over a partially amortized bonds' life only a part of principle payment is made. The remaining principal is paid at maturity as a balloon payment.

Let us consider, similar to the previous examples, a 5-year, 1000 USD bond with 6% coupon payments made annually. Let as assume that the borrower wants to make the principal payment divided into two parts: the amortizing part and the balloon payment part. For the amortizing part we use the same techniques like in the previous example, but here let us assume that the balloon payment part would be the remaining amortizing part plus 200 USD. In this case $PV = 1000$, $r = 0.06$, $FV = 200$, $coupon = 6\%$.

Following the formula 4.1, we get:

$$1000 = \frac{PMT}{1.06^1} + \frac{PMT}{1.06^2} + \frac{PMT}{1.06^3} + \frac{PMT}{1.06^4} + \frac{PMT + 200 + 0}{1.06^5},$$

therefore

$$PMT = 201.92$$

and the last payment is $201.92 + 200 = 401.92$. Then we get that the first interest payment is 60 and the first principal payment is 141.92. After that, using the same methodology as in case of fully amortizing bonds, we can calculate the periodic payments as in Table (2.3).

Year	Cash Flow	Interest Payment	Principal Repayment
1	201.92	60	141.92
2	201.92	51.48	150.43
3	201.92	42.46	159.46
4	201.92	32.89	169.03
5	201.92	22.75	379.17

Table 2.3: The repayment structure of a partially amortized bond

As these examples illustrate, the interest payments are higher in partially amortized bonds as opposed to fully amortized bonds. In case of a fully amortized bond the interest payments reduce more over time and principal repayments increase at a slower pace as compared to partially amortized bonds.

2.4 Coupon Repayment Structures

In this section we focus on different coupon payment structures.

Fixed periodic coupons

It is the most common coupon payment in which a fixed interest is paid over a fixed period of time.

Floating-rate notes (FRNs)

These notes consists of two components: a reference rate, such as LIBOR, and a spread or margin.

LIBOR is a variable, changing as the market conditions change. Hence on every payment period the reference rate might be different, changing the amount of interest in that said period.

The spread is based on the credit rating of the issuer. The higher the credit rating, the lower the spread will be, therefore the lower will be the amount of interest the issuer has to pay the investor.

One important thing to note about floating rate notes: they make quarterly payments. When calculating the interest payments, we first have to look at the three month (or 90 day) LIBOR rate.

Floating rate notes face less interest rate risk as opposed to fixed interest rate bonds. As the interest rate changes, the price of the bond changes too. In case of FRNs, the interest rates reset periodically after each interest payment, after that they follow up with the new market rate.

Floating rate notes are the most suitable bonds for investors who expect an increase in market interest.

Step-up coupon bonds

Step-up coupon bonds are mostly beneficial for bondholders, if an event of rapid interest rate rising occurs. The main idea behind these bonds is that coupon rates increase by specified margins or amounts on specified dates.

When interest rates are at such a high level bond issuers cannot call the bond back so in that instance bond holders are compensated for higher interest income. On the other hand if interest rates are declining so the step up feature gives incentive to the issuer of the bond to call the bond back before rates goes up a predetermined amount beyond which there is no benefit of calling the bond back.

Credit-linked coupon bonds

Credit-linked coupon bonds have a strong link to the credit rating, as the name suggests. Hence, the coupon rate shifts when the credit ratings fluctuate. The higher the credit rating, the lower the coupon rate will be, and vice versa.

If a company trusts their credit rating, they can choose credit-linked coupon bonds and save on the amount of coupon payment that needs to be paid to the investor.

On the other hand, if a bondholder wants to claim higher coupon payments, they can invest in bonds issued by a company with a tendency towards decreasing credit ratings. In this case the investor receives higher coupon payments, if they take the risk that with continuously decreasing credit rating may result in the bond's default.

Payment-in-kind coupon bonds

In this kind of bond the issuer has the option to give more bonds to the bondholders instead of cash payments. This is beneficial is the bond issuer is looking forward towards expansion and for that requires a good amount of capital. Also beneficial for issuers who are financially distressed and fear liquidity and solvency problems. Investors get benefit by demanding higher yield so as to get compensated for higher credit risk of the issuer.

Deferred coupon bonds

Deferred coupon bonds don't make any coupon payments to the investor for a certain amount of time, but later in time pay higher amount of coupon.

These bonds are usually priced below par, that is below the face value of the bond, therefore investors usually like to invest in.

One can say that a zero coupon bond is a type of deferred coupon bond, since those kinds of bonds don't make coupon payment at all during the bond's life, but at maturity are paid in a single lump-sum.

2.5 Different types of bonds

Bonds, in some cases, come with embedded options built into their structures. Investors have unique needs when investing in bonds, and embedded options provide a variety of options and solutions to fit all needs. These options give room for action that the parties can exercise under certain conditions.

In this section we look into the most common forms of embedded options that may occur.

Callable Bonds

These kinds of bonds give the option to the issuer to 'call back' or redeem the bond before the maturity date. Investors face a risk when buying callable bonds, as they could be called back when the bond is rising in value, hence leaving investors without the possibility to reinvest at the previous higher interest rate. Issuers offer a higher yield and sell at a lower price than non-callable bonds, as a compensation.

There are some important details of callable bonds that have to be included in the contract. These details are listed below:

Call price is the price the issuer pays the bondholder at the time of a call.

Call premium is the amount of money paid as a compensation on top of the face value to the bondholders.

Call schedule includes the dates and prices at which the bond may be called.

Call protection period is a period of time during which the bond may not be called.

Call date is the earliest date the bond can be called.

As an example, let us assume a 20-year bond issued on 1 December 2012 at a price of 97.315 USD (as a percentage of the face value). The original face value is 100 USD. The bond can be called in whole or in part every 1 December from 2017. The callable prices are shown in Table 2.4. In this case the call protection period is between 2012 and 2017, since the bond was issued in 2012 and the bonds' earliest call date is in 2017.

Callable bonds are beneficial for the issuer for the following reasons:

- Callable bonds protect issuers from market interest drops.
- Callable bonds give the issuer the opportunity to call old bonds and replace them with new, cheaper bonds. This saves them interest expenses.
- Issuers can use callable bonds to 'advertise' their credit quality.

Year	Call price	Year	Call price
2017	103.78	2023	101.47
2018	103.54	2024	101.21
2019	103.10	2025	100.68
2020	102.81	2026	100.32
2021	102.23	after 2026	100.00
2022	101.59		

Table 2.4: A callable bonds' call prices structure

Putable Bond

A putable bond allows the bondholders to 'put' or sell the bond back to the issuer before maturity. This is valuable for investors who are worried that a bond may fall in value, or if they think interest rates will rise and they want to get their principal back before the bond falls in value.

The bond issuer may include a put option in the bond that benefits the bondholders in return for a lower coupon rate or just to induce the bond sellers to make the initial loan. A putable bond usually trades at a higher value than a bond without a put option, since it is more valuable to the bondholders.

The possible combinations of embedded puts, calls, and convertibility rights in a bond are endless and each one is unique. There is not a strict standard for each of these rights and some bonds will contain more than one kind of "option" which can make comparisons difficult. Generally, individual investors rely on bond professionals to select individual bonds or bond funds that meet their investing goals.

The following details are included when contracting:

- Redemption dates;
- Selling price;
- The number of puts the issuer allows the bondholder.

Based on the number of selling points the issuer allows to the bondholder, we can classify putable bonds into two categories:

One-time put gives investors a single selling opportunity.

Multiple put allows investors multiple sellback opportunities, hence are priced higher.

Furthermore, different bonds allow the bondholder to sellback the bond any time after the put date, only on the put date or only on specified dates, these exercise styles are called American-style put, European-put and Bermuda-style put, respectively.

Convertible Bonds

Convertible bonds are debt instruments with an embedded option that allows bondholders to convert their debt into stock (equity) at some point, depending on certain conditions like the share price. For example, imagine a company that needs to borrow $1 million to fund a new project. They could borrow by issuing bonds with a 12% coupon that matures in 10 years. However, if they knew that there were some investors willing to buy bonds with an 8% coupon that allowed them to convert the bond into stock if the stock's price rose above a certain value, they might prefer to issue those.

The convertible bond may be the best solution for the company because they would have lower interest payments while the project was in its early stages. If the investors converted their bonds, the other shareholders would be diluted, but the company would not have to pay any more interest or the principal of the bond.

The investors who purchased a convertible bond may think this is a great solution because they can profit from the upside in the stock if the project is successful. They are taking more risk by accepting a lower coupon payment, but the potential reward if the bonds are converted could make that trade-off acceptable.

A few terms associated terms with convertible bonds are:

Conversion price Price per share at which the convertible bond can be converted into shares.

Conversion ratio is the number of shares that the bonds can be converted into.

$$Conversion\ ratio = \frac{Par\ value}{Conversion\ price}$$

Conversion value Current share price multiplied by the conversion ratio.

$$Conversion\ value = current\ share\ price \cdot conversion\ ratio$$

Conversion premium is the difference between the convertible bonds' price and the conversion value.

$$Conversion\ premium = convertible\ bond's\ price - conversion\ value$$

Chapter 3

Asset-Backed Securities

In this section we take a look at asset-backed securities, the concept of securitization and its benefits generally, by the 2023 CFA Program Curriculum Level I book [1] .

Asset-backed securities or ABS are based on the securitization principle.

Definition 12 (Securitization)**.** *Securitization is a process during which debt obligations, such as loans or bonds get bundled in a pool of debts, and uses the cash flows from the pool of debts to pay off the bonds created during the process.*

Definition 13 (Securitized Assets)**.** *The instruments which become part of the pool are called securitized assets.*

For example, a bank sells loans to borrowers (or in this example homeowners), then bundles each and every individual loan into a pool of loans. This pool is then sold to a Special Purpose Vehicle (SPV), which is a legal entity. Following that the special purpose vehicle issues bonds to investors.

Remark. *The term mortgage-backed securities (MBS) refer to securities that are backed by real estate mortgages. Asset-backed securities refer to a broader term, that are backed by non-mortgages.*

3.1 The benefits of securitization

In this part, we look into the benefits of the securitization concept from three points of view: the borrowers' or loan originators' (or as in the previous example, homeowners'), the investors' point of view, and the party that connects these two, a bank or financial institution.

The benefits of securitization are as follows:

Benefits to the investors

- during securitization an illiquid asset becomes a liquid security;
- gives direct access to the payment streams of the underlying loans;
- loan pooling results in diversification and lower risk for investors;
- gives opportunity to buy a small part of the borrowers' loans in the form of a security;
- gives exposure to the market, without directly investing in it.

Benefits to the loan borrowers

- lowers the risk;
- the lower risk decreases the cost of borrowing.

Benefits to the bank or financial institution

- enables banks to increase loan origination and monitoring;
- have the ability to lend more money;
- MBSes trade actively in the secondary market, increasing efficiency, liquidity and profitability.

3.2 The role of the special purpose vehicle

The advantage of issuing ABS versus corporate bonds is that the overall funding cost would be lower in the case of ABS. In case of issuing corporate bonds, the investors are exposed to the entire credit risk of the issuing company, meaning any risk that the company is exposed to may lead to a bankruptcy. In this case the bond investors would have to follow the absolute priority rules when trying to recover their investment, together with other creditors of the company. In general, this means that the bondholders may be repaid after other parties. As a result, they would demand higher return for their investment.

In case of issuing an ABS, investors are only exposed to the credit risk of the underlying pool of assets. These assets are usually placed in a Special Purpose Vehicle (SPV). An SPV is a bankruptcy-remote entity from the parent company, meaning the SPV owns the underlying assets. In case the parent company defaults, other creditors do not have a claim on the underlying ABS assets owned by the SPV. As a result, this decreased credit risk means lower funding cost for the issuer.

Chapter 4

Fixed Income Valuation

In this chapter we are going to explore the world and possibilities of bond pricing. We will look into the many factors that can determine the price and the value of a bond. First of all we have to understand the time value of money: how time affects the money's worth.

For better understanding, let us take an example. Let us assume we have 100 USD. If we invest, we can earn 110 USD in a year, with 10% interest rate. We could interpret that a payment of 110 USD in one year equals to 100 USD at the moment. This is why we usually would use the present value of future payments.

In the next section we would explore bond pricing with a market discount rate, yield to maturity,flat price, accrued interest, full price and we would discuss pricing on floating-rate bonds as well.

4.1 Bond pricing with a market discount rate

Definition 14 (Bond's price)**.** *A bond's price can be interpreted as the present value of future cash flows at the market discount rate.*

Definition 15 (Market discount rate)**.** *The discount rate refers to an interest rate required by the investors given the risk of investment in the bond. It is also called the required rate of return or required yield.*

We can express the price of a bond with the following formula:

End of year	Type of cash flow	Money amount	Present value
1	coupon	4	3.77
2	coupon	4	3.56
3	coupon	4	3.36
4	coupon	4	3.17
5	coupon + principal	104	77.71
Total			91.58

Table 4.1: Cash flow structure in case of market discount rate pricing

$$PV\ of\ a\ bond = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + \ldots + \frac{PMT+FV}{(1+r)^N}, \tag{4.1}$$

where

PV = present value;

PMT = coupon payment per period;

FV = face value of the bond at maturity;

r = market discount rate;

N = number of periods until maturity.

For example, let us consider a 5-year, 100 USD bond with a 6% market discount rate and coupon payment 4 USD annually. In this case if we use the 4.1 formula above, we get the present value of said bond at time $t = 0$.

$$PV = \frac{4}{1.06} + \frac{4}{1.06^2} + \frac{4}{1.06^3} + \frac{4}{1.06^4} + \frac{104}{1.06^5} = 91.575.$$

We can also illustrate the cash flows as in Table 4.1.

4.2 Yield measures for floating rate notes

As discussed at section (2.4) of chapter (2) , floating rate notes are divided into two components: a reference rate, such as LIBOR, and a spread, a certain number of basis points. The interest payments change from period to period. The specified spread over the reference rate is often called a **quoted margin** or **discount margin**. This quoted margin is based on the issuers credit rating.

The price of a floating-rate bond can be expressed as a formula:

$$PV = \frac{(Index + QM) \cdot \frac{FV}{m}}{(1 + \frac{Index+DM}{m})^1} + \frac{(Index + QM) \cdot \frac{FV}{m}}{(1 + \frac{Index+DM}{m})^2} + \dots + \frac{(Index + QM) \cdot \frac{FV}{m} + FV}{(1 + \frac{Index+DM}{m})^N}, \tag{4.2}$$

where

PV = present value of the floating rate note;

Index = reference rate;

QM = quoted margin;

FV = face value paid at maturity;

m = periodicity of the floating rate note;

DM = discount margin;

N = number of evenly spaced periods to maturity.

We have to note that the periodicity of a floating rate note is understood as the number of payment periods per year. In this formula $(Index + QM) \cdot \frac{FV}{m}$ is the interest payment per period or similarly to the 4.1 formula, the coupon payment per period. In the 4.1 formula, the coupon payments are divided by $(1 + r)$. In the floating rate case the formula is similar to that, but instead of r, we have the $\frac{Index+DM}{m}$.

Let as assume, as an example, a 3-year floating-rate note pays 3-month Euribor plus 0.75%. Assuming that the FRN is priced at 99 USD, if the 3-month Euribor is at constant 1%, then we can calculate the interest payments for each period and the discount margin as well. In this example $PV = 99$, $FV = 100$, $m = 4$, $N = 12$, $Index = 0.01$, $QM = 0.0075$, and $Index + QM = 0.0175$.

By using the formula (4.2) above, we get

$$-99 = \frac{0.0175 \cdot \frac{100}{4}}{(1 + \frac{0.01+DM}{4})^1} + \frac{0.0175 \cdot \frac{100}{4}}{(1 + \frac{0.01+DM}{4})^2} + \ldots + \frac{0.0175 \cdot \frac{100}{4} + 100}{(1 + \frac{0.01+DM}{m})^{12}}.$$

Therefore the coupon payments are $\frac{0.0175 \cdot 100}{4} = 0,4375$. Solving the above equation for the DM, we get that the discount margin is $2.09\% - 1\% = 1.09\%$.

Chapter 5

Residential Mortgage-Backed Securities

In this chapter we discuss mortgage-backed securities, tranches, SPVs based on the MBSD source book [3].

Mortgage-backed securities

A mortgage-backed security represents an ownership interest in mortgage loans made by financial institutions to finance a borrower's purchase of a home or other real estate. mortgage-backed securities are created when mortgage loans are packaged, or "pooled", by issuers or servicers, and securities are issued for sale to investors. As the underlying mortgage loans are paid off by the borrowers, the investors in the securities receive payments of interest and principal.[3]

Mortgage-backed securities play a crucial role in the availability and cost of housing in the United States. The ability to securitize mortgage loans enables mortgage lenders and mortgage bankers to access a larger reservoir of capital, which makes financing available to home buyers at lower costs and spreads the flow of funds to areas of the country where capital may be scarce.

Asset securitization began when the first mortgage pass-through security was issued in 1970, with a guarantee by the Government National Mortgage Association (GNMA or Ginnie Mae). The most basic mortgage-backed securities, known as pass-throughs or participation certificates (PSs), represent a direct ownership in a pool of mortgage loans. Shortly after this issuece, both the Federal Home Loan Mortgage Corporation (FHMLC or Freddie Mac) and Federal NAtional Mortgage Association(FNMA or Fannie Mae) began issuing mortgage-backed securities.

Although mortgage-backed securities are fixed-income securities that entitle investors to payment of principal and interest, they differ from corporate and Treasury securities in significant ways. With a mortgage-backed security, the ultimate borrower is the homeowner who takes on a mortgage loan. Because the homeowner's monthly payments include both interest and principal, the mortgage-backed security investor's principal is returned over the life of the security, or amortized, rather than repaid in a single lump sum at maturity.

MBS provide payments to investors that include varying amounts of both principal and interest, due to the flexibility that the homeowner has in being able to pay more than the minimum payment required by the loan agreement. As the principal is repaid, or prepaid, the interest payments become smaller because the payments are based on a lower amount of outstanding principal. In addition, while most bonds pay interest semi annually, MBS may pay interest and principal monthly, quarterly or semi annually, depending on the structure and terms of the issue. Most mortgage pass through securities are based on fixed rate mortgage loans with an original maturity of 30 years, but typically most of these loans will be paid off much earlier. (generic criteria maturity, coupon, agency, settlement month and par are determined at the time of the trade).

5.1 Tranches

A portofolio of income-producing assets such as mortgage loans in this case is sold by the originating banks to a Special Purpose Vehicle (SPV) and the cash flows from the assets are then allocated to tranches.These loans are classified by risk, time to maturity, or other characteristics in order to be marketable to different investors. Each portion or tranche of a securitized or structured product is one of several related securities offered at the same time, but with varying risks, rewards and maturities to appeal to a diverse range of investors.[4]

For example we can divide these tranches into three categories: senior tranches, mezzanine tranches(the middle tranches) and junior tranches. Senior tranches typically contain assets with higher credit ratings than junior tranches. The senior tranches have first lien on the assets—they're in line to be repaid first, in case of default. Junior tranches have a second lien or no lien at all.[4]

Investors receive monthly cash flow based on the MBS tranche in which they invested. They can either try to sell it and make a quick profit or hold onto it and realize small but long-term gains in the form of interest payments. These monthly payments are bits and pieces of all the interest payments made by homeowners whose mortgage is included in a specific MBS.[4]

5.2 The conditional default rate

The conditional default rate (CDR) is the percentage of mortgages within a pool of loans in which the borrowers have fallen more than 90 days behind in making payments to their lenders.[5]

The CDR evaluates losses within mortgage-backed securities. The CDR is calculated on a monthly basis and is one of several measures that those investors look at in order to place a market value on an MBS. The method of analysis emphasizing the CDR can be used for adjustable-rate mortgages as well as fixed-rate mortgages. [5]

The CDR can be expressed as a formula:

$$CDR = 1 - \left(1 - \frac{D}{NDP}\right)^N, \tag{5.1}$$

where $D = Amount\ of\ new\ default\ during\ the\ period$,

$NDP = Non - defaulted\ pool\ balance\ at\ the\ beginning\ of\ the\ period$,

$N = Number\ of\ periods\ per\ year.$

To protect investors from loosing money on default, the security usually contains a *recovery rate*. Recovery rate is the extent to which principal and accrued interest on defaulted debt can be recovered, expressed as a percentage of face value.

The recovery rate enables an estimate to be made of the loss that would arise in the event of default, which is calculated as $(1 - \textit{Recovery Rate})$. Thus, if the recovery rate is 60%, the loss given default or LGD is 40%.

The type of instrument and its seniority within the tranches are among the most important determinants of the recovery rate. The recovery rate is directly proportional to the instrument's seniority, which means that an instrument that is more senior in the tranches will usually have a higher recovery rate than one that is lower down in the tranches.[5]

5.3 The conditional prepayment rate

A conditional prepayment rate (CPR) is an estimate of the percentage of a loan pool's principal that is likely to be paid off prematurely. The estimate is calculated based on a number of factors, such as historical prepayment rates for previous loans similar to the ones in the pool and future economic outlooks. [6]

The CPR can help investors measure the likely return on an investment and their prepayment risk, especially in changing economic conditions.[6]

For example, in a time of declining interest rates, homeowners often prepay their mortgages to refinance them at a lower rate. When that occurs, the mortgage-backed security that their mortgage is packaged into may be paid back sooner than expected, with the proceeds released back to the investor. The investor then needs to choose a new security to invest in, which is likely to have a lower rate of return since interest rates overall have dropped since their original investment.[6]

One can measure prepayments in two ways: single month mortality rate, which is a monthly measure, and conditional prepayment rate, which is an annual measure. The CPR can be expressed as a formula:

$$CPR = 1 - (1 - SMM)^{12}, \tag{5.2}$$

where *SMM (or Single Monthly Mortality) = In effect, the amount of principal on mortgage-backed securities that is prepaid in a given month.*

In case of a prepayment, the borrower has to pay additional fees in addition the principal, like a punishment for paying back the loan early. Therefore the investors gain a little compensation for their loss.

Chapter 6

Valuation of Mortgage-Backed Securities

Throughout the years, different types of models have been developed to deal with prepayment risk in any MBS.These models can be divided into two main classes: **structural** and **reduced-form** models.

When valuing mortgage-backed securities, we need to take into account the different factors that can influence a mortgagor's decision to prepay. In structural models, the prepayment is a call option exercised by the mortgagor to minimize the cost of his mortgage. The prepayment decision is entirely determined within the model by option pricing approaches. In these models, differences in prepayment costs drive the heterogeneity among mortgagors making them prepay at different times. Although this endogenous approach can yield considerate insights into the workings of idealized mortgages, it is usually difficult to employ these models for the purpose of empirical estimation. Examples for these kind of models are [7], [8], [9].

In contrast, in case of reduced-form models, the prepayment is not determined internally, but derived from a function of some random processes. These kinds of models are easily estimated by observed prepayment data, since their lack of dependence on unobservable factors, and the MBS prices are derived using Monte Carlo simulations, finite difference methods or lattice methods. Examples for these kinds of models can be found in [10] and [11]. A few years ago reduced-form models appeared that use techniques from credit-risk modeling. Goncharov [12] presented a general **intensity-based** prepayment model with an endogenously defined mortgage rate process. Rom-Poulsen [13] obtained semi-analytical solutions for the value of fixed-rate callable MBSs by presuming the mortgage pool size taking an intensity-base model.Brunel and Jribi [14] modeled the prepayment risk by introducing a prepayment factor $(Q_t)_{t\leq 0}$ which represents the percentage of the initial loan still outstanding at time t. Brunel's model's big advantage is that details of the debt pool are not needed for purposes of calibration. Brunel and Jribi [14] only considered the situation where short term risk-free interest rate and the prepayment rate are all constants, not taking into account the fact that these rates do not behave like that in practice. In general, prepayment rates would rise when interest rates fall because the mortgagors would have more incentive to refinance their mortgages, whereas prepayment rates would fall when interest rates rise because the mortgagors would like to preserve their mortgages in this case. Based on the ideas presented in [13] and [14], Qian, Jiang, Xu and Wu [15] introduced a stochastic process $Q_t = e^{-\int_0^t \lambda_s ds}$ to model the prepayment factor, while assuming that the prepayment rate λ_t is inversely proportional to the stochastic interest rate r_t, which follows a Cox-Ingersoll-Ross (CIR) process [16]. In addition, they also explored the dependence of MBS prices on the mortgage parameters.

In this chapter we carefully go through the description of a possible valuation model for mortgage backed securities (6.1), after that we discuss the estimation of the interest rate processes (6.2). Finally, we talk about the Monte-Carlo simulation of the processes (6.3). This chapter is based on on the work of Schwartz and Torous[11], Boyle [17] and Qian, Jiang, Xu and Wu [15].

An important aspect to note is that we assume a default-free fixed-rate, fully amortizing mortgage.

Let us denote

- by $P(0)$ the amount of principal at the mortgage's origination;
- by c the continuous contract rate and

- by T the term to maturity from the origination.

Therefore, we can express the **total payout rate** by the following formula:

$$A = \frac{cP(0)}{1 - e^{-cT}}, \tag{6.1}$$

then the **principal outstanding at time** t can be written as:

$$P(t) = \frac{A}{c} \cdot (1 - e^{-c(T-t)}). \tag{6.2}$$

6.1 Valuation model

The valuation of an RMBS bond proceeds in the following main steps:

1. Project the values of the interest rates until the bond's maturity.
2. Project the bond cash flows, which depend on the CPR and CDR values. Depending on the type of valuation model, the CPR and CDR values can either be estimated by an analytical function or a separate model can be built to simulate their values based on economic factors.
3. The bond valuation is calculated based on the discounted cash flow method (4.1).

Assumptions:

1. All information about of the term of interest rates can be summarized by two state variables: r, the instantaneous risk-free rate of interest, and l, the yield on a default free consol [18].

2. The dynamics of the state variables r and l are modeled using the Cox-Ingersoll-Ross (CIR) model [15]:

$$dr = a_1(b_1 - r)dt + \sigma_1\sqrt{r}dz_1 \tag{6.3}$$

$$dl = a_2(b_2 - l)dt + \sigma_2\sqrt{l}dz_2 \tag{6.4}$$

where z_1 and z_2 are standardized Wiener-processes, with correlated increments:

$$dz_1 dz_2 = \rho dt, \tag{6.5}$$

with ρ as the correlation coefficient and parameters $a_1, b_1, \sigma_1, a_2, b_2, \sigma_2$ determined based on historical interest rate data. In a CIR model the standard deviation factor $\sigma\sqrt{r}$ ensures rates remain positive for all positive values $a,\ b$.

3. The mortgages are prepaid based on the conditional prepayment rate (CPR) function, π, defined following Schwarz and Tourus's proportional hazard model [11] as follows:

- first, we derive from the cumulative distribution function (CDF) of the logistic distribution the probability density function (PDF) and cumulative distribution function of the log-logistic distribution.
- from the log-logistic PDF and CDF, the log-logistic suvival function and hazard function can be obtained.
- then we can calculate where the log-logistic hazard function takes its maximum value.

A random variable T has a log-logistic distribution, if its logarithm has logistic distribution.

Generally speaking, let us assume that the CDF of $\log T$ is $G(t)$, its PDF is $g(t)$, therefore we can calculate the CDF of $T = P(T < t) = P(\log T < \log t) = G(\log t)$, $t > 0$. After differentiating with respect to t, we get that the PDF of T is:

$$f(t) = \frac{1}{t} g(\log t). \tag{6.6}$$

Generally, if here g is part of a shape-parametric family, that is $g(t) = h(t - \nu)$, then in place of the ν shape parameter we can write $-\log \lambda$, because this way in the PDF of T, λ would be the inverse of the scale parameter:

$$f(t) = \frac{1}{t} h(\log t + \log \lambda) = \frac{\lambda}{\lambda t} h(\log \lambda t). \tag{6.7}$$

The CDF and PDF of the logistic distribution ($Logist(a, b)$) are as follows:

$$G(t) = \frac{1}{1 + e^{-\frac{t-a}{b}}}, \tag{6.8}$$

$$g(t) = \frac{e^{-\frac{t-a}{b}}}{b[1 + e^{-\frac{t-a}{b}}]^2}. \tag{6.9}$$

If $\log T \sim Logist(-\log\lambda, \frac{1}{\alpha})$, where $\lambda,\ \alpha > 0$, then for T we obtain:

$$f(t) = \frac{\alpha\lambda(\lambda t)^{\alpha-1}}{[1+(\lambda t)^{\alpha}]^2}, \tag{6.10}$$

where $\lambda > 0$ is the scale parameter (also the median of the distribution), and $\alpha > 0$ is the shape parameter. From the survival functions' and the hazard functions' formula:

$$\bar{F}(t) = 1 - F(t) = P(T \geq t) \tag{6.11}$$

$$R(t) = -\log\bar{F}(t), \tag{6.12}$$

we can get the hazard rate formula, if F is absolutely continuous:

$$r(t) = R'(t) = \frac{f(t)}{\bar{F}(t)}. \tag{6.13}$$

It is clear that

$$\bar{F}(t) = e^{-R(t)} = e^{-\int_0^t r(s)ds}. \tag{6.14}$$

We can apply these formulas for the previously derived log-logistic distribution, then the survival function of the log-logistic distribution can be expressed as:

$$\bar{F}(t) = \frac{1}{1+(\lambda t)^{\alpha}}, \tag{6.15}$$

then, from equations (6.10)-(6.15), we can derive the hazard rate:

$$r(t) = \frac{\alpha\lambda(\lambda t)^{\alpha-1}}{1+(\lambda t)^{\alpha}} \tag{6.16}$$

The motivation for using survival functions and hazard rates for pricing MBS bonds is that one can think about a mortgage's state as in survival analysis. The question of prepayment can be expressed: what is the probability that a mortgage will not prepay (survive) until time t, which is equal to the survival function above, $P(T > t)$.

Schwartz and Tourus [11] define their prepayment function as a hazard function, depending on $\underline{v} = (v_1, v_2, \ldots v_s)$ a vector of explanatory variables or covariates, and $\underline{\theta} = (\theta_1, \ldots \theta_k)$ a vector of parameters to be estimated:

$$\pi(t;\underline{v},\underline{\theta}) = \frac{f(t;\underline{v},\underline{\theta})}{\bar{F}(t;\underline{v},\underline{\theta})}, \tag{6.17}$$

where $f(t; \underline{v}, \underline{\theta})$ is the PDF and $\overline{F}(t; \underline{v}, \underline{\theta})$ is the survival function. The prepayment function $\pi(t; \underline{v}, \underline{\theta})$ describes the instantaneous rate of prepayment at $T = t$, if the mortgage has not been prepayed prior to time t.

Schwartz and Tourus model the prepayment function (6.17) above by a proportional hazards model:

$$\pi(t; \underline{v}, \underline{\theta}) = \pi_0(t; \gamma, p) exp(\underline{\beta v}), \tag{6.18}$$

where β and v are coefficients on which the prepayment may depend on, $\pi_0(t; \gamma, p)$ is given by a the log-logistic hazard function, which we derived above, substituting α by p and λ by γ:

$$\pi_0(t; \gamma, p) = \frac{\gamma p (\gamma t)^{p-1}}{1 + (\gamma t)^p}. \tag{6.19}$$

If we take the derivative of equation (6.19), we can see that it takes its maximum value at:

$$t^* = \frac{(p-1)^{\frac{1}{p}}}{\gamma}, \tag{6.20}$$

for $p > 1$, that is the probability of prepayment increases from zero till t^* and decreases to zero after, like the Figure (6.1) shows. This is consistent with the fact that conditional repayment rates (CPR) are low in the early life of a mortgage, increasing afterwards.

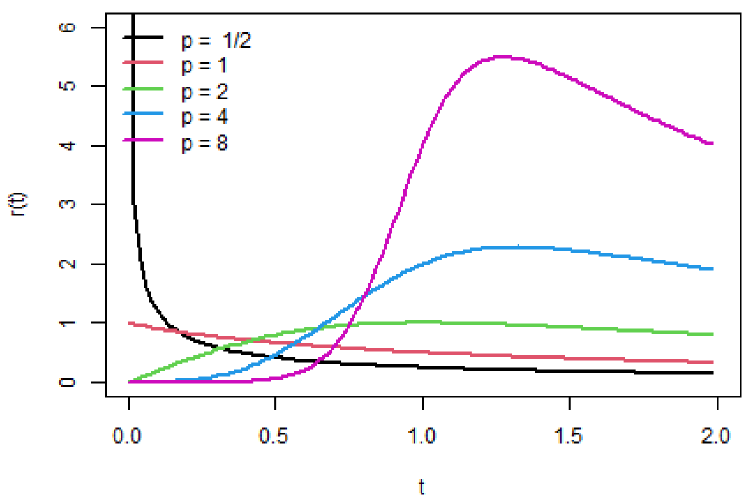

Figure 6.1: Hazard rate function for different shape parameters

After taking into account every aspect that could determine the prepayment function, such as seasonality, economic aspects, refinancing costs etc., we can model it as follows:[2]

$$\pi = \pi(l, x, y, t, c), \tag{6.21}$$

therefore π depends upon the following variables:

- $l(t)$ = prevailing consol yield;
- c = mortgage's contract rate;
- $x(t)$ = the history of past interest rates;
- $y(t)$ = the relative proportion of the pool previously prepaid;
- t = time.

The state variable $x(t)$ can be determined in the following way [19]:

$$x(t) = \alpha \int_{\infty}^{0} e^{-as} l(t-s)\ ds, \tag{6.22}$$

as an exponential average of the past consol yields.

As mentioned above, $y(t)$ defines the relative proportion of the pool, that has been prepaid previously, and we can give its dynamics by:

$$dy = -y(\pi + A \cdot P^{-1} - c)dt. \tag{6.23}$$

Given these assumptions, we can determine the value or price of a mortgage backed security as

$$B = B(r, l, x, y, t). \tag{6.24}$$

which has to satisfy the following partial differential equation:

$$\begin{aligned} &\frac{1}{2} r^2 \sigma_1^2 B_{rr} + rl\rho\sigma_1\sigma_2 B_{rl} + 1/2 l^2 \sigma_2^2 B_{ll} (a_1 + b_1(l-r) - \lambda_1\sigma_1 r) B_r + l(\sigma_2^2 + l - r) B_l \\ &\quad + \alpha(l-x)B_x - y(\pi + A \cdot P^{-1} - c)B_y + B_t - (r+\pi)B + \pi P(t) + A = 0, \end{aligned} \tag{6.25}$$

[2]For more detailed explanation see Schwarz and Tourus [11]

where λ_1 is none other than the market price of short-term interest rate risk. Because of the mortgage's fully amortizing structure, B also has to satisfy the terminal boundary condition:

$$B(r, l, x, 0, T) = 0. \tag{6.26}$$

The value of the mortgage-backed security reflects the fact that at each point in time there exists a possibility of prepaying, this depending upon the current state of economy as summarized by the model's state variables.

6.2 Interest rate process estimation

The coefficients from the partial differential equation (6.25) depend on the parameters of the interest rate processes (6.3)-(6.4). By estimating these parameters and the market price of the short-term interest rate risk, we can implement the mbs valuation method with the prepayment function.

To estimate the parameters of the interest rate processes we need sufficient data on r and l.

The parameter estimation can be done by given discrete approximations to the interest rate processes. Brennan and Schwartz [18] use an iterative Aitken procedure to evaluate the resultant system and to provide a maximum likelihood approximation.

The estimated parameters a_1 and b_1 from the drift of the short-term interest rate process are statistically significant, while the estimated parameters a_2, b_2, and c_2 from the drift of the long-term interest rate processes are statistically insignificant. The estimated standard deviation of proportional changes in short-term interest rates exceeds the estimated standard deviation of proportional changes in long-term interest rates, $\hat{\sigma}_1 > \hat{\sigma}_2$. Finally, the estimated correlation coefficient is consistent with unanticipated proportional changes in r and 1 being positively correlated.[11]

6.3 Monte Carlo method

We use Monte Carlo simulation for solving the partial differential equation (6.25) with the boundary condition (6.26). Since we have five variables - $r(t)$, $l(t)$, $x(t)$ and $y(t)$, and the deterministic state variable t- it would be complicated to estimate the partial differential equations by finite difference methods.

We need to generate r and l by correlated CIR processes[16] as discussed previously :

$$dr = a_1(b_1 - r)dt + \sigma_1\sqrt{r}dZ_1 \tag{6.27}$$

$$dl = a_2(b_2 - l)dt + \sigma_2\sqrt{l}dZ_2 \tag{6.28}$$

The solution process is the following:

1. Firstly, we generate the correlated random variables r and l at every month until maturity or until prepayment.
2. Next we calculate the probability that the pool will be prepaid in a given month, using the prepayment function π, which enables us to derive the MBS cash flows.
3. Thereafter, we can calculate the present value of the cash flows, which can be interpreted as the mortgage backed security's value or price.
4. Finally, if we repeat these steps for different realizations of the interest rate paths, and taking the average of the values we calculated in the above point, we get the solution of the partial differential equations.

Chapter 7

Numerical implementation

In this chapter we calibrate the interest rate model using historical data for short term interest rates (3-month US Treasury Bills) and long term interest rates (10-year US Government Bonds). Next, we evaluate the prepayment model presented in section 6.1 and calculate the MBS bond valuation. In addition, we investigate the dependence of bond valuation on interest rate volatility and prepayment hazard rate assumptions.

7.1 Interest rate simulation

We begin with calibration of the interest rate model using historical data available from the US Federal Reserve Bank of St. Louis for short term interest rates (3-month US Treasury Bills) [20], and long term interest rates (10-year US Government Bonds) [21]. Figure 7.1 below shows the historical values of interest rates starting in 2010.

Given the historical time series of interest rates r_i, $i = 1, \ldots n$, the calibration procedure for the CIR interest rate model uses the following steps:

1. Calibrate the long term mean(b) by taking the average of the absolute interest rates, $b = (\sum_{i=0}^{n} r_i)/n$.
2. Calculate the interest rates relative to the long term average as: $\overline{r}_i = r_i - b$

3. Calibrate the discrete drift parameter (d), by minimizing the sum of residuals:

$$R(d) = \frac{(\overline{r}_i - d \cdot \overline{r}_{i-1})^2}{r_i} \tag{7.1}$$

4. Next we calculate the speed of reversion (a), as $a = -\ln(d)$

5. Compute the discrete volatility by:

$$\sigma_d = \sqrt{\frac{R(d)}{n-1}} \tag{7.2}$$

6. Finally, we estimate the continuous volatility (σ):

$$\sigma = \sigma_d \cdot \sqrt{\frac{2a}{1-e^{-2a}}} \tag{7.3}$$

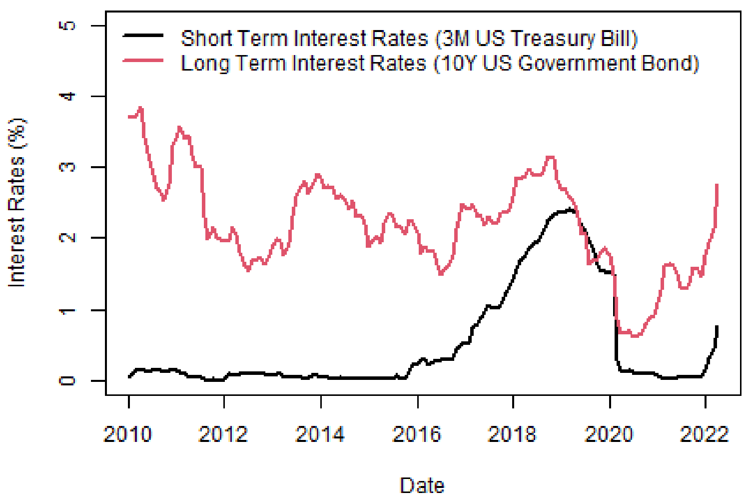

Figure 7.1: Historical short and long term interest rate values

Table (7.1) presents the calibrated parameters for short term (r) and long term (l) interest rates.

Parameter	Short term rates (r)	Long term rates (l)
b	0.005832	0.0204258
d	0.99914	0.96630
a	0.00086085	0.034283
σ_d	0.023582	0.014722
σ	0.023592	0.014975

Table 7.1: Calibrated parameters for the short and long term interest rate processes.

After the parameter calibration, we can generate different interest rate paths using the equations (6.27)-(6.28) with the calibrated parameter values. Figure 7.2 shows 30 sample interest rate paths generated via Monte Carlo simulation.

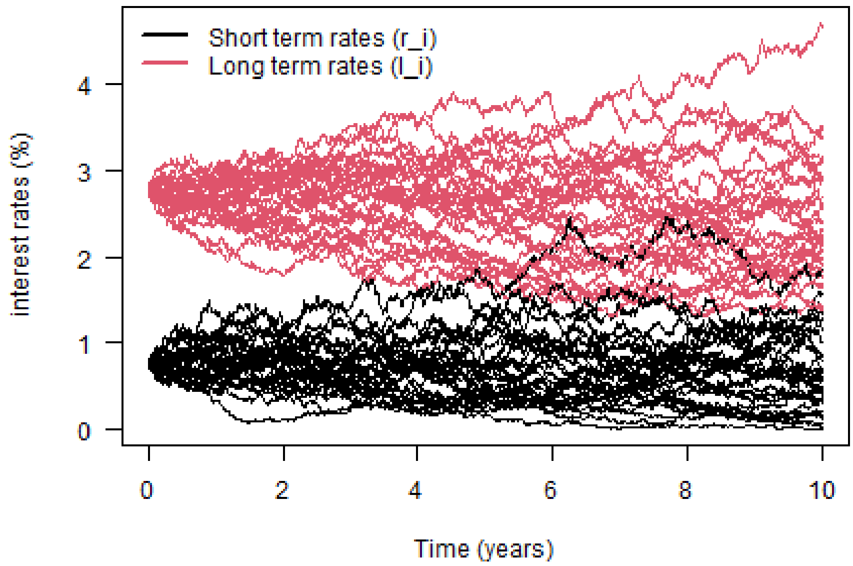

Figure 7.2: Sample short and long term interest rate paths

We integrated numerically the CIR stochastic differential equations using Euler's method with integration step $dt = 0.01$ as implemented in the *Sim.DiffProc* R package [22].

7.2 Prepayment modeling

In general, the cash flows of mortgage backed security are derived from a pool (set) of mortgages that have similar characteristics in terms of the principal balance, coupon rate, underlying property value and borrower characteristics (income, credit score, total assets etc.).

The prepayment behavior of a mortgagor can be influenced by a number of factors, as discussed in section 6.1, including economic factors such as interest rates, cost of refinancing, mortgage age, demographic characteristics of mortgagors and other geographic factors. In the proportional hazard rate model of Schwartz and Tourous [11] given by equation (6.18), these covariates have a comparable effect at all mortgage ages. This model in choice is supported in the literature. Given that there is relatively little publicly available information related to the mortgage prepayment data, in this thesis we will use the parameter values calibrated in reference [11], presented in Table 7.2.

Parameter	Value
γ	0.01572
p	2.35014
β_1	0.39678
β_2	0.00356
β_3	3.74351

Table 7.2: Proportional hazard rate model parameters

Given the interest rate simulation from Section 7.1, we can proceed to evaluate the prepayment hazard rate:

$$\pi(t; \underline{v}, \underline{\theta}) = \pi_0(t; \gamma, p) \cdot exp(\underline{\beta} \underline{v}), \tag{7.4}$$

$$= \frac{\gamma p (\gamma t)^{p-1}}{1 + (\gamma t)^p} \cdot exp(\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3), \tag{7.5}$$

with the covariates v_1, v_2, v_3 defined as

- $v_1(t) = c - l(t-s),\ s \geq 0$ – measuring the incentive to refinance as the difference between the mortgage coupon rate c and the lagged long term interest rate l.

- $v_2(t) = (c - l(t-s))^3,\ s \geq 0$ – the cubic power of v_1 introduced to further accelerate prepayments when the refinancing rates are significantly lower than than the mortgage coupon rate c.
- $v_3(t) = \ln(AO_t/AO_t^*)$ – is the ratio of the outstanding principal at time t, AO_t, divided by the outstanding principal at time t in the absence of prepayment, AO_t^*,

where the lag time s was determined to be 3 months during the parameter fitting.

At any given month, an individual mortgage in an MBS pool can have one of the following states: (i) payed the scheduled principal and interest amount, (ii) prepay partially or in full the remaining mortgage balance, and (iii) default. Investors do not carry the default risk for mortgages in pools issued by the US Federal Housing Agencies (Government National Mortgage Association - GNMA, Federal Home Loan Mortgage Corporation - FHMLC, Federal National Mortgage Association - FNMA), which guarantee interest and principal payments for MBS bonds. As a result, for agency mortgages we only model prepayment and not default risk.

We simulated the prepayment times from the CPR function above using the flexible-hazard method implemented in the *coxed* R package [23].

7.3 Valuation of MBS bonds results

We calculate the price of MBS bonds by applying the discounted cash flow method presented in equation (4.1). The mortgage cash flows are generated based on the mortgage coupon rate c and the simulated prepayment function described in Section 7.2. The discount factors are calculated based on the interest rates simulated in Section 7.1. Finally, we calculate the MBS bond price by averaging the prices evaluated on each interest rate path.

Table 7.3 shows the prices of MBS bonds with a 10 year maturity and the fixed coupon rate $c = 3\%$ under a number of interest rate and volatility conditions. The first row presents the pricing using under the interest rate values at the end of April 2022 with the volatility parameters $\sigma_1,\ \sigma_2$ calibrated as shown in Section 7.1. The second and third rows of the table show the MBS prices with the same interest values, but with the volatilities scaled by 1.5 and 2. We note that the increased volatility decreases the MBS bond prices, but with a rather small value given the combination of impacts on prepayments and discount factors. In the last four rows we fixed the volatility parameters and increased the interest rate values to conditions observed in market environments when interest rates peaked in: Jan 2019, March 2007, Nov 2000 and March 1989. The MBS bond prices in these cases significantly decreased as the interest rate increased. High interest rate environments lead to a decrease in prepayments, since mortgagors have less incentive to refinance their loans.

#	r	l	σ_1	σ_2	Price
1	0.75	2.75	0.023592	0.014975	99.252
2	0.75	2.75	0.035388	0.022462	99.237
3	0.75	2.75	0.047184	0.029950	99.222
4	2.5	2.71	0.023592	0.014975	97.532
5	4.8	5	0.023592	0.014975	95.314
6	6	5.72	0.023592	0.014975	94.174
7	9	9.36	0.023592	0.014975	91.383

Table 7.3: MBS prices on different interest rate and volatility conditions

We investigated the convergence of the simulated prices by varying the number of Monte Carlo paths. Table 7.4 below shows that for the parameters from the current interest rate environment shown in the first row of Table 7.3, the mean price converges starting at 100 paths.

N paths	Mean price	Standard deviation
50	99.256	0.225
100	99.252	0.188
200	99.253	0.192
300	99.259	0.185

Table 7.4: Convergence of the Monte Carlo price estimation

In this thesis we proposed a valuation methodology for MBS bonds using a hazard rate approach to model prepayment, rather than using individual mortgage characteristics. This approach has the advantage of being easier to calibrate and capturing directly the empirical observations that mortgagors do not always follow optimal refinancing strategies [11].

Chapter 8

Conclusion

This thesis presented a Monte Carlo pricing model of Mortgage Backed Securities based on simulation of interest rate and prepayment processes. Given a set of assumptions about the evolution of interest rates and their influence on prepayment rates, we investigated the impact of interest rate volatility and mean reversion speed on the valuation of residential MBS bonds. We conclude that this pricing model captures empirical observations about the evolution of mortgage prices with different economic variables.

Bibliography

[1] CFA Institute. *2023 CFA Program Curriculum Level I*. Vol. 5. USA: CFA Institute, 2022. ISBN: 9781953337177. URL: https://ift.world/booklets/fixed-income-introduction-to-asset-backed-securities-part1/.

[2] R.A. Brealey et al. *Principles of Corporate Finance*. 14th. New York: McGraw-Hill, 2022. ISBN: 9781266032806. URL: https://www.mheducation.com/highered/product/1264080948.html.

[3] DTCC. *MBSD Source Book*. USA: MDTCC, 2012. URL: https://www.dtcc.com/clearing-services/ficc-mbsd/ficc-mbsd-user-documentation.

[4] James Chen. *Tranches*. 2020. URL: https://www.investopedia.com/terms/t/tranches.asp.

[5] Marshall Hargrave. *Constant Default Rate (CDR)*. 2021. URL: https://www.investopedia.com/terms/c/constant-default-rate.asp.

[6] Adam Hayes. *Conditional Prepayment Rate (CPR)*. 2022. URL: https://www.investopedia.com/terms/c/cpr.asp.

[7] Richard Stanton. "Rational Prepayment and the Valuation of Mortgage-Backed Securities". In: *The Review of Financial Studies* 8.3 (1995), pp. 677–708. ISSN: 08939454, 14657368. URL: http://www.jstor.org/stable/2962236.

[8] James B. Kau and Donald C. Keenan. "An Overview of the Option-Theoretic Pricing of Mortgages". In: 2001, pp. 217–244.

[9] Lishang Jiang, Baojun Bian, and Fahuai Yi. "A parabolic variational inequality arising from the valuation of fixed rate mortgages". In: *Jnl of Applied Mathematics* 16 (June 2005), pp. 361–383. DOI: 10.1017/S0956792505006297.

[10] John Mcconnell and Manoj Kumar Singh. "Valuation and Analysis of Collateralized Mortgage Obligations". In: *Management Science* 39 (1993), pp. 692–709.

[11] Eduardo S. Schwartz and Walter N. Torous. "Prepayment and the Valuation of Mortgage-Backed Securities". In: *The Journal of Finance* 44.2 (1989), pp. 375–392. ISSN: 00221082, 15406261. URL: http://www.jstor.org/stable/2328595.

[12] Yevgeny Goncharov. "An Intensity-Based Approach To The Valuation Of Mortgage Contracts And Computation Of The Endogenous Mortgage Rate". In: *International Journal of Theoretical and Applied Finance (IJTAF)* 9.06 (2006), pp. 889–914. DOI: 10.1142/S0219024906003871. URL: https://ideas.repec.org/a/wsi/ijtafx/v09y2006i06ns0219024906003871.html.

[13] Niels Rom-Poulsen. "Semi-analytical MBS Pricing". In: *The Journal of Real Estate Finance and Economics* 34.4 (2007), pp. 463–498. DOI: 10.1007/s11146-007-9020-3. URL: https://ideas.repec.org/a/kap/jrefec/v34y2007i4p463-498.html.

[14] V. Brunel and F. Jribi. "Model-independant ABS duration approximation formulas". In: *working paper* (2008).

[15] Xiao song Qian et al. "Explicit formulas for pricing of callable mortgage-backed securities in a case of prepayment rate negatively correlated with interest rates". In: *Journal of Mathematical Analysis and Applications* 393.2 (2012), pp. 421–433. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2012.03.057. URL: https://www.sciencedirect.com/science/article/pii/S0022247X12002612.

[16] John Cox, Jonathan Ingersoll, and Stephen Ross. "A Theory of the Term Structure of Interest Rates". In: *Econometrica* 53 (Feb. 1985), pp. 385–407. DOI: 10.2307/1911242.

[17] Phelim P. Boyle. "Options: A Monte Carlo approach". In: *Journal of Financial Economics* 4.3 (1977), pp. 323–338. URL: https://ideas.repec.org/a/eee/jfinec/v4y1977i3p323-338.html.

[18] Michael J. Brennan and Eduardo S. Schwartz. "A continuous time approach to the pricing of bonds". In: *Journal of Banking & Finance* 3.2 (1979), pp. 133–

155. ISSN: 0378-4266. DOI: `https://doi.org/10.1016/0378-4266(79)90011-6`. URL: `https://www.sciencedirect.com/science/article/pii/0378426679900116`.

[19] Krishna Ramaswamy and Suresh M. Sundaresan. "The valuation of floating-rate instruments : Theory and evidence". In: *Journal of Financial Economics* 17.2 (1986), pp. 251–272. URL: `https://ideas.repec.org/a/eee/jfinec/v17y1986i2p251-272.html`.

[20] Board of Governors of the Federal Reserve System (US). *3-Month Treasury Bill Secondary Market Rate [TB3MS]*. 2022. URL: `https://fred.stlouisfed.org/series/TB3MS`.

[21] Board of Governors of the Federal Reserve System (US). *10-year: Main (Including Benchmark) for the United States [IRLTLT01USM156N]*. 2022. URL: `https://fred.stlouisfed.org/series/IRLTLT01USM156N`.

[22] Arsalane Chouaib Guidoum and Kamal Boukhetala. *Sim.DiffProc: Simulation of Diffusion Processes*. version 4.8, 2022. URL: `https://rdrr.io/cran/Sim.DiffProc/`.

[23] Jonathan Kropko and Jeffrey J. Harden. *coxed: Duration-Based Quantities of Interest for the Cox Proportional Hazards Model*. version 0.3.3, 2022. URL: `https://rdrr.io/cran/coxed/`.

List of Figures

List of Tables