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EÖTVÖS LORÁND UNIVERSITY

FACULTY OF SCIENCE

Division algebras and their applications

Thesis

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1 Introduction

When we add, subtract, multiply or divide the **real numbers** used in everyday life, we always get another real number. Three generalizations of real numbers are also behave in this way. These four number systems are called ‘normed division algebras’. The real number system is but one example. The set of all real numbers forms a line, so we say that the collection of real numbers is one-dimensional. Conversely, the line is one-dimensional because specifying a point on it requires one real number.

The next most familiar division algebra is the system of **complex numbers**. The square root of -1 was introduced as a kind of secret weapon to solve more complex forms of equations by the Italian mathematician and physicist, Gerolamo Cardano. Mathematicians followed in Cardano’s footsteps and began working with complex numbers, numbers of the form $a + bi$, where a and b are ordinary real numbers. The rules for addition, subtraction, multiplication, and division of complex numbers were developed by the Italian mathematician Rafael Bombelli. The complex numbers are two-dimensional, one more than the reals. This is because it takes two real numbers to specify a point on a plane, one more than it takes to specify a point on a line. So the complex numbers behave like coordinates on a two-dimensional plane. The extra dimension comes from having another number: the number i. It was Jean-Robert Argand who popularized the idea that complex numbers describe points on the plane around 1805. He also showed how to think of the operations of complex numbers as geometric manipulations on the plane.

The signifcance of complex numbers for two-dimensional geometry led the mathematician and physicist William Rowan Hamilton to seek a similar system of numbers to play the same role in three-dimensions. This problem vexed him for many years, and Hamilton’s breakthrough came only when he began to think of even higher dimensions. He discovered how to treat complex numbers as pairs of real numbers in 1835. Hamilton noted that we are free to think of the number $a+bi$ as just a peculiar way of writing a list of two real numbers, for instance (a, b). This notation makes it very easy to add and subtract complex numbers; just add or subtract each number in the second (complex) list to the corresponding number in the first (real) list. Hamilton also came up with rules for how to multiply and divide complex numbers so that they maintained the nice geometrical meaning discovered by Argand.

After Hamilton invented this algebraic system for complex numbers that had a geometric meaning, he tried for many years to invent a bigger algebra of triplets that would play a similar role in three-dimensional geometry. He figured out a solution on the 16th of October, 1843. He was walking with his wife along the Royal Canal to a meeting of the Royal Irish Academy in Dublin when he had a sudden revelation. Rotations in three dimensions couldn’t be described with just three numbers. He needed a fourth number, thereby generating a four-dimensional set called **quaternions** that take the form $a + bi + cj + dk$. Like the complex numbers which owe their two-dimensions to 1 and a single square root of -1, the quaternions owe their four dimensions to 1 and three unique square roots of -1, called i, j and k respectively. And in a famous act of mathematical vandalism, he carved these equations into the stone of the Brougham Bridge:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Just as we think of the complex numbers as points in a two-dimensional plane, we can also think of the quaternions as points in a four-dimensional space. So the quaternions behave like coordinates in four-dimensional space.

One reason this story is so well-known is that Hamilton spent the rest of his life obsessed with the quaternions, and found many practical uses for them. Today, in many of these applications the quaternions have been replaced by their simpler cousins: vectors, which can be thought of as quaternions of the special form $ai + bj + ck$ (the first number is just zero).

The quaternions are noncommutative:

$$xy \neq yx.$$

Which means that the order of multiplication matters. Order is important because quaternions describe rotations in three dimensions, and for such rotations the order makes a difference to the outcome.

Inspired by Hamilton's work, his friend John Graves went on to discover the fourth and most mysterious of the division algebras: the **octonions**. This is an 8-dimensional number system, with seven square roots of -1. They behave like coordinates in eight-dimensional space. However, the octonions break two familiar law of arithmetic: not only are they noncommutative, but also, they are nonassociative:

$$(xy)z \neq x(yz).$$

Meanwhile the young Arthur Cayley, fresh out of Cambridge, published a paper where as an afterthought, he attached a brief description of the octonions. In fact, this paper was so full of errors that it was omitted from his collected works, except for the part about octonions. Since Cayley preceded Graves in publication, we often refer to the octonions as Cayley numbers.

While somewhat neglected due to their nonassociativity, octonions stand at the crossroads of many interesting fields of mathematics. They are still closely related to the geometry of 7 and 8 dimensions, and we can still describe rotations in those dimensions using the multiplication of octonions. It's just that, because rotations are associative and octonions are not, the relationship is more subtle than it is for the other division algebras.

There was some speculation that the octonions could be extended further, but a celebrated theorem by Hurwitz concluded that the sequence of normed division algebras over the real numbers contains only the reals, $\mathbb{R}$, themselves; the complexes, $\mathbb{C}$; the quaternions, $\mathbb{H}$; and the octonions, $\mathbb{O}$.

It would take the development of modern particle physics, and string theory in particular, to see how the octonions might be useful in the real world. And indeed, if string theory is a correct representation of the universe, they may explain why the universe has the number of dimensions it does. [1–3]

To put it differently, there is a certain hierarchy of algebras. Its very foundation is the algebra of real numbers. Its closest neighbor is the algebra of complex numbers in which multiplication retains the most important properties of the multiplication of real numbers such as commutativity, associativity, invertibility (this is an allusion to the possibility of division), and the existence of a multiplicative identity. Then comes the algebra of quaternions, in which multiplication is no longer commutative. Then comes the algebra of octonions, in which the multiplication is alternative rather than associative, but which is still a division algebra with a multiplicative identity. Other algebras do not enjoy such a "minimal package" of properties. [4]

2 Preliminaries

Definition. A set $\mathcal{V} \neq 0$ is a **vector space** over a field $\mathcal{F}$, if the following axioms are satisfied.

- $\mathcal{V}$ is equiped with an operation called addition, such that for all $u, v \in \mathcal{V}$ there exist an element in $\mathcal{V}$, denoted by $u + v$.
- Addition is associative: $(u + v) + w = u + (v + w)$ for all $u, v, w \in \mathcal{V}$.
- Addition is commutative: $u + v = v + u$ for all $u, v \in \mathcal{V}$.
- There exists an element called the zero element, $0 \in \mathcal{V}$, such that $v + 0 = 0 + v = v$ for all $v \in \mathcal{V}$.
- For every element $v \in \mathcal{V}$, there exists an element $-v \in \mathcal{V}$, called the additive inverse of v, such that $v + (-v) = (-v) + v = 0$.
- $\mathcal{F}$ and $\mathcal{V}$ are equiped with an operation called scalar multiplication, such that for all $\lambda \in \mathcal{T}$ and $u \in \mathcal{V}$ there exist an element in $\mathcal{V}$, denoted by λu.
- $(\lambda + \mu)v = \lambda v + \mu v$ for all $\lambda, \mu \in \mathcal{F}$ and $v \in \mathcal{V}$
- $\lambda(u + v) = \lambda u + \lambda v$ for all $\lambda \in \mathcal{F}$ and $u, v \in \mathcal{V}$
- $(\lambda\mu)v = \lambda(\mu v)$ for all $\lambda, \mu \in \mathcal{F}$ and $v \in \mathcal{V}$
- There exists an element called the the multiplicative identity in $\mathcal{F}$, $1 \in \mathcal{F}$, such that $1v = v$ for all $v \in \mathcal{V}$.

Definition. $v_1, \ldots v_n \in \mathcal{V}$ vectors are **linearly independent**, if $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$, where $\lambda_1, \ldots, \lambda_n \in \mathcal{T}$, if and only if, every $\lambda_i = 0$, where $i = 1, \ldots, n$.
Definition. A set of vectors $v_1, \ldots v_n \in \mathcal{V}$ is a **generating set** in $\mathcal{V}$ if every element of $\mathcal{V}$ is a linear combination of the vectors v_i, where $i = 1, \ldots, n$.
Definition. A set of vectors in a vector space is a **basis** if its a linearly independent generating set. [5]
Definition. An **algebra** $\mathcal{A}$ is a vector space that is equipped with a bilinear map $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ called 'multiplication' and a nonzero element $1 \in \mathcal{A}$ called the 'identity of $\mathcal{A}$' or 'unit' such that $m(1, a) = m(a, 1) = a$. As usual, we abbreviate $m(a, b)$ as ab.
Given an algebra, we can think of real numbers as elements of this algebra via the map $\alpha \to \alpha 1$. [6]
To see why an n-dimensional algebra is completely determined by its 'multiplication table', here

is an equivalent definition:
Definition. By an n-dimensional algebra we mean the set of expressions of the form

$$a_1 i_1 + a_2 i_2 + \ldots + a_n i_n$$

(where $a_1, \ldots, a_n$ are arbitrary real numbers and the set of $i_1, \ldots, i_n$ vectors form a basis) with the following operations:

- Multiplication by a real number: $k(a_1 i_1 + \ldots + a_n i_n) = ka_1 i_1 + \ldots + a_n i_n$.
- Addition: $(a_1 i_1 + \ldots + a_n i_n) + (b_1 i_1 + \ldots + b_n i_n) = (a_1 + b_1) i_1 + \ldots + (a_n i_n) i_n$.
- Multiplication given in terms of a table of products: $i_\alpha i_\beta = p_{\alpha\beta,1} i_1 + \ldots + p_{\alpha\beta,n} i_n$, where α and β are integers from 1 to n.

The multiplication table is used to find the product

$$(a_1 i_1 + \ldots + a_n i_n)(b_1 i_1 + \ldots + b_n i_n).$$

So the choice of n^3 numbers $p_{\alpha\beta,\gamma}$, which are the elements of the multiplication table, completely determines an n-dimensional algebra. [4]
Therefore we can say that an n-dimensional algebra is an n-dimensional vector space with a multiplication table of the basis elements.
Definition. An algebra is **commutative** if for any two of its elements a and b we have

$$ab = ba.$$

Definition. An algebra is **associative** if for any three of its elements a, b, c we have

$$(ab)c = a(bc).$$

Definition. An algebra $\mathcal{A}$ is called a **division algebra** if each of the equations

$$ax = b$$

and

$$ya = b,$$

where a and b are any elements of $\mathcal{A}$ and $a \neq 0$, is uniquely solvable.
Definition. Equivalently, $\mathcal{A}$ is a division algebra if the operations of left and right multiplication by any nonzero element are invertible.
Definition. Equivalently, $\mathcal{A}$ is a division algebra if $ab = 0$ implies that either a or b is zero.

Definition. A **normed division algebra** is an algebra $\mathcal{A}$ that is also a normed vector space with $||ab|| = ||a|| \cdot ||b||$.
This implies that $\mathcal{A}$ is a division algebra and that $||1|| = 1$.
Definition. A set $\mathcal{P}$ of elements of an algebra $\mathcal{A}$ is called **subalgebra** of $\mathcal{A}$ if

- $\mathcal{P}$ is a subspace of the vector space $\mathcal{A}$;
- $\mathcal{P}$ is closed under the multiplication in $\mathcal{A}$, that is, if $a \in \mathcal{P}$ and $b \in \mathcal{P}$, then $ab \in \mathcal{P}$.

There are three levels of associativity.
Definition. An algebra is **power-associative** if the subalgebra generated by any one element is associative.
Definition. An algebra is **alternative** if the subalgebra generated by any two elements is associative.
Definition. An algebra is **associative** if the subalgebra generated by any three elements is associative.
Theorem. By a theorem of Emil Artin, an algebra $\mathcal{A}$ is alternative, if and only if for all $a, b \in \mathcal{A}$ we have

$$(bb)a = b(ba), \qquad (ba)b = b(ab), \qquad (ab)b = a(bb).$$

Any two of these equations implies the remaining one, so we can take the first and last as the definition of 'alternative'. [6]
Properties of the **scalar product**:

$$(x, x) \geq 0. \quad (x, x) = 0 \text{ only if } x = 0;$$

$$(x, y) = (y, x);$$

$$(x, ky) = k(x, y) \text{ or } (kx, y) = k(x, y), \text{ where } k \text{ is any real number};$$

$$(x, y + z) = (x, y) + (x, z) \text{ or } (x + y, z) = (x, z) + (y, z).$$

Definition. Suppose that with any two vectors x and y in the space $\mathcal{A}_n$ there is associated a number (x, y) such that the above properties hold. Then (x, y) is the scalar product of the vectors x and y.
Let

$$i_1, i_2, \ldots, i_n$$

be a basis in $\mathcal{A}_n$. With any two vectors

$$x = x_1 i_1 + x_2 i_2 + \ldots + x_n i_n,$$

$$y = y_1 i_1 + y_2 i_2 + \ldots + y_n i_n,$$

in $\mathcal{A}_n$ we associate the number

$$(x, y) = \sum_{j,k} x_j y_k (i_j, i_k)$$

where (x, y) is the scalar product of the vectors x and y.

Definition. The length, or **norm**, of an n-dimensional vector is the number

$$|x| = \sqrt{(x, x)}.$$

Definition. Two vectors, x and y are perpendicular, or **orthogonal**, in symbols $x \perp y$, if their scalar product is zero, $(x, x) = 0$.

Theorem. Let

$$y_1, y_2, \ldots, y_p$$

be p given vectors in the space $\mathcal{A}_n$. If $p < n$, then there exists a nonzero vector x perpendicular to all the given vectors.

Corollary. If $\mathcal{P}$ is a subspace of the space $\mathcal{A}_n$ and $\mathcal{P} \neq \mathcal{A}_n$, then there exists a nonzero vector $x \in \mathcal{A}_n$ orthogonal to all vectors in $\mathcal{P}$.

Assertion. Let i be a nonzero vector. Any vector a can be decomposed into a sum of two vectors of which one is a multiple of i and the other is perpendicular to i.

$$a = ki + u, \qquad u \perp i.$$

Proof. To prove this, we must prove the existence of a number k such that the vector $u = a - ki$ is orthogonal to i, that is, such that

$$(a - ki, i) = 0.$$

Equivalently,

$$(a, i) = k(i, i).$$

But then

$$k = \frac{(a, i)}{(i, i)}.$$

(Note that $i \neq 0$, so that $(i, i) \neq 0$.) □

Definition. A basis

$$i_1, i_2, \ldots, i_n$$

is said to be **orthonormal** if any two of its vectors are orthogonal,

$$(i_j, i_k) = 0 \quad (j, k = 1, ..., n; \quad j \neq k),$$

and each of its vectors has length 1,

$$(i_j, i_j) = 1 \quad (j = 1, ..., n).$$

If the basis is orthonormal, then the expression

$$(x, y) = \sum_{j,k} x_j y_k (i_j, i_k)$$

for the scalar product of two vectors reduces to

$$(x, y) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

that is, the scalar product of two vectors expressed in an orthonormal basis reduces to the sum of the products of the corresponding coordinates of the two vectors. [4]

3 Constructions

What are the division algebras and how can we construct them?

3.1 Multiplication table and Fano plane

Complex multiplication

Think of the real numbers as one-dimensional vectors. Interpret ordered pairs of real numbers as complex numbers.

Definition. For real a and b,

$$(a; b) = a + bi.$$

Definition. For real a_1, a_2 and b_1, b_2, complex multiplication is defined by:

$$(a_1; a_2) \cdot (b_1; b_2) = ([a_1 b_1 - a_2 b_2]; [a_1 b_2 + a_2 b_1]).$$

And it satisfies

$$||(a_1; a_2) \cdot (b_1; b_2)|| = ||(a_1; a_2)|| \cdot ||(b_1; b_2)||.$$

In the case of the complex numbers the multiplication table consists of the single equality

$$i \cdot i = -1 + 0i.$$

Quaternion multiplication

Think of the complex numbers as two-dimensional vectors. Interpret ordered pairs of complex numbers as quaternions.

Definition. For complex $c = a_1 + a_2 i$ and $d = b_1 + b_2 i$,

$$\begin{aligned}(c; d) &= c + dj \\ &= a_1 + a_2 i + b_1 j + b_2 ij \\ &= a_1 + a_2 i + b_1 j + b_2 k.\end{aligned}$$

where $ij = k$.

Definition. For complex $c = a_1 + a_2 i$, $\bar{c} = a_1 - a_2 i$ is the conjugate of c.

Definition. For complex numbers $c_1 = a_1 + a_2 i$, $c_2 = a_3 + a_4 i$, $d_1 = b_1 + b_2 i$ and $d_2 = b_3 + b_4 i$, quaternion multiplication is defined by:

$$\begin{aligned}(c_1; c_2) \cdot (d_1; d_2) &= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4) \\ &+ (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)i \\ &+ (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)j \\ &+ (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)k \\ &= (a_1b_1 - a_2b_2) + (a_1b_2 + a_2b_1)i \\ &- [(a_3b_3 + a_4b_4) + (-a_3b_4 + a_4b_3)i] \\ &+ [(a_1b_3 - a_2b_4) + (a_1b_4 + a_2b_3)i]j \\ &+ [(a_3b_1 + a_4b_2) + (-a_3b_2 + a_4b_1)i]j \\ &= (c_1d_1 - c_2\bar{d}_2) + (c_1d_2 + c_2\bar{d}_1)j \\ &= ([c_1d_1 - c_2\bar{d}_2]; [c_1d_2 + c_2\bar{d}_1]).\end{aligned}$$

And it satisfies

$$||(c_1; c_2) \cdot (d_1; d_2)|| = ||(c_1; c_2)|| \cdot ||(d_1; d_2)||.$$

Quaternion multiplication is completely determined by this table for the multiplication of i, j, and k:

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Proposition. Quaternion multiplication is not commutative, since $ij = k$ and $ji = -k$.
Proposition. Quaternion multiplication is associative. That is

$$(q_1 \cdot q_2) \cdot q_3 = q_1 \cdot (q_2 \cdot q_3). \tag{1}$$

Proof. Each of the quaternions q_α ($\alpha = 1, 2, 3$) is the sum of four terms ($q_\alpha = a_\alpha + b_\alpha i + c_\alpha j + d_\alpha k$). It follows that the left side of the previous equation is the sum of 4 x 4 x 4 = 64 terms of the form

$$(u_1u_2)u_3, \tag{2}$$

where u_1 is one of the summands in q_1, u_2 in q_2 and u_3 in q_3. Similarly, the right side is the sum of 64 terms

$$u_1(u_2u_3). \tag{3}$$

If we can show that each of the terms (2) is equal to some term (3), then we'll have proved (1). Thus, to verify (1) it suffices to verify it for the special case when q_1, q_2, q_3 are any three of the four quaternions a, bi, cj, dk. Since we can pull out numerical coefficients, we need only verify (1) for the four quaternions $1, i, j, k$. For example, instead of showing that

$$((bi)(cj))(b'i) = (bi)((cj)(b'i)),$$

it suffices to show that

$$(ij)i = i(ji).$$

If one of the quaternions q_1, q_2, q_3 is 1, then (1) is true. Thus it suffices to verify (1) when q_1, q_2, q_3 are any of the quaternions i, j, k.
There are 27 such equalities. Using the multiplication table we can check all of them. This proves the associativity of the multiplication of quaternions. □

Since the quaternions, $\mathbb{H}$, are a 4-dimensional algebra with basis $1, i, j, k$, to describe the product instead of a multiplication table, it is easier to remember that:

- 1 is the multiplicative identity,
- i, j, and k are square roots of -1,
- we have $ij = k$, $ji = -k$, and all identities obtained from these by cyclic permutations of (i, j, k).

We can summarize the last rule by using Fano plane:

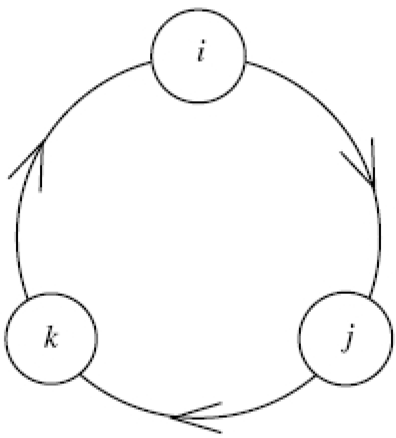

Figure 1: Fano plane for quaternion multiplication

When we multiply two elements going clockwise around the circle we get the next one: for example, $ij = k$. But when we multiply two going around counterclockwise, we get minus the next one: for example, $ji = -k$. Here, it is even easier to see that this multiplication rule is not commutative; the outcome depends on the order of the factors.

Octonion multiplication

Think of the quaternions as four-dimensional vectors. Interpret ordered pairs of quaternions as octonions.

Definition. For quaternions $q = a_1 + a_2 i + a_3 j + a_4 k$ and $r = b_1 + b_2 i + b_3 j + b_4 k$,

$$\begin{aligned}(q; r) &= q + rl \\ &= a_1 + a_2 i + a_3 j + a_4 k \\ &+ b_1 l + b_2 il + b_3 jl + b_4 kl \\ &= a_1 + a_2 i + a_3 j + a_4 k \\ &+ b_1 l + b_2 I + b_3 J + b_4 K.\end{aligned}$$

where $il = I$, $jl = J$, and $kl = K$.

Definition. For quaternion $q = a_1 + a_2 i + a_3 j + a_4 k$, $\bar{q} = a_1 - a_2 i - a_3 j - a_4 k$ is the conjugate of q.

Remark. If q' is a "pure imaginary" quaternion, that is, if $q' = bi + cj + dk$, then

$$q'^2 = b(b^2 + c^2 + d^2) \leq 0.$$

Conversely, if the square of a quaternion is real and less than or equal to zero, then that quaternion is pure imaginary. (In fact, if $q = a + bi + cj + dk$, then $q^2 = (a + q')(a + q') = a^2 + q'^2 + 2aq' = a^2 - b^2 - c^2 - d^2 + 2aq'$. If the last expression were a real number and $a \neq 0$, then $q' = 0$. But then $q = a$ and q^2 is not ≤ 0.)

It follows that quaternions of the form $bi + cj + dk$, and only such quaternions, can be characterized by the condition that their squares are real numbers ≤ 0. With this in mind, we can give the following alternate description of the operation of conjugation:

Let q be a quaternion and let $q = a + q'$ be its unique representation such that q'^2 is real and ≤ 0. Then $\bar{q} = a - q'$.

Definition. For quaternions q_1, q_2, r_1 and r_2, octonion multiplication is defined by:

$$\begin{aligned}(q_1; q_2) \cdot (r_1; r_2) &= (q_1 + q_2 l) \cdot (r_1 + r_2 l) \\ &= (q_1 r_1 - \bar{r}_2 q_2) + (r_2 q_1 + q_2 \bar{r}_1) l \\ &= ([q_1 r_1 - \bar{r}_2 q_2]; [r_2 q_1 + q_2 \bar{r}_1]).\end{aligned}$$

And it satisfies

$$||(q_1; q_2) \cdot (r_1; r_2)|| = ||(q_1; q_2)|| \cdot ||(r_1; r_2)||.$$

Octonion multiplication is completely determined by this table for the multiplication of i, j, k, l, I, J and K, which describes the result of multiplying the element in the n-th row by the element in the m-th column:

	i	j	k	l	I	J	K
i	-1	k	$-j$	I	$-l$	$-K$	J
j	$-k$	-1	i	J	K	$-l$	$-I$
k	j	$-i$	-1	K	$-J$	I	$-l$
l	$-I$	$-J$	$-K$	-1	i	j	k
I	l	$-K$	J	$-i$	-1	$-k$	j
J	K	l	$-I$	$-j$	k	-1	$-i$
K	$-J$	I	l	$-k$	$-j$	i	-1

Proposition. Since the octonions contain the quaternions, octonion multiplication is also not commutative.
Proposition. Octonian multiplication is not associative, since $l \cdot (I \cdot J) = K$ and $(l \cdot I) \cdot J = -K$.
Proposition. The octonions are alternative.

Proof. We will show that the following equalities hold for any two octonions u, v:

$$(uv)v = u(vv), \tag{4}$$

and

$$v(vu) = (vv)u. \tag{5}$$

We can regard formulas (4) and (5) as a weak form of associativity. Systems in which these two formulas hold are called alternative systems.
Remark. Instead of proving (4) and (5) it suffices to prove

$$(uv)\bar{v} = u(v\bar{v}), \tag{6}$$

and

$$\bar{v}(vu) = (\bar{v}v)u. \tag{7}$$

Indeed, if we replace $\bar{v}$ in these equalities by $-v + 2a$, where a is the real part of the octonion v, then we can obtain (4) and (5).
Lemma. We prove (6). Put $u = q_1 + q_2 e, \quad v = r_1 + r_2 e$. Then

$$\begin{aligned}
(uv)\bar{v} &= ((q_1 + q_2 e)(r_1 + r_2 e))(\bar{r}_1 - r_2 e) \\
&= ((q_1 r_1 - \bar{r}_2 q_2) + (r_2 q_1 + q_2 \bar{r}_1)e)(\bar{r}_1 - r_2 e) \\
&= ((q_1 r_1 - \bar{r}_2 q_2) + \bar{r}_2(r_2 q_1 + q_2 \bar{r}_1)) \\
&+ ((-r_2)(q_1 r_1 - \bar{r}_2 q_2) + (r_2 q_1 + q_2 \bar{r}_1) r_1)e \\
&= (|r_1|^2 + |r_2|^2) q_1 + (|r_1|^2 + |r_2|^2) q_2 e \\
&= (|r_1|^2 + |r_2|^2)(q_1 + q_2 e) = |v|^2 u.
\end{aligned}$$

On the other hand, $v\bar{v} = |v|^2$, so that

$$u(v\bar{v}) = |v|^2 u.$$

This implies (6). A similar proof establishes (7). □

Since the octonions are an 8-dimensional algebra with basis i, j, k, l, I, J and K, to describe the product instead of a multiplication table, it is easier to use the Fano plane:

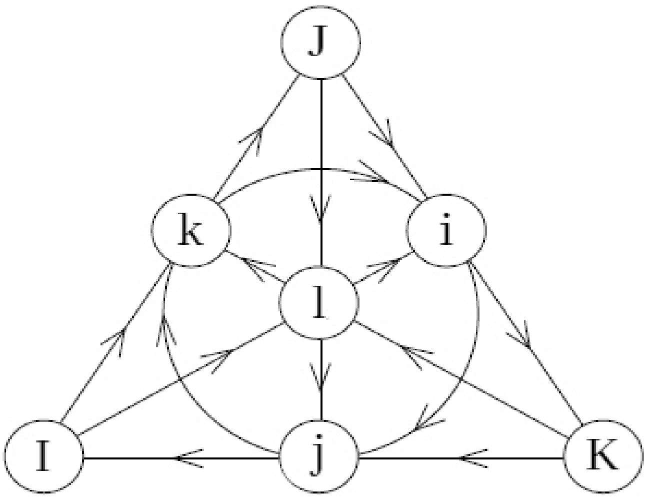

Figure 2: Fano plane for octonion multiplication

The Fano plane is a nice mnemonic with 7 points and 7 lines. The 'lines' are the sides of the triangle, its altitudes, and the circle containing all the midpoints of the sides. Each pair of distinct points lies on a unique line. Each line contains three points, and each of these triples has has a cyclic ordering shown by the arrows. If e_i, e_j, and e_k are cyclically ordered in this way then

$$e_i e_j = e_k, \qquad e_j e_i = -e_k.$$

Together with these rules:

- 1 is the multiplicative identity,
- i, j, k, l, I, J and K are square roots of -1,

the Fano plane completely describes the algebra structure of the octonions. [4, 6, 7]

3.2 The Cayley-Dickson Construction

Definition. As Hamilton noted, the complex number $a + bi$ can be thought of as a pair (a, b) of real numbers. Addition is done component-wise, and multiplication goes like this:

$$(a, b)(c, d) = (ac - db, ad + cb).$$

Definition. We can also define the conjugate of a complex number by:

$$(\overline{a, b}) = (a, -b).$$

Definition. We can define the quaternions in a similar way. A quaternion can be thought of as a pair of complex numbers. Addition is done component-wise, and multiplication goes like this:

$$(a, b)(c, d) = (ac - d\bar{b}, \bar{a}d + cb).$$

If we included them in the previous formula nothing would change, since the conjugate of a real number is just itself.
Definition. We can also define the conjugate of a quaternion by:

$$(\overline{a, b}) = (\bar{a}, -b).$$

Definition. We can define an octonion to be a pair of quaternions. As before, we add and multiply them using the two formulas above.
This trick for getting new algebras from old is called the Cayley–Dickson construction. It explaines why each one fits neatly inside the next. It makes it clear why $\mathbb{H}$ is noncommutative and $\mathbb{O}$ is nonassociative. It gives an infinite sequence of algebras, doubling in dimension each time, with the normed division algebras as the first four.
The Cayley-Dickson construction is also called the doubling procedure. Here is why:
Definition. Let $\mathcal{A}$ be an algebra of dimension n whose elements are expressions of the form:

$$u = a_0 + a_1 i_1 + a_2 i_2 + \ldots + a_n i_n$$

Definition. We call the element

$$\overline{u} = a_0 - a_1 i_1 - a_2 i_2 - \ldots - a_n i_n$$

the conjugate of u.
Definition. We define $\mathcal{A}^{(2)}$, the doubled $\mathcal{A}$, as the algebra of dimension $2n$ whose elements are expressions of the form

$$u_1 + u_2 e,$$

where u_1 and u_2 are arbitrary elements in $\mathcal{A}$ and e is a new symbol.
The elements of $\mathcal{A}^{(2)}$ are added according to the natural rule

$$(u_1 + u_2 e) + (v_1 + v_2 e) = (u_1 + v_1) + (u_2 + v_2)e,$$

and multiplied in accordance with the rule

$$(u_1 + u_2 e)(v_1 + v_2 e) = (u_1 v_1 - \overline{v}_2 u_2) + (v_2 u_1 + u_2 \overline{v}_1)e.$$

The usual form of an element of $\mathcal{A}^{(2)}$ is

$$a_0 + a_1 i_1 + a_2 i_2 + \ldots + a_n i_n + a_{n+1} i_{n+1} + \ldots + a_{2n+1} i_{2n+1}$$

This determines the pair of elements u_1, u_2 in $\mathcal{A}$ given by

$$u_1 = a_0 + a_1 i_1 + a_2 i_2 + \ldots + a_n i_n,$$

$$u_2 = a_{n+1} + a_{n+2} i_1 + \ldots + a_{2n+1} i_n,$$

and thus the element $u_1 + u_2 e$, and conversely. [4, 6]

Now that we have defined the doubling procedure it is easy to see that what we did in the previous subsections is obtain the complex numbers by doubling the real numbers, obtain the quaternions by doubling the complex numbers and obtain the octonions by doubling the quaternions.

But as we saw, when we double the dimensions in every step we lose a property. Real numbers can be ordered from smallest to largest, for instance, whereas in the complex plane there's no such concept. Next, quaternions lose commutativity, that is swapping the order of elements changes the answer. This makes sense, since multiplying higher-dimensional numbers involves rotation, and when you switch the order of rotations in more than two dimensions you end up in a different place. Much more bizarrely, the octonions are nonassociative, which means it matters how they are grouped. [3]

If we keep applying the Cayley–Dickson process to the octonions we get a sequence of algebras of dimension 16, 32, 64, and so on. The first of these is called the sedenions, presumably alluding to the fact that it is 16-dimensional. It follows from the above results that all the algebras are not division algebras, since an explicit calculation demonstrates that the sedenions, and thus all the rest, have zero divisors. [4]

4 Division

So far we have defined three of the four arithmetic operations, namely, addition, subtraction, and multiplication in each of these number systems. But what about the fourth?

Definition. An algebra $\mathcal{A}$ is called a division algebra if each of the equations

$$ax = b$$

and

$$ya = b,$$

where a and b are any elements of $\mathcal{A}$ and $a \neq 0$, is uniquely solvable.

4.1 Division of Complex Numbers

Definition. Let z_1 and z_2 be any two complex numbers and $z_2 \neq 0$. The quotient z_1/z_2 is the solution of the equation

$$z_2 x = z_1.$$

Proposition. The complex numbers are a division algebra.

Proof. Multiplying both sides of this equation by $\bar{z}_2$ we obtain $\bar{z}_2 z_2 x = \bar{z}_2 z_1$, so that

$$|z_2|^2 x = \bar{z}_2 z_1.$$

Multiplying both sides of the last equation by $1/|z_2|^2$ we have

$$x = \frac{1}{|z_2|^2} \bar{z}_2 z_1.$$

Substitution shows that this expression is a solution. □

4.2 Division of Quaternions

Definition. Let q_1 and q_2 be any two quaternions and $q_2 \neq 0$. Since multiplication of quaternions is noncommutative, it is necessary to consider two equations:

$$q_2 x = q_1$$

and

$$x q_2 = q_1.$$

We call the solution of the first of these equations the *left quotient* of q_1 by q_2 and denote it by x_l. Similarly, we call the solution of the second equation the *right quotient* of q_1 by q_2 and denote it by x_r.

Proposition. The quaternions are a division algebra.

Proof. Multiplying both sides of the first equation on the left by $\bar{q}_2$ and then by $1/|q_2|^2$ we have

$$x = \frac{1}{|q_2|^2}\bar{q}_2 q_1.$$

Substitution shows that this expression is a solution. Hence

$$x_l = \frac{1}{|q_2|^2}\bar{q}_2 q_1.$$

Similarly,

$$x_r = \frac{1}{|q_2|^2} q_1 \bar{q}_2.$$

□

4.3 Division of Octonions

Definition. Let u and v be any two octonions and $v \neq 0$. The left quotient of u by v is the solution of the equation

$$vx = u,$$

and the right quotient of u by v is the solution of the equation

$$xv = u.$$

Proposition. The octonions are a division algebra.

Proof. Just as in the case of the quaternions, we multiply both sides of the first equation on the left by $\bar{v}$. This yields

$$\bar{v}(vx) = \bar{v}u,$$

or, in view of (7),

$$|v|^2 x = \bar{v}u.$$

Hence

$$x = \bar{v}u/|v|^2.$$

Substitution and the use of (7) shows that this value of x satisfies the first equation. In other words, the left quotient of u by v is

$$x_l = \bar{v}u/|v|^2.$$

A similar argument shows that the right quotient is

$$x_r = u\bar{v}/|v|^2,$$

using the formula (6). □

[4]

4.4 Frobenius' Theorems

One of the classical problems of the theory of algebras is finding all division algebras. In spite of the fundamental nature of the problem, it is still not completely solved. An important result was obtained rather recently. It is to the effect that the dimension of such an algebra must be equal to one of the numbers 1, 2, 4, 8. While this shows that the dimensions of division algebras are small, we still have no complete overview of these algebras.
A considerably simpler problem is that of finding the division algebras satisfying additional natural conditions. In 1878, the German mathematician, Ferdinand Georg Frobenius established the following remarkable result.

Frobenius' Theorem. Every associative division algebra is isomorphic to one of the following: the algebra of real numbers, the algebra of complex numbers, and the algebra of quaternions.

The Generalized Frobenius' Theorem. Every alternative division algebra is isomorphic to one of the following four algebras: the real numbers, the complex numbers, the quaternions, and the octonions.

Since every associative algebra is alternative, the Frobenius' Theorem follows from the Generalized Frobenius Theorem. On the other hand, the algebra of octonions is alternative but not associative, so that the two theorems are different. [4]

5 The Problem of The Sum of Squares

For what values of n are there identities stating that "the product of a sum of n squares by a sum of n squares is a sum of n squares"?
For $n = 1$ we have the immediate answer

$$a^2b^2 = (ab)^2.$$

But what about $n = 2, 3, 4, 5, 6$ and so on?
German mathematician Adolf Hurwitz showed in 1898 that identities of the required kind exist for $n = 1, 2, 4, 8$ and for no other values of n.
Exploiting the connection between these identities and multiplicative norms, this result can be turned around to show that normed algebras only exist for dimension $n = 1, 2, 4, 8$. An example of an algebra in each dimension is given by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively.

5.1 General Formulation of The Problem of the Sum of Squares

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two set of numbers. By a bilinear form of these numbers we mean a sum such that each summand is a product of a number from the first set and a number from the second set.
"The problem of the sum of squares" can be stated precisely as follows. For what values of n one can find n bilinear forms $\phi_1, \phi_2, \ldots, , \phi_n$, where

$$\begin{aligned}
&\phi_1(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n),\\
&\phi_2(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n),\\
&.\\
&.\\
&.\\
&\phi_n(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n)
\end{aligned}$$

such that

$$({a_1}^2 + {a_2}^2 + \ldots + {a_n}^2)({b_1}^2 + {b_2}^2 + \ldots + {b_n}^2) = {\phi_1}^2 + {\phi_2}^2 + \ldots + {\phi_n}^2. \tag{8}$$

5.2 The Connection between The Problem of the Sum of Squares and a Certain Algebra

Proposition. With every identity (8) there is associated a certain algebra defined in the following manner: We consider the n-dimensional vector space whose elements are the vectors

$$a_1i_1 + a_2i_2 + \ldots + a_ni_n.$$

Definition. The product of two elements

$$a = a_1 i_1 + a_2 i_2 + \ldots + a_n i_n \quad \text{and} \quad b = b_1 i_1 + b_2 i_2 + \ldots + b_n i_n$$

in that space is defined by the formula

$$ab = \phi_1 i_1 + \phi_2 i_2 + \ldots + \phi_n i_n. \tag{9}$$

In view of the linearity of the forms $\phi_1, \phi_2, \ldots, \phi_n$ with respect to the variables $a_1, a_2, \ldots, a_n$ as well as the variables $b_1, b_2, \ldots, b_n$ the following equalities hold:

$$ka \cdot b = k(ab) \qquad a \cdot kb = k(ab),$$
$$(a_1 + a_2)b = a_1 b + a_2 b, \qquad a(b_1 + b_2) = ab_1 + ab_2.$$

But then the multiplication rule (9) defines a certain algebra (see in Preliminaries). Let this algebra be denoted by $\mathcal{A}$. Thus, it follows that the algebra $\mathcal{A}$ is completely determined by the identity (8).

5.3 Introducing a Norm in the Algebra

We wish to find out what property of the algebra $\mathcal{A}$ is a reflection of the fact that forms $\phi_1, \phi_2, \ldots, \phi_n$ are not entirely arbitrary but satisfy the identity (8).

Definition. We introduce in the algebra $\mathcal{A}$ a scalar product (a, b) defined in terms of the coordinates of the vectors a and b relative to the basis $i_1, i_2, \ldots, i_n$ by means of the rule

$$(a, b) = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.$$

In particular,

$$(a, a) = {a_1}^2 + {a_2}^2 + \ldots + {a_n}^2.$$

Remark. By defining the scalar product in this way we make the basis $i_1, i_2, \ldots, i_n$ orthonormal. Indeed,

$$(i_\alpha, i_\alpha) = 1,$$
$$(i_\alpha, i_\beta) = 0,$$

for $\alpha, \beta = 1, \ldots, n$, $\alpha \neq \beta$. This is so because the only nonzero coordinate of the vector i_α is its α-th coordinate (it has the value 1), and the only nonzero coordinate of i_β is its β-th coordinate.

Corollary. Using the scalar product we can write (8) as

$$(ab, ab) = (a, a)(b, b) \tag{10}$$

Definition. The norm of an element a is

$$|a| = \sqrt{(a, a)}.$$

Corollary. (10) can be rewritten as

$$|ab| = |a||b| \tag{11}$$

Definition. We say that an algebra $\mathcal{A}$ is normed if we can define in it a scalar product such that the identity (10) holds.

Examples of normed algebras are the complex numbers, the quaternions, and the octonions. That these are normed algebras follows from the fact that formula (11) holds in them.

In order to satisfy all the requirements of the definition of a normed algebra we need only introduce a scalar product such that $|a| = \sqrt{(a, a)}$.

5.4 The identity of Two-Squares

The Absolute Value of a Complex Number

Definition. Let $z = a + bi$ be any complex number. The nonnegative real number $\sqrt{a^2 + b^2}$ is called the absolute value or norm of z and is denoted by $|z|$, that is,

$$|z| = \sqrt{a^2 + b^2}.$$

Corollary. We have

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.$$

Proposition. The absolute value of a product of complex numbers is the product of the absolute values of the factors.

Proof. Let z_1 and z_2 be two complex numbers, then

$$|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2}) = z_1z_2\bar{z}_1\bar{z}_2 = z_1\bar{z}_1 \cdot z_1\bar{z}_2 = |z_1|^2|z_2|^2,$$

so that

$$|z_1z_2|^2 = |z_1|^2|z_2|^2, \tag{12}$$

and therefore

$$|z_1z_2| = |z_1||z_2|.$$

□

The identity of Two-Squares

Let

$$z = a + bi, \qquad z' = a' + b'i.$$

Then

$$zz' = (a + bi)(a' + b'i) = (aa' - bb') + (ab' + a'b)i.$$

Proposition. The product of a sum of two squares by a sum of two squares is a sum of two squares.

Proof. Rewriting equation (12) as:

$$(a^2+b^2)(a'^2+b'^2) = |z|^2|z'|^2 = |zz'|^2 = |(aa' - bb') + (ab' + a'b)i|^2 = (aa' - bb')^2+(ab' + a'b)^2.$$

□

If we would prefer to use the general form we have to make a minor change of notation:
Let

$$z = a_1 + a_2 i, \qquad z' = b_1 + b_2 i.$$

Then the previous equation can be rewritten as

$$({a_1}^2 + {a_2}^2)({b_1}^2 + {b_2}^2) = (a_1 b_1 - a_2 b_2)^2 + (a_1 b_2 + a_2 b_1)^2.$$

5.5 The identity of Four-Squares

The Absolute Value of a Quaternion

Definition. Let $q = a+bi+cj+dk$ be any quaternion. The nonnegative number $\sqrt{a^2+b^2+c^2+d^2}$ is called the absolute value or norm of q and is denoted by $|q|$, that is,

$$|q| = \sqrt{a^2+b^2+c^2+d^2}.$$

Corollary. We have

$$q\bar{q} = (a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2 = |q|^2.$$

This formula is the same as the one for complex numbers.
Proposition. The absolute value of a product of quaternions is the product of the absolute values of the factors.

Proof. Let q_1 and q_2 be two quaternions, then

$$|q_1 q_2|^2 = (q_1 q_2)(\overline{q_1 q_2}) = (q_1 q_2)(\bar{q}_2 \bar{q}_1) = q_1 (q_2 \bar{q}_2) \bar{q}_1 = |q_1|^2 |q_2|^2,$$

so that

$$|q_1 q_2|^2 = |q_1|^2 |q_2|^2, \tag{13}$$

and therefore

$$|q_1 q_2| = |q_1||q_2|.$$

□

The identity of Four-Squares

Let

$$q = a + bi + cj + dk, \qquad q' = a' + b'i + c'j + d'k.$$

Then

$$\begin{aligned} qq' &= (aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')i \\ &\quad + (ac' + ca' + db' - bd')j + (ad' + da' + bc' - cb')k. \end{aligned}$$

Proposition. The product of the sum of four squares by a sum of four squares is a sum of four squares.

Proof. Rewriting equation (13) as:

$$\begin{aligned} (a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2) &= |q|^2|q'|^2 = |qq'|^2 \\ &= |(aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')i \\ &\quad + (ac' + ca' + db' - bd')j + (ad' + da' + bc' - cb')k|^2 \\ &= (aa' - bb' - cc' - dd')^2 + (ab' + ba' + cd' - dc')^2 \\ &\quad + (ac' + ca' + db' - bd')^2 + (ad' + da' + bc' - cb')^2. \end{aligned}$$

□

If we would prefer to use the general form we have to make a minor change of notation:
Let

$$q = a_1 + a_2 i + a_3 j + a_4 k, \qquad q' = b_1 + b_2 i + b_3 j + b_4 k.$$

Then the previous equation can be rewritten as

$$\begin{aligned} &({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)({b_1}^2 + {b_2}^2 + {b_3}^2 + {b_4}^2) = \\ &\qquad (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)^2 + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)^2 \\ &\qquad + (a_1b_3 + a_3b_1 + a_4b_2 - a_2b_4)^2 + (a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)^2. \end{aligned}$$

5.6 The identity of Eight-Squares

The Operation of Conjugation of Octonions

Definition. Let

$$u = a + bi + cj + dk + Al + BI + CJ + DK$$

be any octonion. By its conjugate we mean the octonion

$$\bar{u} = a - bi - cj - dk - Al - BI - CJ - DK.$$

Definition. Equivalently, if we use the short representation

$$u = q_1 + q_2 e, \tag{14}$$

where

$$q_1 = a + bi + cj + dk, \qquad q_2 = A + Bi + Cj + Dk,$$

then the conjugate octonion is given by

$$\bar{u} = \bar{q}_1 - q_2 e.$$

The Absolute Value of an Octonion

Definition. Let $u = a+bi+cj+dk+Al+BI+CJ+DK$ be any octonion. The nonnegative number $\sqrt{a^2 + b^2 + c^2 + d^2 + A^2 + B^2 + C^2 + D^2}$ is called the absolute value or norm of the octonion u and is denoted by $|u|$, that is,

$$|u| = \sqrt{a^2 + b^2 + c^2 + d^2 + A^2 + B^2 + C^2 + D^2}$$

Note that if u is given in the form (14), then

$$|u| = \sqrt{|q_1|^2 + |q_2|^2}.$$

Corollary. The product of any octonion u and its conjugate $\bar{u}$, just as in the case of complex numbers and quaternions, is a real number (that is, an octonion of the form $a+0i+0J+\ldots+0K$). In fact,

$$u\bar{u} = (q_1 + q_2 e)(\bar{q}_1 - q_2 e) = (q_1\bar{q}_1 + \bar{q}_2 q_2) + (-q_2 q_1 + q_2 q_1)e = q_1\bar{q}_1 + q_2\bar{q}_2 = |q_1|^2 + |q_2|^2.$$

Remark. Since the squares of the absolute values of the octonionss u and $\bar{u}$ are equal, then we have

$$\bar{u}u = |u|^2 = u\bar{u}.$$

Proposition. The absolute value of the product of octonions is the product of the absolute values of the factors.

Proof. Let $u = q_1 + q_2 e$ and $v = r_1 + r_2 e$ be two octonions, then

$$uv = (q_1 + q_2 e)(r_1 + r_2 e) = (q_1 r_1 - \bar{r}_2 q_2) + (r_2 q_1 + q_2 \bar{r}_1)e.$$

Let's compute $|uv|^2$ and $|u|^2|v|^2$:

$$\begin{aligned} |uv|^2 &= (q_1 r_1 - \bar{r}_2 q_2)(\overline{q_1 r_1 - \bar{r}_2 q_2}) + (r_2 q_1 + q_2 \bar{r}_1)(\overline{r_2 q_1 + q_2 \bar{r}_1}) \\ &= (q_1 r_1 - \bar{r}_2 q_2)(\bar{r}_1 \bar{q}_1 - \bar{q}_2 r_2) + (r_2 q_1 + q_2 \bar{r}_1)(\bar{q}_1 \bar{r}_2 + r_1 \bar{q}_2), \end{aligned}$$

$$|u|^2|v|^2 = (q_1\bar{q}_1 + q_2\bar{q}_2)(r_1\bar{r}_1 + r_2\bar{r}_2).$$

If we compare the two expressions, then we see that they differ by the sum S of four terms,

$$S = r_2 q_1 r_1 \bar{q}_2 + q_2 \bar{r}_1 \bar{q}_1 \bar{r}_2 - q_1 r_1 \bar{q}_2 r_2 - \bar{r}_2 q_2 \bar{r}_1 \bar{q}_1.$$

Therefore we must show that $S = 0$ for any four quaternions q_1, q_2, r_1, r_2.
If r_2 is real, $S = 0$.
On the other hand, if r_2 is a pure imaginary quaternion, then $\bar{r}^2 = -r_2$ and

$$S = r_2(q_1 r_1 \bar{q}_2 + q_2 \bar{r}_1 \bar{q}_1) - (q_1 r_1 \bar{q}_2 - q_2 \bar{r}_1 \bar{q}_1) r_2.$$

The expressions in parentheses are a sum of two conjugate quaternions and therefore equal to some real number c. Hence

$$S = r_2 c - c r_2 = 0.$$

If $S = 0$ for $r_2 = a$ and $r_2 = b$, then it is also 0 for $r_2 = a + b$. Since every quaternion is a sum of a real number and a pure imaginary quaternion and for each of these $S = 0$, it follows that S is always equal to zero.
Therefore

$$|uv|^2 = |u|^2 |v|^2, \tag{15}$$

so that

$$|uv| = |u||v|.$$

□

The identity of Eight-Squares
Let

$$u = a + bi + cj + dk + Al + BI + CJ + DK, \qquad v = a' + b'i + c'j + d'k + A'l + B'I + C'J + D'K.$$

Then

$$uv = \phi_1 + \phi_2 i + \phi_3 j + \phi_4 k + \phi_5 l + \phi_6 I + \phi_7 J + \phi_8 K.$$

Proposition. The product of the sum of eight squares by a sum of eight squares is a sum of eight squares.

Proof. Rewriting equation (15) as

$$(a^2 + \ldots + D^2)(a'^2 + \ldots + D'^2) = |u|^2 |v|^2 = |uv|^2 = {\phi_1}^2 + {\phi_2}^2 + \ldots + {\phi_8}^2.$$

Expressing $\phi_1, \phi_2, \ldots, \phi_8$ in terms of $a, \ldots, D$, $a', \ldots, D'$, we get the identity:

$$\begin{aligned}
&(a^2+b^2+c^2+d^2+A^2+B^2+C^2+D^2)\\
&\quad\cdot(a'^2+b'^2+c'^2+d'^2+A'^2+B'^2+C'^2+D'^2)\\
&\quad=(aa'-bb'-cc'-dd'-AA'-BB'-CC'-DD')^2\\
&\quad+(ab'+ba'+cd'-dc'-A'B+B'A+C'D-D'C)^2\\
&\quad+(ac'+ca'-bd'+db'-A'C+C'A-B'D+D'B)^2\\
&\quad+(ad'+da'+bc'-cb'-A'D+D'A+B'C-C'B)^2\\
&\quad+(A'a-B'b-C'c-D'd+Aa'+Bb'+Cc'+Dd')^2\\
&\quad+(A'b+B'a+C'd-D'c-Ab'+Ba'-Cd'+Dc')^2\\
&\quad+(A'c+C'a-B'd+D'b-Ac'+Ca'+Bd'-Db')^2\\
&\quad+(A'd+D'a+B'c-C'b-Ad'+Da'-Bc'+Cb')^2.
\end{aligned}$$

□

If we would prefer to use the general form we have to make a minor change of notation:
Let

$$u=a_1+a_2i+a_3j+a_4k+a_5l+a_6I+a_7J+a_8K, \qquad v=b_1+b_2i+b_3j+b_4k+b_5l+b_6I+b_7J+b_8K.$$

Then the previous equation can be rewritten as

$$\begin{aligned}
&({a_1}^2+{a_2}^2+{a_3}^2+{a_4}^2+{a_5}^2+{a_6}^2+{a_7}^2+{a_8}^2)\\
&\quad({b_1}^2+{b_2}^2+{b_3}^2+{b_4}^2+{b_5}^2+{b_6}^2+{b_7}^2+{b_8}^2)\\
&\quad=(a_1b_1-a_2b_2-a_3b_3-a_4b_4-a_5b_5-a_6b_6-a_7b_7-a_8b_8)^2\\
&\quad+(a_1b_2+a_2b_1+a_3b_4-a_4b_3-b_5a_6+b_6a_5+b_7a_8-b_8a_7)^2\\
&\quad+(a_1b_3+a_3b_1-a_2b_4+a_4b_2-b_5a_7+b_7a_5-b_6a_8+b_8a_6)^2\\
&\quad+(a_1b_4+a_4b_1+a_2b_3-a_3b_2-b_5a_8+b_8a_5+b_6a_7-b_7a_6)^2\\
&\quad+(b_5a_1-b_6a_2-b_7a_3-b_8a_4+a_5b_1+a_6b_2+a_7b_3+a_8b_4)^2\\
&\quad+(b_5a_2+b_6a_1+b_7a_4-b_8a_3-a_5b_2+a_6b_1-a_7b_4+a_8b_3)^2\\
&\quad+(b_5a_3+b_7a_1-b_6a_4+b_8a_2-a_5b_3+a_7b_1+a_6b_4-a_8b_2)^2\\
&\quad+(b_5a_4+b_8a_1+b_6a_3-b_7a_2-a_5b_4+a_8b_1-a_6b_3+a_7b_2)^2.
\end{aligned}$$

5.7 Conclusion

All n-tuples of forms $\phi_1, \phi_2, \ldots, \phi_n$ satisfying the identity (8) can be obtained in the following manner: We take any normed n-dimensional algebra $\mathcal{A}$ and choose in it an orthonormal basis $i_1, i_2, \ldots, i_n$. Then we write down the law of multiplication in the algebra $\mathcal{A}$ in the form (9).
It follows that the problem of determining all identities (8) reduces to two problems:

- Finding all normed algebras.
- Writing down the multiplication law for each of these algebras relative to all orthonormal bases.

6 Hurwitz's Theorem

In the previous chapter, we concluded that in order to find all identities (8), we need to determine all normed algebras. Turns out that the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only algebras with an identity in which it is possible to define a scalar product such that the norm of a product is the product of the norms of the factors. [4]

Hurwitz's Theorem. Every normed algebra with an identity is isomorphic to one of following four algebras: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), and octonions ($\mathbb{O}$).

Definition. We recall that a normed algebra is an algebra in which one can define a scalar product such that

$$(ab, ab) = (a, a)(b, b). \tag{16}$$

Let $\mathcal{A}$ be a normed algebra with an identity denoted by 1.

Proposition. Every element $a \in \mathcal{A}$ can be uniquely represented as a sum of two terms one of which is proportional to 1 and the other orthogonal to 1. Thus

$$a = k_1 + a',$$

where k is a real number and $a' \perp 1$.

Definition. We introduce in the algebra an operation of conjugation:

$$\bar{a} = k_1 - a'.$$

In particular, if a is proportional to 1, then $\bar{a} = a$, and if a is orthogonal to 1, then $\bar{a} = -a$. We note,

$$\bar{\bar{a}} = a$$

and

$$\overline{a + b} = \bar{a} + \bar{b}.$$

Let $\mathcal{U}$ be a subalgebra of the algebra $\mathcal{A}$ containing 1 and different from $\mathcal{A}$.

Let $1, i_1, i_2, \ldots, i_n$ be a basis of $\mathcal{U}$ such that $i_1, i_2, \ldots, i_n$ are orthogonal to 1. Then the conjugate of an element $a_0 1 + a_1 i_1 + \ldots + a_n i_n$ is the element $a_0 1 - a_1 i_1 - \ldots - a_n i_n$. This shows that if u is an element of $\mathcal{A}$, then so is its conjugate $\bar{u}$.

Remark. There exists a nonzero vector orthogonal to $\mathcal{U}$ (see Preliminaries). And a suitable numerical multiple of it is a unit vector e. We show that the set of elements of the form

$$u_1 + u_2 e \qquad (u_1 \in \mathcal{U}, u_2 \in \mathcal{U}) \tag{17}$$

is closed under multiplication, and thus a subalgebra of $\mathcal{U}$. Let $\mathcal{U} + \mathcal{U}e$ denote this subalgebra.

Assertion 6.1 The representation of an element of $\mathcal{U} + \mathcal{U}e$ in the form (17) is unique.

Assertion 6.2 The product of two elements of the form (17) is given by

$$(u_1 + u_2 e)(v_1 + v_2 e) = (u_1 v_1 - \bar{v}_2 u_2) + (v_2 u_1 + u_2 \bar{v}_1)e. \tag{18}$$

Juxtaposing these facts and the doubling procedure, we arrive at the conclusion that the subalgebra $\mathcal{U} + \mathcal{U}e$ is isomorphic to the doubled subalgebra $\mathcal{U}$.
Corollary. Since $\mathcal{A}$ contains an identity element 1, it contains the subalgebra of elements of the form k_1. This subalgebra is isomorphic to the algebra of real numbers. We denote it by $\mathbb{R}$. If in the preceding argument we replace $\mathcal{U}$ by $\mathbb{R}$, then e will be a unit vector orthogonal to 1.
By formula (18)

$$e^2 = (0 + 1e)(0 + 1e) = -1.$$

This implies that the square of a vector a' orthogonal to 1 is λ_1, where $\lambda \le 0$. Conversely, if the square of an element is λ_1 and $\lambda \le 0$, then this element is orthogonal to 1. Thus the elements orthogonal to 1 , and only these elements, are characterized by the fact that their squares are equal to λ_1, where $\lambda \le 0$.
This enables us to give the following alternative description of conjugation in $\mathcal{A}$:
Definition. Let

$$k_1 + a', \text{ where } a'^2 = \lambda_1, \lambda \le 0,$$

be the unique representation of an element $a \in A$. Then $\bar{a} = k_1 - a'$.
Proposition. If the subalgebra $\mathbb{R} \ne \mathcal{A}$, then there is a unit vector e orthogonal to $\mathbb{R}$. Consider the subalgebra $\mathbb{C} = \mathbb{R} + \mathbb{R}e$, the result of doubling $\mathbb{R}$. This algebra isomorphic to the algebra of complex numbers. From the characterization of conjugation in $\mathcal{A}$ it follows that for the elements of $\mathbb{C}$ conjugation coincides with the conjugation of complex numbers.
Proposition. If the subalgebra $\mathbb{C} \ne \mathcal{A}$, then there is a unit vector e' orthogonal to $\mathbb{C}$. Consider the subalgebra $\mathbb{H} = \mathbb{C} + \mathbb{C}e'$, the result of doubling $\mathbb{C}$. This algebra is isomorphic to the algebra of quaternions. From the characterization of conjugation in $\mathcal{A}$ it follows that for the elements of $\mathbb{H}$ conjugation coincides with conjugation in the algebra of quaternions.
Proposition. If the subalgebra $\mathbb{H} \ne \mathcal{A}$, then there is a unit vector e'' orthogonal to $\mathbb{H}$. Consider the subalgebra $\mathbb{O} = \mathbb{H} + \mathbb{H}e''$, the result of doubling $\mathbb{H}$. This algebra is isomorphic to the octonions. Since multiplication of octonions is not associative, the subalgebra $\mathbb{O}$ must coincide with the whole algebra $\mathcal{A}$. Conversely, if the algebra $\mathcal{A}$ is not isomorphic to one of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, then it is isomorphic to the algebra $\mathbb{O}$.
Assertion 6.3 Every subalgebra containing 1 and not equal to $\mathcal{A}$ is associative.
Hurwitz's theorem will have been proved if we prove the *Assertions 6.1*, *6.2* and *6.3*.
Lemma 6.1 The following identity holds in any normed algebra:

$$(a_1b_1, a_2b_2) + (a_1b_2, a_2b_1) = 2(a_1, a_2)(b_1, b_2). \tag{19}$$

We note that this identity connects four elements a_1, a_2, b_1, b_2 of the algebra $\mathcal{A}$.

Proof. Put for a in (16) the sum $a_1 + a_2$.
We have

$$(a_1b + a_2b, a_1b + a_2b) = (a_1 + a_2, a_1 + a_2)(b, b)$$

or

$$(a_1b, a_1b) + (a_2b, a_2b) + 2(a_1b, a_2b) = (a_1, a_1)(b, b) + (a_2, a_2)(b, b) + 2(a_1, a_2)(b, b).$$

By (16), the first and second terms on the left are equal, respectively, to the first and second terms on the right. Hence

$$(a_1b, a_2b) = (a_1, a_2)(b, b). \tag{20}$$

To obtain the required result we replace b in (20) by $b_1 + b_2$.
Then we have

$$(a_1b_1 + a_1b_2, a_2b_1 + a_2b_2) = (a_1, a_2)(b_1 + b_2, b_1 + b_2),$$

or

$$\begin{aligned}(a_1b_1, a_2, b_1) + (a_1b_2, a_2b_2) + (a_1b_1, a_2b_2) + (a_1b_2, a_2b_1)\\ = (a_1, a_2)(b_1, b_1) + (a_1, a_2)(b_2, b_2) + 2(a_1, a_2)(b_1, b_2).\end{aligned}$$

By (20), the first and second summands on the left are equal, respectively, to the first and second summands on the right. Cancellation yields the identity (19). □

Lemma 6.2 The following identity holds in a normed algebra:

$$(ab)\bar{b} = (b, b)a. \tag{21}$$

In other words, the element $(ab)\bar{b}$ is always proportional to a and the proportionality coefficient is (b, b).

Proof. We note that it suffices to prove the identity (21) for the case when $b \perp 1$.
Let b' be an element of the algebra A. If we represent it in the form

$$b' = k_1 + b,$$

with $b \perp 1$, then $\bar{b} = -b$, and

$$(ab')\bar{b}' = (a(k_1 + b))(k_1 - b) = k^2a - (ab)b = k^2a + (ab)\bar{b}.$$

If we assume that formula (21) holds for the vector b, then we have

$$(ab')\bar{b}' = k^2a + (b, b)a = [k^2 + (b, b)]a = (b', b')a,$$

that is, formula (21) holds for b'.
Now we prove (21) under the assumption that $b \perp 1$ (or, equivalently, $\bar{b} = -b$). We write λ for (b, b).
Consider the element

$$c = (ab)\bar{b} - \lambda a.$$

We must show that $c = 0$ or, equivalently, that

$$(c, c) = 0.$$

In view of the properties of scalar products we have

$$(c, c) = ((a, b)\bar{b}, (ab)\bar{b}) + \lambda^2(a, a) - 2\lambda((ab)\bar{b}, a). \tag{22}$$

The right side is a sum of three terms. Using the fundamental identity (16) we can simplify the first summand:

$$((ab)\bar{b}, (ab)\bar{b}) = (ab, ab)(\bar{b}, \bar{b}) = (a, a)(b, b)^2 = \lambda^2(a, a).$$

To simplify the third summand we use the identity (19). First we write it as

$$(a_1b_1, a_2b_2) = 2(a_1, a_2)(b_1, b_2) - (a_1b_2, a_2b_1).$$

In the last identity we put

$$a_1 = ab, \quad b_1 = \bar{b}, \quad a_2 = a, \quad b_2 = 1$$

and obtain

$$((ab)\bar{b}, a) = 2(ab, a)(\bar{b}, 1) - (ab, a\bar{b}).$$

Since $b \perp 1$, the first summand on the right is zero, and the second is

$$-(ab, a\bar{b}) = (ab, ab) = (a, a)(b, b) = \lambda(a, a).$$

Hence

$$((ab)b, a) = \lambda(a, a).$$

Now we can rewrite (22) and obtain

$$(c, c) = \lambda^2(a, a) + \lambda^2(a, a) - 2\lambda^2(a, a) = 0,$$

which is what we wished to prove. □

A Consequence of Lemma 6.2. We now deduce from the identity (21) another identity that will use in what follows.

If we replace b in (21) by $x + y$, then we obtain

$$(a(x + y))(\bar{x} + \bar{y}) = (x + y, x + y)a,$$

or

$$(ax)\bar{x} + (ay)\bar{y} + (ax)\bar{y} + (ay)\bar{x} = (x, x)a + (y, y)a + 2(x, y)a.$$

In view of (21), the first and second summands on the left are equal, to the first and second summands on the right. Hence

$$(ax)\bar{y} + (ay)\bar{x} = 2(xy)a. \tag{23}$$

This is the identity we wished to establish.
Putting $a = 1$ in (21) we obtain

$$b\bar{b} = (b, b)1.$$

This and (21) yield

$$(ab)\bar{b} = a(b\bar{b}).$$

Hence

$$(ab)b = a(bb).$$

A similar argument proves that

$$b(ba) = (bb)a.$$

The last two formulas show that the algebra $\mathcal{A}$ is alternative.
Proposition. The subspaces $\mathcal{U}$ and $\mathcal{U}e$ are orthogonal, that is, $u_1 \perp u_2 e$ for any two elements $u_1 \in \mathcal{U}$, $u_2 \in \mathcal{U}$, where $\mathcal{U}$ denotes a subalgebra of the algebra $\mathcal{A}$ that contains 1 and does not coincide with $\mathcal{A}$, and e is a unit vector orthogonal to $\mathcal{U}$.

Proof. We use *Lemma 6.1*. If we put in (19) $a_1 = u_1$, $b_1 = u_2$, $a_2 = e$, $b_2 = 1$, then we obtain

$$(u_1 u_2, e) + (u_1, u_2 e) = 2(u_1, e)(u_2, 1).$$

Since $\mathcal{U}$ is a subalgebra, so that u_1, u_2 is in $\mathcal{U}$, $u_1 \perp e$, $u_1 u_2 \perp e$. It now follows from the last equality that

$$(u_1, u_2 e) = 0,$$

that is, $u_1 \perp u_2 e$. This means that the subspaces $\mathcal{U}$ and $\mathcal{U}e$ are orthogonal. □

It remains to prove the *Assertions 6.1*, *6.2* and *6.3*.

Proof. Assertion 6.1. The representation of any element in $\mathcal{U} + \mathcal{U}e$ in the form $u_1 + u_2 e$ is unique. Suppose that

$$u_1 + u_2 e = u_1' + u_2' e.$$

Then

$$u_1 - u_1' = (u_2' - u_2)e.$$

This means that the element $v = u_1 - u_1'$ is in the subspaces $\mathcal{U}$ and $\mathcal{U}e$.
Since these subspaces have just been shown to be orthogonal, $(v, v) = 0$, and therefore $v = 0$.

This implies that $u_1 - u_1' = 0$, and $(u_2' - u_2)e = 0$. Also, in view (16), $ab = 0$ implies that $a = 0$ or $b = 0$. In our case $(u_2' - u_2)e = 0$ and $e \neq 0$ imply that $u_2' - u_2 = 0$.
Hence $u_1 = u_1'$ and $u_2 = u_2'$. □

Proof. Assertion 6.2. The correctness of formula (18). We shall prove that if u and v are elements of the subalgebra $\mathcal{U}$, then

$$(ue)v = (u\bar{v})e, \tag{24}$$

$$u(ve) = (vu)e, \tag{25}$$

$$(ue)(ve) = -\bar{v}u. \tag{26}$$

With these relations we can prove formula (18). In fact,

$$(u_1 + u_2e)(v_1 + v_2e) = u_1v_1 + (u_2e)(v_2e) + (u_2e)v_1 + u_1(v_2e).$$

If we transform the last three terms on the right in accordance with the formulas (24), (25), and (26), then we obtain the equality

$$(u_1 + u_2e)(v_1 + v_2e) = (u_1v_1 - \bar{v}_2u_2) + (v_2u_1 + u_2\bar{v}_1)e,$$

that is, formula (18).
To prove (24), (25), and (26) we use the identity (23):

$$(ax)\bar{y} + (ay)\bar{x} = 2(x, y)a.$$

To prove (24), put in (23)

$$a = u, \quad x = e, \quad y = \bar{v},$$

and bear in mind that $\bar{v} \perp e$, then we have

$$(ue)v + (u\bar{v})\bar{e} = 0.$$

Since $\bar{e} = -e$ (for $e \perp 1$), we obtain the formula (24).
To prove (25), put in (23)

$$a = 1, \quad x = u, \quad y = \overline{ve}.$$

Since $\overline{ve} = -ve$ ($ve \perp \mathcal{U}$, so that $ve \perp 1$), it follows that

$$u(ve) - (ve)\bar{u} = 0.$$

Using (24) we obtain

$$u(ve) = (ve)\bar{u} = (vu)e.$$

To prove (26), we use the following remark:

Remark. If (26) holds for $v = c$ and $v = d$, then it also holds for $v = c + d$. Since every element v can be written as a sum of two terms one of which is proportional to 1 and the other orthogonal to 1, it suffices to prove (26) in two cases: when $v = k1$ and when $v \perp 1$.
If $v = k1$, then formula (26) becomes

$$k(ue)e = -ku,$$

an identity whose validity is implied by the identity (21).
If $v \perp 1$, so that $v = -v$, putting in (23)

$$a = u, \quad x = e, \quad y = -ve,$$

we have

$$(ue)(ve) - (u(ve))\bar{e} = -2(e, ve)u.$$

By the identity (20), (e, ve) equals $(1, v)(e, e)$, that is, zero. Further, by (25), the second term on the left equals $-((vu)e)\bar{e} = -vu = \bar{v}u$. But then

$$(ue)(ve) = -\bar{v}u,$$

which is what we wished to prove. □

Proof. Assertion 6.3. Every subalgebra $\mathcal{U}$ of the algebra $\mathcal{A}$ that contains 1 and is not $\mathcal{A}$ is associative, that is,

$$(uv)w = u(vw)$$

for any three elements u, v, w in $\mathcal{U}$.
Putting in (23)

$$a = ve, \quad x = \bar{w}, \quad y = \bar{u}e,$$

we have

$$((ve)\bar{w})(-\bar{u}e) + ((ve)(\bar{u}e))w = 0,$$

or, using (24) and (26),

$$u(vw) - (uv)w = 0.$$

□

7 Applications

7.1 Using the Quaternions to Perform Three-dimensional Rotations

Introduction to Group Theory

A group collects all transformations that should not change the theory if the symmetry is respected.

Definition. A group is a set $G \neq 0$ with an operation $*$ such that:

- Associativity: For all $a, b, c \in G$, we have $(a * b) * c = a * (b * c)$.
- Identity element: There is an element $e \in G$ such that $e * a = a * e = a$ for all $a \in G$.
- Inverse element: For each element $e \in G$, there is an element a^{-1} such that $a * a^{-1} = a^{-1} * a = e$.

Definition. Let G be a group. If H is a subset of G and H also forms a group under the operation of G, then H is called a subgroup of G.

Definition. The symmetric group $S(n)$ is the group of bijections from a set with n elements, to itself. An element of this group is called a permutation.

Definition. Let F be a field and $n \geq 1$. Then the group of $n \times n$ invertible matrices over F together with the operation of matrix multiplication, is called the general linear group. We denote it by $GL(n, F)$.

Definition. Let be $n \geq 1 \in \mathbb{Z}$. Then $O(n)$ denotes the orthogonal matrices that form a subgroup in $GL(n, F)$. That is $O(n, F) = \{A \in GL(n, F), \text{ where } A^T A = AA^T = I\}$.

Definition. The special orthogonal group, denoted by $SO(n)$, is a subgroup of orthogonal matrices in the general linear group $GL(n, F)$ with determinant 1. That is $O(n, F) = \{A \in GL(n, F), \text{ where } A^T A = AA^T = I \text{ and } detA = 1\} = \{A \in O(n), \text{ where } det(A) = 1\}$.

Remark. $SO(n)$ is also called the rotation group because:

- When $n = 2$: $SO(2)$'s elements are the usual rotations around a point on a plane.
- When $n = 3$: $SO(3)$'s elements are the usual rotations around an axis through the origin in space.

Definition. The matrices with determinant 1 form a subgroup in $GL(n, F)$. We call this the special linear group and denote it by $SL(n, F)$.

Definition. A complex square matrix A is unitary if its conjugate transpose A^* is also its inverse, that is, if

$$A^* A = AA^* = AA^{-1} = I,$$

where I is the identity matrix.

Definition. The unitary group, denoted by $U(n)$, is a subgroup of unitary matrices in the general linear group $GL(n, \mathbb{C})$.

Remark. When $n = 1$: $U(1)$ is the group of all complex numbers with absolute value 1, that is $U(1) = \{z \in \mathbb{C}, \text{ where } |z| = 1\}$. $U(1)$ is also called the circle group because it is the rotations of a circle about its axis. Here we can easily see that it is a group because adding angles together, which represents rotations, is associative. There is an identity element, the rotation of 0 degrees. And there is an inverse element, the rotation in the opposite direction.

Definition. The special unitary group, denoted by $SU(n)$, is a subgroup of unitary matrices in the general linear group $GL(n, \mathbb{C})$ with determinant 1.

Remark.

- When $n = 1$: $SU(1)$ is the trivial group.
- When $n = 2$: $SU(2)$ is isomorphic to the group of quaternions of norm 1, that is $SU(2) = \{A \in Matrix_{2\times 2}(\mathbb{C}), \text{ where } detA = 1 \text{ and } A^*A = 1\}$. It is also called the isospin group.
- When $n = 3$: $SU(3) = \{A \in Matrix_{3\times 3}(\mathbb{C}), \text{ where } detA = 1 \text{ and } A^*A = 1\}$. It is the group of unitary transformations in 3-dimensions with determinant 1.

Definition. A ring is a set R equipped with an operations + called addition, and an operation $\times$ called multiplication with the following properties:

- R is an abelian group under addition. Meaning that: the operation + is associative and commutative. There is an element 0 in R such that $a + 0 = a$ for all $a \in R$. And every element has an inverse.
- R is a monoid under multiplication. Meaning that: the operation $\times$ is associative.
- Multiplication is distributive with respect to addition. Meaning that: $a\times(b+c) = a\times b+a\times c$ and $(b + c) \times a = b \times a + c \times a$ for all $a, b, c \in R$.

Definition. A subset I of a ring R is called a left ideal, if it is an additive subgroup of R and for every $a \in I$, $r \in R$, the product $ra \in I$.

Definition. Similarly, a subset I is called a right ideal of R, where R is a ring, if it is an additive subgroup of R and for every $a \in I$, $r \in R$, the product $ar \in I$.

Definition. I is an ideal if it is a left ideal that is also a right ideal. [8]

An interesting connection between groups and the quaternions

The group of rotations $SO(2)$ is isomorphic to the group $U(1)$ of complex numbers $e^{i\theta} = cos\theta + isin\theta$ of unit length. This follows from the observation that $U(1)$ is the unit circle. We can identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, letting $z = x + iy \in \mathbb{C}$ represent $(x, y) \in \mathbb{R}^2$. Then every plane rotation R_θ by an angle θ is represented by multiplication by the complex number $e^{i\theta} \in U(1)$, in the sense that for all $z, z' \in \mathbb{C}$, $z' = R_\theta(z)$ if and only if $z' = e^{i\theta} z$. And since $e^{i\theta} = cos(\theta) + sin(\theta)i$, $z' = cos(\theta) + sin(\theta)iz$.

In some sense, the quaternions generalize the complex numbers in such a way that rotations of $\mathbb{R}^3$ are represented by multiplication by quaternions of unit length. However, quaternion multiplication is not commutative, and a rotation in $SO(3)$ requires conjugation with a quaternion for its representation. And also, instead of the unit circle, we need to consider the sphere in $\mathbb{R}^4$, and $U(1)$ is replaced by $SU(2)$. [9]

Motivating Examples:

First Example. There is a point in 2-dimensional space, like $p = (4, 1)$, and we want to know the new coordinates after rotating it 30 degrees around the origin.

Solution. Take the complex number that is 30 degrees off the horizontal with magnitude 1, $cos(30°)+sin(30°)i$. Then multiply this by the point, represented as a complex number, $p = 4+1i$. Keeping in mind that $i^2 = -1$.

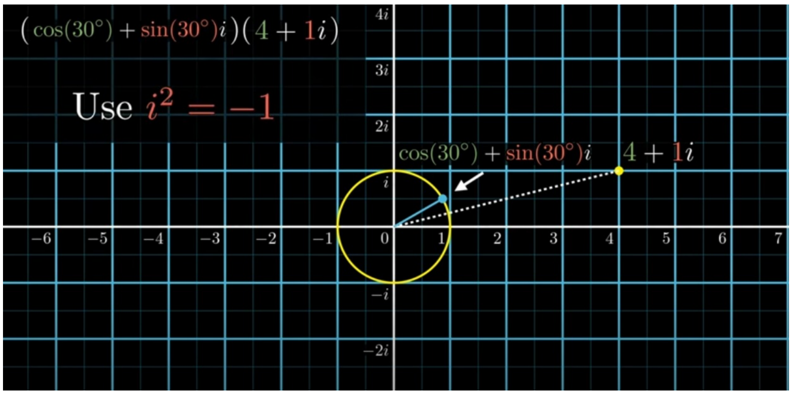

Figure 3: Rotating a point on a plane

The product

$$(cos(30°)+sin(30°)i)(4+1i) = (4cos(30°)-1sin(30°))+(1cos(30°)+4sin(30°))i \approx 2.96+2.87i$$

gives the coordinates of the new point, rotated 30 degrees away from the original. That is (2.96,2.87).

Before giving an example of the rotations in three dimension, we introduce the necessary notions of the quaternions.

Definition. The set of quaternions, together with the two operations of addition and multiplication, form a noncommutative ring. The standard orthonormal basis for $\mathbb{R}^3$ is given by three unit vectors $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$. A quaternion q is defined as the sum of a scalar q_0 and a vector $q' = (q_1, q_2, q_3)$; namely,

$$q = q_0 + q' = q_0 + q_1 i + q_2 j + q_3 k.$$

How can a quaternion, which lives in $\mathbb{R}^4$, operate on a vector, which lives in $\mathbb{R}^3$?

Proposition. First, we note that a vector $v \in \mathbb{R}^3$ is a pure quaternion whose real part is zero. Let us consider a unit quaternion $q = q_0 + q'$. That $q_0{}^2 + ||q'^2|| = 1$ implies that here must exist an angle α such that

$$cos^2\alpha = q_0{}^2,$$
$$sin^2\alpha = ||q'^2||.$$

In fact, there exists a unique $\alpha \in [0, \pi]$ such that $cos\alpha = q_0$ and $sin\alpha = ||q'||$. The unit quaternion can now be written in terms of the angle α and the unit vector $u = q'/||q'||$:

$$q = cos\alpha + sin\alpha u.$$

Theorem. For any unit quaternion

$$q = q_0 + q' = cos\frac{\theta}{2} + sin\frac{\theta}{2}u,$$

and for any vector $v \in \mathbb{R}^3$, the rotation of the vector through an angle θ about u as the axis of rotation is equivalent to qvq^{-1}. [10]

Second Example. Determine the image of the point $(3, 1, -2)$ under the rotation by an angle of 60 degrees about an axis in the xy-plane that is inclined at an angle of 45 degrees to the positive x-axis.

Solution. The unit vector u in the direction of the axis of rotation is $cos45°i+sin45°j = \frac{\sqrt{2}}{2}i+\frac{\sqrt{2}}{2}j$. The quaternion (or vector) corresponding to the point $p = (3, 1, -2)$ is $p = 3i + j - 2k$. To find the image of p under the rotation, we calculate qpq^{-1} where q is the quaternion $cos\frac{\theta}{2} + sin\frac{\theta}{2}u$ and θ the angle of rotation, which is $60°$. The resulting quaternion has no constant term and therefore we can interpret it as a vector, which gives us the image of p. So we have

$$\begin{aligned} q = cos\frac{60°}{2} + sin\frac{60°}{2}u &= cos30° + sin30°u \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}u = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{4}i + \frac{\sqrt{2}}{4}j \\ &= \frac{1}{4}(2\sqrt{3} + \sqrt{2}i + \sqrt{2}j) \end{aligned}$$

Since q is by construction a unit quaternion, its inverse is its conjugate:

$$q^{-1} = \frac{1}{4}(2\sqrt{3} - \sqrt{2}i - \sqrt{2}j).$$

Now we compute qp and qpq^{-1}:

$$\begin{aligned} qp &= \frac{1}{4}(2\sqrt{3} + \sqrt{2}i + \sqrt{2}j)(3i + j - 2k) \\ &= \frac{1}{4}(6\sqrt{3}i + 2\sqrt{3}j - 4\sqrt{3}k + 3\sqrt{2}i^2 + \sqrt{2}ij - 2\sqrt{2}ik + 3\sqrt{2}ji + \sqrt{2}j^2 - 2\sqrt{2}jk) \\ &= \frac{1}{4}(6\sqrt{3}i + 2\sqrt{3}j - 4\sqrt{3}k - 3\sqrt{2} + \sqrt{2}k + 2\sqrt{2}j - 3\sqrt{2}k - \sqrt{2} - 2\sqrt{2}i) \\ &= \frac{1}{4}(-4\sqrt{2} + (6\sqrt{3} - 2\sqrt{2})i + (2\sqrt{3} + 2\sqrt{2})j + (-4\sqrt{3} - 2\sqrt{2})k) \end{aligned}$$

$$
\begin{aligned}
qpq^{-1} =& \frac{1}{4}(-4\sqrt{2} + (6\sqrt{3} - 2\sqrt{2})i + (2\sqrt{3} + 2\sqrt{2})j + (-4\sqrt{3} - 2\sqrt{2})k) \\
& \frac{1}{4}(2\sqrt{3} - \sqrt{2}i - \sqrt{2}j) \\
=& \frac{1}{8}(-8\sqrt{2}\sqrt{3} + 8i + 8j \\
& +(36 - 4\sqrt{2}\sqrt{3})i - (6\sqrt{3}\sqrt{2} - 4)i^2 - (6\sqrt{3}\sqrt{2} - 4)ij \\
& +(12 + 4\sqrt{2}\sqrt{3})j - (2\sqrt{3}\sqrt{2} + 4)ji - (2\sqrt{3}\sqrt{2} + 4)j^2 \\
& -(24 + 4\sqrt{2}\sqrt{3})k + (4\sqrt{2}\sqrt{3} + 4)ki + (4\sqrt{3}\sqrt{2} + 4)kj) \\
=& \frac{1}{8}(-8\sqrt{2}\sqrt{3} + 8i + 8j \\
& +(36 - 4\sqrt{2}\sqrt{3})i + (6\sqrt{3}\sqrt{2} - 4) - (6\sqrt{3}\sqrt{2} - 4)k \\
& +(12 + 4\sqrt{2}\sqrt{3})j + (2\sqrt{3}\sqrt{2} + 4)k + (2\sqrt{3}\sqrt{2} + 4) \\
& -(24 + 4\sqrt{2}\sqrt{3})k + (4\sqrt{2}\sqrt{3} + 4)j - (4\sqrt{3}\sqrt{2} + 4)i) \\
=& \frac{1}{8}(-8\sqrt{2}\sqrt{3} + 8i + 8j \\
& +36i - 4\sqrt{2}\sqrt{3}i + 6\sqrt{2}\sqrt{3} - 4 - 6\sqrt{2}\sqrt{3}k + 4k \\
& +12j + 4\sqrt{2}\sqrt{3}j + 2\sqrt{2}\sqrt{3}k + 4k + 2\sqrt{2}\sqrt{3} + 4 \\
& -24k - 4\sqrt{2}\sqrt{3}k + 4\sqrt{2}\sqrt{3}j + 4j - 4\sqrt{2}\sqrt{3}i - 4i) \\
=& \frac{1}{8}(-8\sqrt{2}\sqrt{3} + 6\sqrt{2}\sqrt{3} - 4 + 2\sqrt{2}\sqrt{3} + 4 \\
& +8i + 36i - 4\sqrt{2}\sqrt{3}i - 4\sqrt{2}\sqrt{3}i - 4i \\
& +8j + 12j + 4\sqrt{2}\sqrt{3}j + 4\sqrt{2}\sqrt{3}j + 4j \\
& -6\sqrt{2}\sqrt{3}k + 4k + 2\sqrt{2}\sqrt{3}k + 4k - 24k - 4\sqrt{2}\sqrt{3}k) \\
=& \frac{1}{8}(0 + 40i - 8\sqrt{2}\sqrt{3}i + 24j + 8\sqrt{2}\sqrt{3}j - 8\sqrt{2}\sqrt{3}k - 16k) \\
=& \frac{1}{8}((40 - 8\sqrt{2}\sqrt{3})i + (24 - 8\sqrt{2}\sqrt{3})j - (16 + 8\sqrt{2}\sqrt{3})k)
\end{aligned}
$$

The point corresponding to the vector on the right hand side in the above equation is the image of $(3, 1, -2)$ under the given rotation. That point is

$$
(\frac{40 - 8\sqrt{2}\sqrt{3}}{8}, \frac{24 + 8\sqrt{2}\sqrt{3}}{8}, \frac{-(16 + 8\sqrt{2}\sqrt{3})}{8}) = (5 - \sqrt{2}\sqrt{3}, 3 + \sqrt{2}\sqrt{3}, -(2 + \sqrt{2}\sqrt{3})).
$$

Third Example. Find which quaternion corresponds to a 270 degree turn of a sphere around the z axis in the counter clockwise direction as you face it from above.

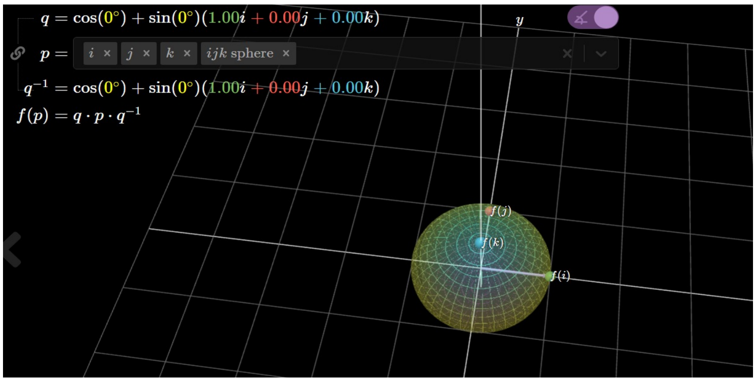

Figure 4: Rotating a point on a plane

Solution. First set the axis to be fully in the z direction, $0i + 0j + 1k$. Then change the angle to $270/2 = 135$ degrees.

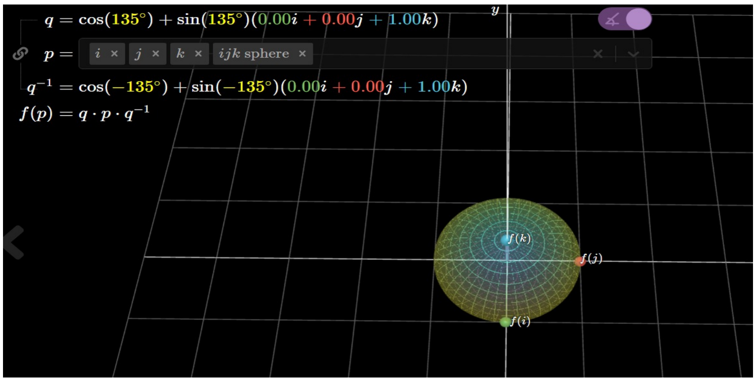

Figure 5: Setting the specified axis and angle

Now look at the quaternionic representation instead of the angle representation. It has equal parts real and k.

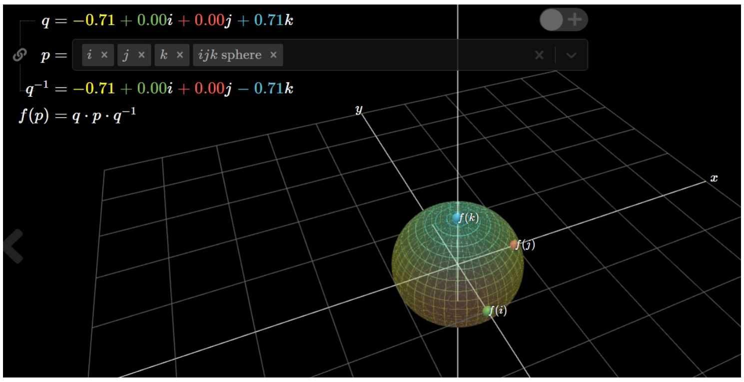

Figure 6: Quaternionic represerntation for a 270 degree rotation

But there is another way we can orient this sphere like this. So far we have roated it three quaters of a turn counter clockwise. But we can also roated it −90 degrees with changing the angle to −45 degrees.

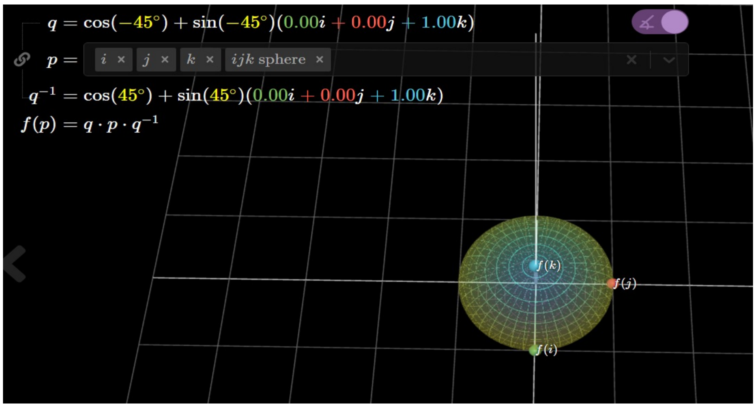

Figure 7: Rotating the sphere to the other direction $(360 - 270)/2$ degrees

Now the quaternions are represented with equal parts real and k but the negative what we had before.

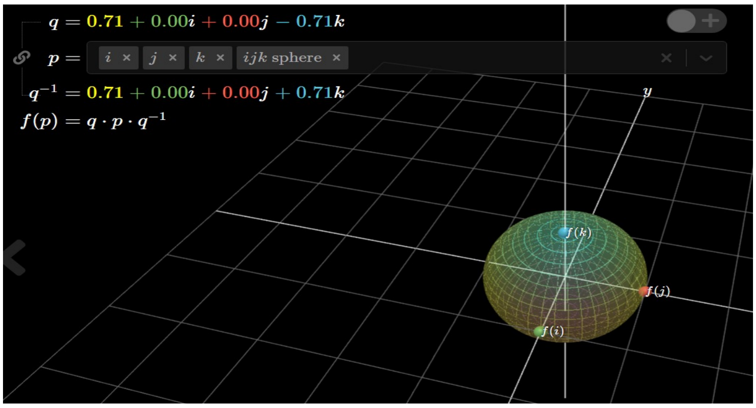

Figure 8: Quaternionic represerntation for a 90 degree rotation

So we have two separate quaternions corresponding to the same orientation in 3-dimensional space. In other words, each rotation can be represented as an orientation about an axis, or, as a negative orientation about an axis pointing in the opposite direction. That is why the quaternions are called a *double cover* for rotations in 3-dimenasional space. Continuing with the group representation from the beginning of this section this means that the unitary group $SU(2)$ double covers the orthogonal group $SO(3)$.
Because of the angle doubling phenomenon, we can also think of the two separate quaternions as two points on the opposite side of a hypersphere in 4-dimensions. Let's look at the last example.

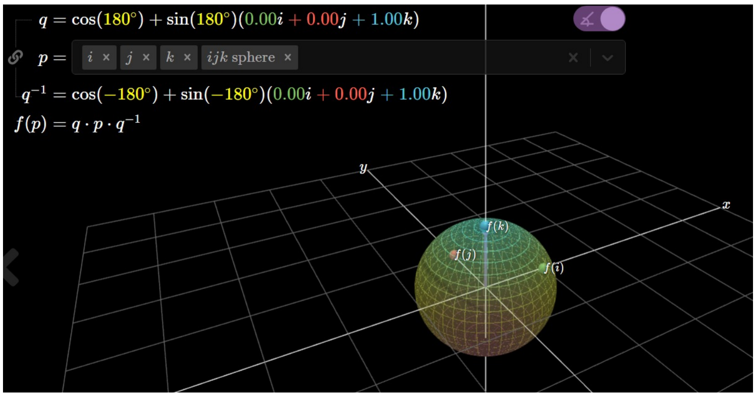

Figure 9: Changing the angle to 180 degrees

As we change the angle up to a 180 degrees the orientation it corresponds rotates 360 degrees, getting back to where it started.

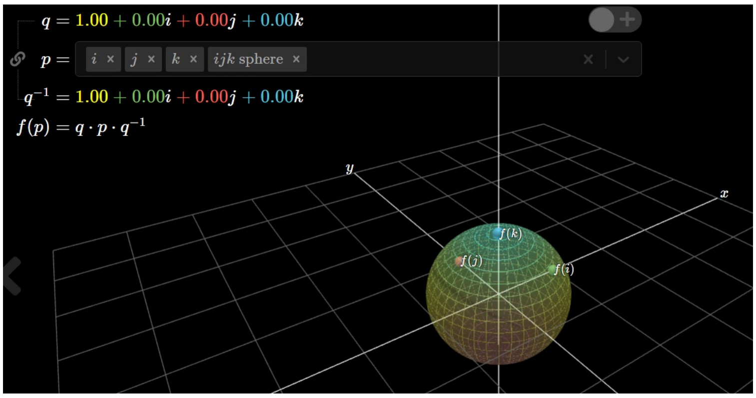

Figure 10: Quaternionic represerntation before a 180 degree rotation

However the quaternion it is representing has rotated a 180 degrees around the hypersphere.

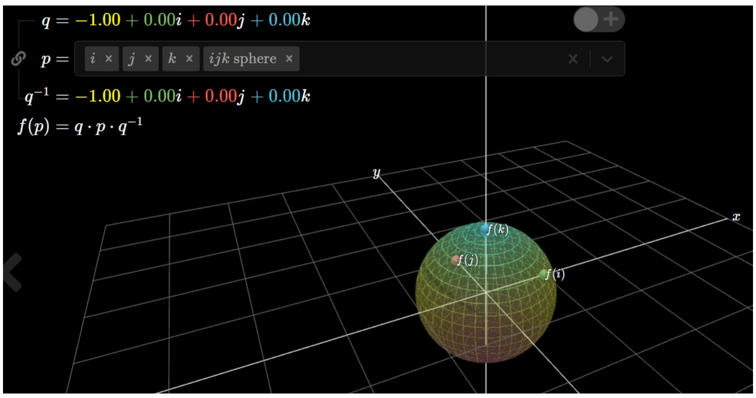

Figure 11: Quaternionic represerntation after a 180 degree rotation

In this case it went from being 1 to being -1. That is the reason for taking half the angle of the rotation in the definition. [11]

7.2 The Standard Model

Everything in the universe is found to be made from fundamental particles, governed by four fundamental forces. Our best understanding of how these particles and three of the forces are related to each other is encapsulated in The Standard Model of particle physics.

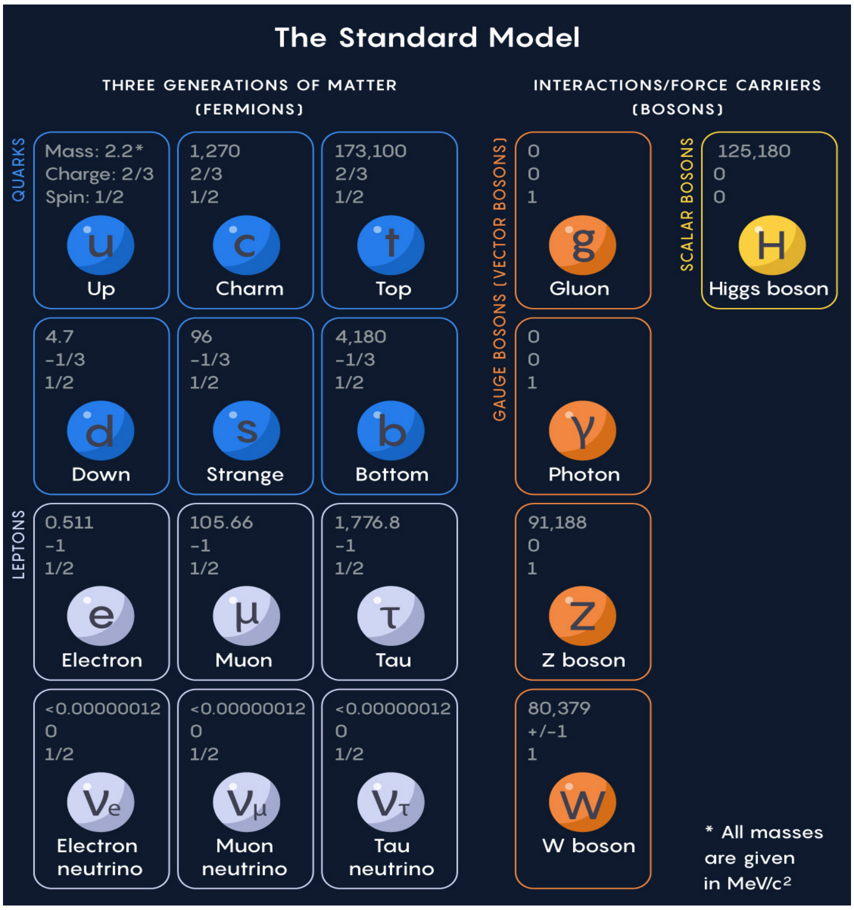

Figure 12: The periodic table of elementary particles

Matter particles

Matter particles, also called as fermions, occur in two types called *quarks* and *leptons*. Each group consists of six particles, which are related in *generation*. The lightest particles make up the first generation, and the heavier particles belong to the second and third generations. The six quarks are paired in three generations, the *up quark* and the *down quark* form the first generation, followed by the *charm quark* and *strange quark*, then the *top quark* and *bottom quark*. Quarks also possess a kind of charge called *color* which makes it sensitive to the strong force. Leptons, however, do not have color charge and do not interact via the strong force. This is the main feature that distinguishes them from quarks. The six leptons are similarly arranged in three generations, the *electron* and the *electron neutrino*, the *muon* and the *muon neutrino*, and the *tau* and the *tau neutrino*. The electron, the muon and the tau all have an electric charge and a sizeable mass, whereas the neutrinos are electrically neutral and have very little mass.

Force particles

There are four fundamental forces in the universe: the strong force, the weak force, the electromagnetic force, and the gravitational force. Three of them result from the exchange of force particles, which belong to the group called *bosons*. Particles of matter transfer discrete amounts of energy by exchanging bosons with each other. Each fundamental force has its own corresponding boson. The strong force is carried by the *gluon*, the electromagnetic force is carried by the *photon*, and the W and Z *bosons* are responsible for the weak force. Although not yet found, the *graviton* should be the corresponding force particle of gravity. [12] As the universe cooled, an event known as electroweak symmetry breaking split the forces in two. This event was marked by the sudden appearance of a field extending throughout space, known as the Higgs field, which is associated with a particle called the Higgs boson. As a particle such as an electron moves through space, it constantly interacts with Higgs bosons. These interactions slow down the electron, and that's what we mean by "mass". In general, the more a particle interacts with the Higgs boson, the more mass it has. [13]

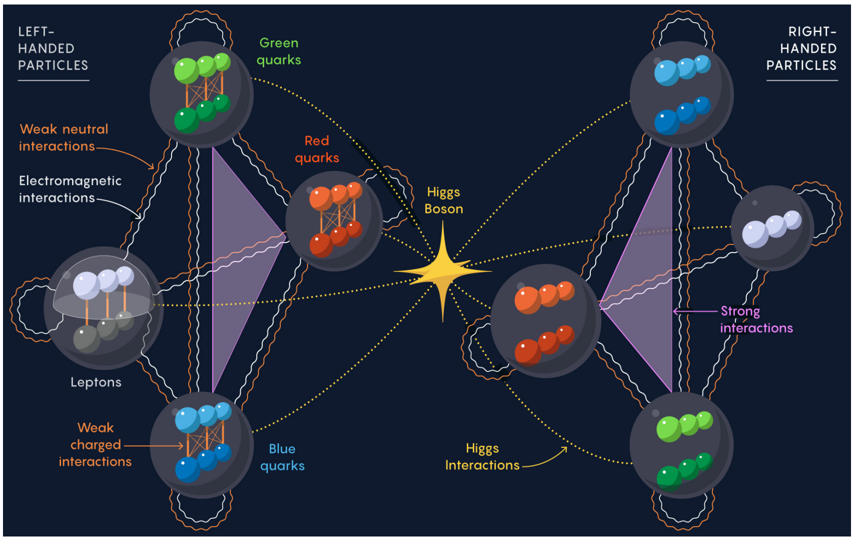

Figure 13: A 2-dimensional version of the Double-Simplex representation of The Standard Model by Chris Quigg

7.3 The symmetries of The Standard Model

In the previous section we have discussed that the fermions of the strong nuclear force are the quarks, of which we have found six. The different types or flavors of quarks are called the up, down, strange, charm, bottom, and top quarks. The quarks' interaction is mediated by eight massless gluons, which are the bosons of the strong force. Their number follows from the symmetry group of the strong force, which is $SU(3)$. Quantum Chromo Dynamics is the theory that describes the quarks, the gluons and the interaction between them. Like the electromagnetic force, the strong force responds to a kind of charge, but in this case there are three different varieties of charge. While electric charge consists of only positive and negative, the charge of the strong force, called color charge, consists of red, anti-red or cyan, blue, anti-blue or yellow, green, and anti-green or magenta. The combination of all color, all anti-collor and a color with its anti color gives a colorless particle. All observed particles are colorless. [14]

The remaining fermions do not participate in the strong interaction and are called leptons. Of these we have also six. The different types or flavors of leptons are called the electron, muon, and tau and their associated neutrinos, the electron neutrino, muon neutrino, and tau neutrino. The electroweak interaction is mediated by the massless, neutral photon and by the massive Z, W+, and W- bosons. The number of bosons follows from the symmetry group, which for the electroweak interaction is $SU(2) \times U(1)$. Electro-weak interaction is described by the thory called Quantum Electro-Flavor Dynamics.

Therefore in The Standard Model, elementary particles are manifestations of three symmetry groups, which are ways of interchanging subsets of the particles that leave the equations unchanged. While particles with color are representations of the symmetry group $SU(3)$, particles with the internal properties of flavor and electric charge are representations of the symmetry groups $SU(2)$ and $U(1)$, respectively.

To summarize, $SU(3)$, $SU(2)$ and $U(1)$, correspond to the strong, weak and electromagnetic forces, respectively, and they act on six types of quarks, two types of leptons, plus their antiparticles, with each type of particle coming in three copies, or generations, that are identical except for their masses. The fourth fundamental force, gravity, is described separately, and incompatibly, by Einstein's general theory of relativity, which casts it as curves or bending in the geometry of space-time.

The question is, why this symmetry group: $SU(3) \times SU(2) \times U(1)$?

The conventional attitude toward such questions has been to treat The Standard Model as a broken piece of some more complete theoretical structure. The first attempt for a unified symmetry used the group $SU(5)$ because it is the smallest group that contains the symmetry groups of The Standard Model. Unified forces like this, however, generically enable new interactions that allow protons to decay. And if protons are unstable, so are all atomic nuclei. The next attempt at unification used a larger group, $SO(10)$, in which the upper bound on the proton lifetime is higher. Besides proton decay, grand unified theories also predict new particles because the large groups contain more than what is in The Standard Model. These new particles, as usual, are assumed to be too heavy to have been detected yet. And so theoretical physicists now have a selection of unified theories that are safe from being experimentally ruled out in the foreseeable future. [3, 15, 16]

Furey's model

"The real numbers are appearing ubiquitously. The complex numbers are providing the math of Quantum Mechanics. The quaternions underlie the structure of Albert Enstein's Special Theory of Relativity. But in the 178 years since the octonions were discovered, they have not been found to be central to any major theory in physics. Why would nature rely so heavily on the first 3 of these number systems and yet that it would forget about the fourth?" [17]

Cohl Furey puts forward the proposal that the group representations of fundamental particles could ultimately come from a single algebra acting on itself. Specifically, they are proposed to arise from $\mathbb{R}\times\mathbb{C}\times\mathbb{H}\times\mathbb{O}$ in the form of generalized *ideals*. $\mathbb{R}\times\mathbb{C}\times\mathbb{H}\times\mathbb{O}$, the four number systems combined, form a 64-dimensional abstract space. Within this space, particles are mathematical ideals: elements of a subspace that, when multiplied by other elements, stay in that subspace, allowing particles to stay particles even as they move, rotate, interact and transform. As the ideal will pull any element into itself, it can be thought of as an algebra's version of a black hole. In other words, an ideal is a special subspace of an algebra because it can survive multiplication by any element in A. Ideals persisting under multiplication bear a striking resemblance to particles persisting under propagation.

This algebra splits cleanly into two parts: $\mathbb{C}\times\mathbb{H}$ and $\mathbb{C}\times\mathbb{O}$. The symmetries associated with how particles move and rotate in space-time come from the quaternionic part, $\mathbb{C}\times\mathbb{H}$. The symmetry group $SU(3)\times SU(2)\times U(1)$, associated with particles internal properties and mutual interactions via the strong, weak and electromagnetic forces, come from the octonionic part, $\mathbb{C}\times\mathbb{O}$. In this case, the symmetries act on all three particle generations and also allow for the existence of particles called sterile neutrinos, candidates for dark matter.

The $SU(3)$ generators identified within $\mathbb{C}\times\mathbb{O}$ breaks down the remaining space into six singlets, six triplets, and their antiparticles, with no extra particles beyond these. These representations suggest the existence of exactly three generations and they relate particles to antiparticles by using only the complex conjugate $i \mapsto -i$.

In conclusion, the division algebras not only represent The Standard model, but they also come very close to deriving it. [3, 17–19]

7.4 Supersymmetry

So far we have found twenty-five different elementary particles. Supersymmetry completes this collection with a set of still undiscovered partner particles, one for each of the known particles, and some additional ones. This supersymmetric completion is appealing because the known particles, as we have seen, are of two different types, fermions and bosons (named after the Italian physicist Enrico Fermi and the Indian mathematician and physicist Satyendra Nath Bose, respectively), and supersymmetry explains how these two types belong together.

Supersymmetry postulates that the laws of physics will remain unchanged if we exchanged all the matter and force particles. The possibility of exchanging bosons with fermions means that every known boson must have a fermionic partner, and every known fermion must have a bosonic partner. But besides differing in their fermionic or bosonic affiliation, partner particles must be identical. So supersymmetry states that at the most fundamental levels, the universe exhibits a symmetry between matter and the forces of nature.
One of the main motivations for supersymmetry is that it avoids the need to fine-tune the mass of the Higgs boson, one of the twenty-five particles of the standard model. The Higgs is the only known particle of its type, and it suffers from a peculiar mathematical problem that the other elementary particles are immune to: quantum fluctuations make a huge contribution to the Higgs's mass. Therefore the Higgs mass requires explanation. A number that seems to require explanation is called *fine-tuned*, while a theory that has no fine-tuned numbers is called *natural*. In The Standard Model, the Higgs mass is not natural, which makes it unpretty. Supersymmetry much improves the situation because it prevents the overly large contributions from quantum fluctuations to the Higgs' mass. It does so by enforcing the required delicate cancellation of large contributions, without the need to fine-tune. Instead there are only more moderate contributions from the masses of the superpartners. Assuming all masses are natural then implies that the first superpartners should appear at energies not too far away from the Higgs itself. That is because if the superpartners are much heavier than the Higgs, their contributions must be canceled by a fine-tuned term to give a smaller Higgs mass. And while that is possible, it seems absurd to fine-tune supersymmetry, since one of the main motivations for it is that it avoids fine-tuning. However, if we cannot find a natural explanation for a number, so the argument goes, then there is not any. Just choosing a parameter is too unattractive. Therefore, if the parameter is not natural, then it can take on any value, and for every possible value there is a universe. This leads to the bizarre conclusion that if we do not see supersymmetric particles at the Large Hadron Collider, then we live in a multiverse.
According to quantum mechanics, particles are also waves. To describe the motion of the wave we use *spinors*, in case of matter particles, and *vectors*, in case of force particles. The properties of vectors and spinors depends on the dimension of spacetime. But imagine a universe with no time, only space. Then if this universe had dimension 1, 2, 4, or 8, both matter and force particles would be waves described by a number in a division algebra. In other words, the vectors and spinors coincide, and simplify: they are each just real numbers, complex numbers, quaternions or octonions. Conversly, division algebras only exist when vectors and spinors coincide, and this only happens in dimensions 1, 2, 4 and 8.
In this universe of dimension 1, 2, 4 or 8, the interaction of matter and force particles would be described by multiplication in these number systems. In physics, such interactions are usually drawn using Feynman diagrams, named for physicist Richard Feynman. We can use the same

diagram to depict multiplication in a division algebra. So, in these universes with no time and special dimensions, nature would have supersymmetry.
Even though physicists have not yet found any concrete experimental evidence in support of supersymmetry, the theory is so beautiful, and has led to so much enchanting mathematics, that many physicists hope and expect that it is real. Moreover, among field theoretical models only a supersymmetric theory has the prospect of unifying the gravitational interaction with the electroweak and strong interaction, since the mediating particles have differing spins. Such a unification of interactions would incorporate a unification of General Relativity and Quantum Field Theory, which are the cornerstones of our modern understanding of the physical world. [15, 20]

7.5 String theory

At any moment in time a string is a 1-dimensional thing, like a curve or a line. But this string traces out a two-dimensional surface as time passes. This changes the dimensions in which supersymmetry naturally arises, by adding two, one for the string, and one for time. Instead of supersymmetry in dimensions 1, 2, 4 or 8, we get supersymmetry in dimensions 3, 4, 6, or 10. The idea is that a string traces out a 2-dimensional surface in spacetime, but it wiggles in 1, 2, 4, or 8 extra spatial dimensions. So for strings in dimensions 3, 4, 6 or 10, wiggling in 1, 2, 4, or 8 spatial directions, the vibrations are described using a division algebra.
Different vibrations of the string describe different kinds of particles. In particular, one of the vibrational modes fits the profile of the *graviton*, the hypothetical particle associated with the force of gravity. So String theory succeeds in quantizing gravity, a problem to which not many solutions are known. Therefore, it is often described as the leading candidate for the Theory of Everything in our universe. A theory that unifies gravity with the rest of the fundamental forces. However, string theory has made no testable predictions about the observable universe, many researchers trust it anyway. Richard Dawid, a physicist-turned-philosopher, identified three non-empirical arguments that generate trust in string theory among its proponents:

- There appears to be only one version of string theory capable of achieving unification in a consistent way. It turns out that only the 10-dimensional string theory is consistent. The rest suffer from glitches called anomalies, where computing the same thing in two different ways gives different answers. This is the theory that uses octonions. So, if superstring theory is right, the 10-dimensionality of spacetime arises from the octonions. Furthermore, no other theory of everything capable of unifying all the fundamental forces has been found, despite immense effort.

- String theory grew out of The Standard Model, the accepted and empirically validated theory incorporating all known fundamental particles and forces, apart from gravity, in a single mathematical structure.

- String theory has unexpectedly delivered explanations for several other theoretical problems aside from the unification problem it was intended to address. It explains the entropy of black holes, for example.

With these arguments Richard Dawid wants to show that to select a good theory, its ability to describe observation is not the only criterion. [15,20,21]
However, different opinions also exist among researchers. Maybe most notably, Feynman expressed this other type of view in 1964: "If a new law disagrees with experiment, it's wrong. That simple statement is the key to science. It doesn't make a difference how beautiful your guess is, it doesn't make a difference how smart you are, who made the guess or what his name is. If it disagrees with the experiment, it's wrong. That's all there is to it."

7.6 M-theory

Strings naturally fit together with supersymmetry and while originally several different types of string theory were found, these different theories turned out to be related to each other by "duality transformations." Such duality transformations identify the objects described by one theory with the objects described by another theory, thereby revealing that both theories are alternative descriptions of what is really the same physics. This led the American mathematical and theoretical physicist Edward Witten to conjecture that there are infinitely many string theories, all related to each other and subsumed by a larger, unique theory, dubbed "M-theory."
Recently physicists have started to go beyond strings to consider *membranes*. Membranes have one more dimension than strings. Thus when we're dealing with membranes we would expect supersymmetry to naturally emerge in dimensions 4, 5, 7 and 11. Researchers tell us that M-theory has 11 dimensions. The reason is very much the same as in string theory, but now these numbers are 3 more than 1, 2, 4 and 8. Which implies that it should naturally make use of octonions. Since nobody understands M-theory well enough to even write down its basic equation, M can also stand for mysterious.
We should emphasize that string theory and M-theory have as of yet made no experimentally testable predictions. Although there's a chance that evidence will turn up at the new particle experiments at CERN. [1,2]
However, a modern framework called Bayesianism, based on the 18th-century probability theory of the English statistician and minister Thomas Bayes, allows for the fact that modern scientific theories typically make claims far beyond what can be directly observed. For example no one has ever seen an atom, and so today's theories often resist a falsified-unfalsified dichotomy. Instead,

trust in a theory often falls somewhere along a continuum, sliding up or down between 0 and 100 percent as new information becomes available. [15, 20]

8 Summary

My thesis aimed to introduce normed division algebras, collect and summarize their properties systematically and show why it is worthwhile to deal with this area of mathematics.
There are exactly four normed division algebras: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), and octonions ($\mathbb{O}$). In Chapter 1, I looked at the history of these number systems. The use of real numbers dates back at least to 500 BC. The set of all real numbers forms a line, so we say that the real numbers are one-dimensional. Conversely, the line is one-dimensional because specifying a point on it requires one real number. The complex numbers were introduced by Gerolamo Cardano in the 16th century. They behave like coordinates on a plane. And since it takes two real numbers to specify a point on a plane, the complex numbers are two-dimensional, one more than the reals. The quaternions were invented by William Rowan Hamilton in 1843. And just as we think of the complex numbers as points in a two-dimensional plane, we can also think of the quaternions as points in a four-dimensional space. Not long after that, the octonions were discovered independently by John Graves and Arthur Cayley. They behave like coordinates in eight-dimensional space. Therefore the octonions form an 8-dimensional number system.
After introducing the necessary definitions and concepts, mainly from linear and abstract algebra in Chapter 2, I considered the construction of these number systems.
Chapter 3 shows three methods for doing that. First, I introduced their multiplication table, since every algebra is determined by that. Then I showed that to describe the product of the basis elements, it is enough to remember a few properties where the most complicated one can be expressed by using the Fano plane. I gave proof to the quaternions and the octonions noncommutative properties, and also to the octonions nonassociative property. The last way I showed how to create these algebras was through The Cayley-Dickson Construction which main concept is that these algebras can be built from each other, forming a sequence of algebras.
In Chapter 4, I looked into one of the properties of these number systems, namely division. I prove that division, except by zero, is always possible in each four of them. Then mentioned The Generalized Frobenius' Theorem, which states that every alternative division algebra is isomorphic to one of the following four algebras: the real numbers, the complex numbers, the quaternions, and the octonions.
In Chapter 5, I investigated the reason we can call these algebras normed. Essentially, I introduced a norm in each of them and also converted the 'question of normability' to the 'problem of the sum of squares'.
Chapter 6 consists of the proof of Hurwitz's Theorem, which states that every normed algebra with an identity is isomorphic to one of the following four algebras: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), and octonions ($\mathbb{O}$).
In the final chapter, I showed a few of the many applications these algebras have, like Super-

symmetry and String theory, with a special focus on the group representation of The Standard Model. And I further looked into how to rotate in 3-dimensions using the quaternions.

"We are still a long way from knowing if the octonions are of fundamental importance in understanding the world we see around us, or are merely a piece of beautiful mathematics. Of course mathematical beauty is a worthy end in itself, but it would be even more delightful if the octonions turned out to be built into the fabric of nature. And as the story of the complex numbers and countless other pieces of mathematics demonstrate, it would hardly be the first time that purely mathematical inventions later provided precisely the tools that physicists need." [6]

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