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BSc THESIS

THE ROBBA RING

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Contents

Introduction

The Robba ring is an essential tool in the theory of p-adic differential equations, see [6], and p-adic Galois representations. We aim to provide a fundamental and compact note on the Robba ring, usually denoted by $\mathcal{R}$, which is mostly studied in the framework of p-adic theory. However, to the maximal general, we prefer the context of discrete valuation fields and rings in this thesis. We work mostly with the note of P. Schneider and several books for each part of the thesis. The thesis is structured as follows.

For starter, the basic concepts and main properties of discrete valuation rings are summarized in Chapter 1, in which we explain how a discrete valuation function induces a norm, namely a non-archimedean norm, which uniquely determines a topology on a discrete valuation field, and be extendable on the finite extension fields as well as the algebraic closure.

One of the most effective approaches is to consider the Robba ring as the union of rings consisting of Laurent series with coefficients in a discrete valuation field over a certain closed or open annulus. Those rings over closed annuli are thoroughly analyzed by Newton polygons, similarly to polynomials with coefficients weighted by a valuation function, and Gauss norms in terms of factorization and roots. Consequently, we learn how the union conveys properties of rings of Laurent series with closed annuli as domains to rings with open annuli as domains.

The main part of the thesis is Chapter 3, where the Robba ring and its sub-rings are examined regarding ring structures, topology, and vector space norm. Once the basic knowledge is presented, we take a concise glimpse of endomorphism on the Robba ring as well as higher dimensional vector spaces over the Robba ring, where we want to construct an automorphism on. All theorems in the thesis are proven based on notes and books in the reference along with modifications to make the thesis itself a great self-contained material. The only statement without proof is Nagata's lemma, which requires the theory of local ring, see details in [11], but only the lemma only plays an insignificant role, then we avoid giving a full proof of the lemma to make the thesis more coherent.

1 Discrete Valuation Rings

This beginning chapter is to introduce discrete valuation rings, which play an essential part as a ring of coefficients in the main object. The main idea is to present fundamental concepts of discrete valuation function and study its behavior on a certain ring, which consequently induces a different absolute value function on the fraction field of the given ring, see [1]. For instant, the p-adic absolute value acts on the p-adic numbers field $\mathbf{Q}_p$ as well as its extension has an important role and wide applications in various brands of mathematics, see [2].

1.1 Discrete valuation functions

Definition 1.1. A discrete valuation function is defined on both fields and rings.

(**1**) Let K be a field. By a discrete valuation ν on the field K, we mean a function ν from $K^\times \to \mathbb{Z}$ satisfying

(**i**) ν is surjective,

(**ii**) $\nu(ab) = \nu(a) + \nu(b) \ \ \forall a, b \in K^\times$,

(**iii**) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\} \ \forall a, b \in K^\times$ with $a + b \neq 0$.

The subring $\{x \in K;\ \nu(x) \geq 0\} \cup \{0\}$ is called the valuation ring of the Discrete Valuation Ring of ν on the field K.

(**2**) An integral domain (i.e a nonzero commutative ring) R is called a Discrete Valuation Ring (D.V.R.) if R is the valuation ring of a discrete valuation ν on the field of fractions of R.

The convention $\nu(0) = +\infty$ is used for the element 0, so the function ν is well-defined on the field K, in which the condition (**i**), (**ii**), and (**iii**) remain satisfied.

The two definitions are identical because the latter of the above definition is interpreted in the manner that: Let ν is a function from an integral domain R to $\mathbb{N}$, in which the conditions (**i**), (**ii**), and (**iii**) are satisfied. One can extend the function ν to a discrete valuation on $\mathrm{Frac}(R)^\times$ by

$$\nu(\frac{a}{b}) = \nu(a) - \nu(b) \ \forall a, b \in R$$

That is a surjective homomorphism from the multiplicative group $\mathrm{Frac}(R)^\times$ to $\mathbb{Z}$ under addition. If $R = \{x \in K;\ \nu(x) \geq 0\}$ follows, then it is so-called a discrete valuation ring.

Definition 1.2. For a discrete valuation ring R with respect to discrete valuation ν, a uniformizer of R is an element of valuation 1, denoted by π.

Because of surjectivity, the uniformizer of a discrete valuation ring always exists, which leads to the following properties to give a deeper perception of D.V.R.

Proposition 1.3. *Given a discrete valuation ring R with respect to a discrete valuation ν, and let π be a uniformizer. Then*

(i) *An element t is a unit (i.e invertible) iff $\nu(t) = 0$.*

(ii) *Every nonzero element t of the field $\operatorname{Frac}(R)$ accepts the form $t = \pi^n u$ for some unit $u \in R^\times$, and some $n \in \mathbb{Z}$. In addition, the integer number n is independent from π, and the element $t \in R$ iff $n \in \mathbb{N}$.*

(iii) *The ring R is a principal ideal ring, every ideal is generated by an element of the form π^n, which again is independent from π. In particular, R has a unique maximal ideal $\langle \pi \rangle$.*

(iv) *For $a, b \in R$ with $\nu(a) \neq \nu(b)$, then $\nu(a+b) = \min\{\nu(a), \nu(b)\}$. Furthermore, let $(a_n)_{n\in\mathbb{N}} \subset R$ and $m = \min\{\nu(a_n);\ n \in \mathbb{N}\}$, then $\langle (a_n)_{n\in\mathbb{N}} \rangle = \langle \pi^m \rangle$.*

Proof. If t is a unit, then $t^{-1} \in R$ and $0 = \nu(1) = \nu(t) + \nu(t^{-1}) \geq 0$ with the range $\mathbb{N}$ of ν on R imply that $\nu(t) = \nu(t^{-1}) = 0$. In reverse, if $\nu(t) = 0$, then $t^{-1} \in \operatorname{Frac}(R)$ and $\nu(t^{-1}) = 0$, by definition, it shows that $t^{-1} \in R$ as required.

In (**ii**), let $\nu(t) = n$, then $\nu(t\pi^{-n}) = 0$, applying (**i**) to obtain the statement, and the independence from π is claimed because changing π by another uniformizer does not affect the valuation of $\nu(t\pi^{-n})$.

Let I be an arbitrary nonzero ideal in R, and let $\min\{\nu(x);\ x \in I\} = n$, then there exists an element t in I such that $\nu(t) = n$, by (**ii**), $t = \pi^n u$, where u is a unit. Hence, π^n is an element of I, which leads to $\langle \pi^n \rangle \subset I$. In other hand, every element of I is of at least valuation n, which means $I \subset \langle \pi^n \rangle$. This proves (**iii**).

The last property is a direct consequence of the results (**ii**) and (**iii**). □

Corollary 1.4. *For any $a, b \in K$ with $\nu(a) \neq \nu(b)$, then $\nu(a+b) = \min\{\nu(a), \nu(b)\}$.*

Proof. Let u be an invertible emelemt, equivalently, $\nu(u) = 0$. Notice that for any element t of positive valuation, the sum $u + v$ is invertible as well, because if it is not, then $u \in \langle \pi \rangle$, which is a contradiction. Combining that with property (**ii**) claims the corollary. □

Example 1.5. (**1**) An important example of valuation fields is the field of integer numbers $\mathbb{Q}$, when p is a fixed prime as well as a uniformizer. The discrete valuation ν is defined on $\mathbb{Q}$ by

$$\begin{aligned} \mathbb{Q}\backslash\{0\} &\longrightarrow \mathbb{Z} \\ t &\mapsto n \text{ where } t = p^n \frac{a}{b} \text{ with } p \nmid ab \end{aligned}$$

is called p-adic valuation on $\mathbb{Q}$. We recall definition of localization. By this, we mean a localization of a commutative ring R by a multiplicatively closed set S is a new ring $S^{-1}R$. It is clear that the localization of $\mathbb{Z}$ given by

$$\{ab^{-1};\ a, b \in \mathbb{Z} \text{ and } p \nmid b\}$$

is the D.V.R of $\mathbb{Q}$ with respect to the p-adic valuation. The example itself also shows that not every integral ring quipped with a valuation function is a D.V.R, in particular, the ring $\mathbb{Z}$ with respect to the p-adic valuation is not a D.V.R.

(**1**) Let K be an arbitrary field, and consider the field $K((T))$ of the formal Laurent series, then the function ν given by

$$\begin{aligned} K((T)) &\longrightarrow \mathbb{Z} \\ \sum_{m=n_0}^{+\infty} a_n T^n &\mapsto \min\{n;\ a_n \neq 0\} \end{aligned}$$

is a discrete valuation, the D.V.R with respect to which is the ring of power series $K[[T]]$.

Note that there is more than one way to define a discrete valuation ring, but the definition referring to discrete valuation function on fraction fields is seemingly the most explicit. Other alternative definitions regard the other types of rings with several different conditions such as Principal Ideal Rings (P.I.D), Unique Factorization Domains (U.F.D), and Noetherian integral domains, see [3]. The connection of those algebra structures will be explicitly described in the following theorem.

Theorem 1.6. *The attributes listed below of a ring R are interchangeable:*

(**1**) *R is a D.V.R,*

(**2**) *R is a P.I.D with a unique non-zero maximal ideal.*

(**3**) *R is a U.F.D with a unique irreducible element π up to unity.*

(**4**) *R is a Noetherian integral domain as well as a local ring, whose unique nonzero maximal ideal is principal. Particularly, the Krull dimension of this local ring is* 1.

Proof. Given a D.V.R R, we recall the proposition 1.3, property (**iii**) and (**ii**) imply (**2**) and (**3**).

If (**2**) holds, then let $\langle \pi \rangle$ be the maximal ideal, which directly shows the element π is a unique irreducible up to unity, hence (**3**) follows.

In order to obtain the last statement from (**3**), we need to show that R is Noetherian, equivalently, every ascending chain of ideals is of finite length. Indeed, let the nonzero starting ideal be I, then by the proposition 1.3, there exists a non-negative number n such that $I = \langle \pi^n \rangle$. Thus, it is clear that the longest ascending chain starting by I is of the length n, which is exactly

$$\langle \pi^n \rangle \subsetneqq \langle \pi^{n-1} \rangle \subsetneqq \ldots \subsetneqq \langle \pi \rangle.$$

In addition, the spectrum of R is $\{0, \langle \pi \rangle\}$.

In reverse, given (**3**), as π is the unique irreducible, every nonzero element of $\mathrm{Frac}(R)$ is a multiple of a (either positive or non-negative) power of π with a unity independently from π. Thus, the function $\nu : \mathrm{Frac}(R)^\times \to \mathbb{Z}$ in the manner $\nu(\pi^n u) = n$ with u is a unit is well-defined. It is straightforward to check the function ν is a discrete valuation, and R is the valuation ring of ν. It is obvious that ν meets all the conditions of a discrete valuation. On other hand, let u be an element of R with valuation 0, then $\langle u \rangle \neq \langle \pi \rangle$, which means $\langle u \rangle = R$, accordingly, u is invertible. Consequently, every element of non-negative valuation is R.

Finally, suppose (**4**) is given, let t be an arbitrary element of R, and π be the element generating the unique nonzero maximal ideal. Hence, the local ring property implies that either $\langle t \rangle$ is the whole ring or $\langle t \rangle \subset \langle \pi \rangle$. Equivalently, either an element is invertible or there exists another element t_1 in R such that $t = \pi t_1 \Rightarrow \langle t \rangle \subset \langle t_1 \rangle$. One applies the Noetherian chain condition to that result, property (**ii**) in the proposition 1.3 is claimed, which results back to (**2**). □

1.2 Topology on Discrete Valuation Rings

The main role of an absolute value is to equip a "size" to elements of a given field, by which we are able to measure distance of distinct points, as well as topology on it, that regards the properties of the induced metric, namely convergence, open and closed set, and completeness. In this section, we shall briefly study the topology of a discrete valuation field K with a discrete valuation ν, the discrete valuation ring R as defined in the last section, see [4].

Definition 1.7. An absolute value $|\cdot|$ is called a non-archimedean absolute value if the following condition holds

$$|x + y| \leq \max\{|x|, |y|\} \ \forall x, y \in K$$

Otherwise, it is archimedean.

Definition 1.8. Let $|\cdot|_\nu : K \to \mathbb{R}_{\geq 0}$ be a function defined by

$$|x|_\nu = \begin{cases} e^{-\nu(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This function is called the absolute value associated to the discrete valuation ν.

It is obvious to show that $|\cdot|_\nu$ is a non-archimedean absolute value by using properties of exponential function and discrete valuation. For simplicity, from now on, unless explicitly stated otherwise the notation $|\cdot|$ implies the absolute value associated to the given discrete valuation. Consequently, let define a metric by $d(x,y) = |x-y|$, then the non-archimedean property is equivalent to a condition, so-called "strong triangle inequality":

$$d(x,y) \leq \max\{d(x,z), d(y,z)\} \; \forall x,y,z \in K$$

The metric (K,d) is an ultra-metric space that has several noticeable properties listed in the following proposition.

Proposition 1.9. *Let $B(a,r)$ and $\overline{B}(a,r)$ be open and closed balls of radius r and center a respectively.*

- (**i**) *All triangles are isosceles.*
- (**ii**) *Every interior point of a ball is a center of that ball.*
- (**iii**) *Every point that is contained in a closed ball is also a center of that ball.*
- (**iv**) *Every open ball is closed and has empty boundary.*
- (**v**) *Every closed ball of positive radius is also open and has empty boundary.*
- (**vi**) *Any two open or closed with positive radius balls are either disjoint or contained in one another.*

Proof. (**i**) Let x, y and z be three arbitrary distinct points, and consider the strong triangle inequality. If the equality holds, thus the triangle consisting of x, y and z is isosceles. Otherwise, $\nu(x-y) > \nu(x-z) + \nu(y-z)$, which reveals that $\nu(x-z) = \nu(y-z) \Rightarrow |x-z| = |y-z|$ by making use of corollary 1.4.

(**ii**) Let $b \in B(a,r)$, and let c be an arbitrary point in the ball $B(a,r)$, one estimates the distance between two points b and c

$$|b-c| \leq \max\{|b-a|, |c-a|\} < r$$

so that $c \in B(b,r)$, this show $B(a,r) \subset B(b,r)$, and switching a and b makes the two balls identical.

(iii) It is the same as the part **(ii)**.

(iv) Let observe the complement of the open ball $B(a, r)$

$$B^c(a, r) = \{x \in K;\ |x - a| \geq r\}$$

For any element x of C and $0 < s < r$, the such number s exists because if $r = 0$ then there is nothing to discuss. Let $y \in B(x, s)$, then $|x - y| < s < r \leq |x - a|$. By corollary 1.4, we obtain

$$|y - a| = \max |x - y|, |x - a| \geq r$$

so that $B(x, s) \subset B^c(a, r)$, this proves the complement is open as well, thus, $B(a, r)$ is closed and has empty boundary.

(v) It is similar to **(iv)**.

(vi) The statement is a corollary of **(ii)** and **(iii)**.

$\square$

Definition 1.10. Given a discrete valuation ring R, the maximal ideal is denoted by $\mathfrak{m}$, and the field induced by the quotient ring $R/\mathfrak{m}$ is called the Residue Field with notation $\mathbf{k}$.

In terms of topology, we can rewrite those definitions in the manner

- The discrete valuation ring $R = \overline{B}(0, 1) = \{x \in K;\ |x| \leq 1\}$.
- The maximal ideal, also as known as the valuation ideal $\langle \pi \rangle = B(0, 1) = \{x \in K;\ |x| < 1\}$.
- The residue field $\mathbf{k} = \overline{B}(0, 1) \backslash B(0, 1) = \{x \in K;\ |x| = 1\}$.

Moreover, the absolute value induces the casual terms for consequence in the field K. In particular, given a sequence $(a_n)_{n \in \mathbb{N}} \subset K$ and an element a in K. The convergence can be expressed both in languages of topology and valuation.

- $(a_n)_{n \in \mathbb{N}}$ converges to a if every open ball of center a contains all but finitely many elements of $(a_n)_{n \in \mathbb{N}}$, equivalently, $\lim_{n \to \infty} |a_n - a| = 0$, equivalently, $\lim_{n \to \infty} \nu(a_n - a) = +\infty$.
- $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence if for $\varepsilon > 0$, there exists a positive integer N such that for all $m, n > N$, $|a_m - a_n| < \varepsilon$, equivalently, for any $C > 0$, there exists a positive integer N such that for all $m, n > N$, $\nu(a_m - a_n) > C$.

Definition 1.11. The discrete valuation field K is complete if the metric space with distance function associated with the discrete valuation is complete.

Theorem 1.12. ***(Uniqueness of completeness)*** *Let K be a discrete valuation field with a discrete valuation ν, there is exactly one complete discrete valuation field $\hat{K}$ up to isometric isomorphism with respect to a discrete valuation ν' such that, the K is dense in $\hat{K}$, and the restriction of ν' on $K^\times$ is ν.*

Proof. The uniqueness is obviously provided by the properties of Cauchy sequences and the identity of the restriction function of a discrete valuation on a dense set K. The remaining part is to prove there is a complete discrete valuation field, which can be constructed as follows.

Let $\mathcal{C}$ be a subset of $K^{\mathbb{N}}$ consisting of all Cauchy sequences over the field K. On $\mathcal{C}$, we equip component-wise multiplication and addition, with respect to which, the set $\mathcal{C}$ is kindly closed. Moreover, $(\mathcal{C}, +, \cdot)$ is a ring with additive identity element $\mathbf{0} = (0, 0, \dots)$ and multiplicative identity element $\mathbf{1} = (1, 1, \dots)$. Therewith, we define an equivalence relation

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}} \Leftrightarrow \lim_{n\to\infty} (a_n - b_n) = 0$$

Thus, the quotient ring $\mathcal{C}/\sim$ is a commutative ring. In addition, it is a field by following the argument: for any nonzero equivalence class $\left[(a_n)_{n\in\mathbb{N}}\right]$, from some intermediate index, components are nonzero, then there exists a sequence $(b_n)_{n\in\mathbb{N}}$ given by

$$b_n = \begin{cases} a_n^{-1} & \text{if } a_n \neq 0 \\ 0 & \text{if } a_n = 0 \end{cases}$$

Thus

$$(a_n)_{n\in\mathbb{N}} \cdot (b_n)_{n\in\mathbb{N}} \in \left[(c_n = 1)_{n\in\mathbb{N}}\right] \Rightarrow \left[(a_n)_{n\in\mathbb{N}}\right] \cdot \left[(b_n)_{n\in\mathbb{N}}\right] = \left[(c_n = 1)_{n\in\mathbb{N}}\right]$$

Let use the notation $\hat{K}$ to represent the field $\mathcal{C}/\sim$, and let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence, and $A_n = \left[(a_{n,k})_{k\in\mathbb{N}}\right]$, then it is clear that $\left[(a_{n,n})_{n\in\mathbb{N}}\right]$ is a Cauchy sequence, which is the limit of $(A_n)_{n\in\mathbb{N}}$ as well. That means $\left(\hat{K}, \nu'\right)$ is complete with respect to the absolute value associated to the valuation ν' given by $\nu'\left[(a_n)_{n\in\mathbb{N}}\right] = \lim_{n\to\infty} \nu(a_n)$.

Finally, the inclusion

$$\begin{aligned} \mathbf{i} : K &\to \hat{K} \\ x &\mapsto \left[(x_n = x)_{n\in\mathbb{N}}\right] \end{aligned}$$

is an injective homomorphism, and $\mathrm{Im}(\mathbf{i}) = \left\{ \left[(x_n = x)_{n\in\mathbb{N}}\right];\ x \in K \right\}$ is a dense subset of

$\hat{K}$. That proves $\left(\hat{K}, \nu'\right)$ is the unique complement of K up to isometric isomorphism. $\square$

Corollary 1.13. *For any uniformizer π of K, the element $\left[(x_n = \pi)_{n\in\mathbb{N}}\right]$ is a uniformizer of the discrete valuation field $\hat{K}$, and the residue field is isomorphic to the residue field of the original field K.*

Proof. The first one follows from $\nu'\left(\left[(x_n = \pi)_{n\in\mathbb{N}}\right]\right) = 1$. For the latter, the inclusion $\mathbf{i}$ also induces an inclusion $K/\langle\pi\rangle \to \hat{K}/\left\langle \left[(x_n = \pi)_{n\in\mathbb{N}}\right] \right\rangle$. Meanwhile, since $\mathrm{Im}(\mathbf{i})$ is dense in $\hat{K}$, for every $\left[(a_n)_{n\in\mathbb{N}}\right]$ of valuation 0, there exists an element $a \in \mathbf{k}$ such that $\nu'\left(\left[(a_n - a)_{n\in\mathbb{N}}\right]\right) > 0$, this shows that $\hat{K}/\left\langle \left[(x_n = \pi)_{n\in\mathbb{N}}\right] \right\rangle \subset K/\langle\pi\rangle$. $\square$

Example 1.14. A familiar example is the field of integer $\mathbb{Q}$ is not complete with respect to p-adic valuation, and its complement is the field of p-adic number $\mathbb{Q}_p$.

1.3 Expanding valuation to finite extension fields

Unless explicitly stated, otherwise, we assume that R is a complete D.V.R of a complete discrete valuation field K. The goal is to provide an accurate description of finite extension fields by ramification index and residue degree, that shall be introduced in this section. For simplicity, the notation congruence $a \equiv b \mod (\pi^n)$ accounts for $a - b \in \langle\pi^n\rangle$.

Theorem 1.15. *Let $(\pi_n)_{n\in\mathbb{Z}}$ be a fixed sequence with property: for any $n \in \mathbb{Z}$, $\nu(\pi_n) = n$. In addition, let $\mathcal{A}$ be a system of representative for the residue field. Then the followings are claimed.*

(**i**) *Every series $\sum_{n\geq m_0} a_n\pi_n$ with coefficients in $\mathcal{A}$ is in K.*

(**ii**) *Every element $x \in K$ accepts a uniquely determined representation which is a series*

$$\sum_{n\geq m_0} a_n\pi_n$$

with coefficients in $\mathcal{A}$ and $\nu(x) = m_0$.

In other words, $K = R\left[\frac{1}{\pi}\right]$.

Proof. (**i**) Note that the partial sums of the series $S_m = \sum_{n=m_0}^{m} a_n\pi_n$ make a Cauchy sequence, which converges to an element in K because of its completeness.

(**ii**) The uniqueness is trivial since any two elements, that share the same series shall consequently have valuation of their difference is infinite, which implies the unity element zero.

The remaining is the existence, which is provided as following: By definition, for every unit, there exists exactly one element in $\mathcal{A}$, that is congruent with the unit.

According to that, we utilize induction to show for any $n \in Z$, the sequence $(S_m)_{m \geq m_0}$ given by

$$a_{m_0} \equiv \frac{x}{\pi_{m_0}} \Rightarrow S_{m_0} = a_{m_0}\pi_{m_0}$$
$$\ldots$$
$$a_m \equiv \frac{x - S_{m-1}}{\pi_m} \Rightarrow S_m = S_{m-1} + a_m\pi_m$$
$$\ldots$$

is a Cauchy sequence. As a result of part (**i**), $\lim_{m\to\infty} S_m = \sum_{n\geq m_0} a_n\pi_n$, which is an element in K, and $\lim_{m\to\infty} S_m = x$. Hence, $x = \sum_{n\geq m_0} a_n\pi_n$.

□

Definition 1.16. Let E/F be a finite extension field of a field F, and let $[E : F] = d \in \mathbb{Z}_{\geq 1}$. A Norm Mapping $\mathbf{N}_{E/F} : E \to F$ is a function satisfying one of the following conditions:

(**i**) Let α be an arbitrary element of E. Hence, the bijection $\varphi_\alpha : E \to E : x \mapsto \alpha x$ is alternatively interpreted as a F-linear operator over the F-vector space of dimension d. Therewith, let $\mathbf{N}_{E/F}(\alpha)$ be the determinant of the matrix associated to the operator φ_α.

(**ii**) Let α be an arbitrary element of E, r be the degree of E as an extension of the field $F(\alpha)$, and the minimal polynomial of α over F given by

$$f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in F[X]$$

Then we define $\mathbf{N}_{E/F}(\alpha) = (-1)^{nr}a_0^r$.

(**iii**) Let α be an arbitrary element of E, without loss of generality, we can assume E/F is normal, because if it is not, we can define the map over the normal closure of E instead. Then let define

$$\mathbf{N}_{E/F}(\alpha) = \prod_{\sigma\in\mathbf{Aut}(E/F)} \sigma(\alpha).$$

Remark 1.17. Note that the "Norm mapping" appears to have the same name as "norm". It needs to be highlighted that the function defined as above is not a norm. In particular, the mapping is a surjective from a finite extension of a field onto itself, which surprisingly possesses many interesting properties, that can be used to expand a discrete valuation as well as its associated absolute value to any finite extension fields in the following method.

In order to make sense of this definition, the next proposition shall prove the connection of the three conditions

Proposition 1.18. *The conditions* (**i**), (**ii**), *and* (**iii**) *are equivalent.*

Proof. Notice that for any $\sigma \in \mathbf{Aut}(E/F)$, $\sigma(\alpha)$ is also a root of the polynomial f. Meanwhile, every F-automorphism σ on E/F is extended from some F-isomorphism form $F(\alpha)$ to $F(\sigma(\alpha))$. In reverse, because E/F is normal, thus every F-isomorphism form $F(\alpha)$ to $F(\sigma(\alpha))$ is extended to exactly r F-automorphism on E/F. That makes $(-1)^{nr}a_0^r = \prod_{\sigma\in\mathbf{Aut}(E/F)} \sigma(\alpha)$, equivalently, (**ii**), and (**iii**) are interchangeable.

Let $\{1, \alpha, \dots, \alpha^{n-1}\}$ be a basis of the F-vector space $F(\alpha)$, and let $\{b_1, \dots, b_r\}$ be a basis of $F(\alpha)$-vector space E. Thus, $\{\alpha^i b_j;\ i = \overline{1, n-1},\ j = \overline{1, r}\}$ be a basis of F-vector space E. By using $\alpha^n = -(a_{n-1}X^{n-1} + \dots + a_0)$, the associated matrix of φ_α with respect to that basis is given by $\mathbf{Diag}(\mathbf{A}, \dots, \mathbf{A})$, where $\mathbf{A}$ is a $n \times n$ matrix defined by

$$\mathbf{A} = \begin{bmatrix} 0 & \dots & \dots & 0 & -a_0 \\ 1 & \dots & \dots & 0 & -a_1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & -a_{n-1} \end{bmatrix} \Rightarrow \det(\mathbf{A}) = (-1)^n a_0$$

Hence

$$\det(\mathbf{Diag}(\mathbf{A}, \dots, \mathbf{A})) = (-1)^{nr} a_0^r$$

That proves (**i**) and (**ii**) are equivalent as well. □

Corollary 1.19. *The norm mapping is multiplicative, and let three fields $L/E/F$, then*

$$\mathbf{N}_{L/F} = \mathbf{N}_{E/F} \circ \mathbf{N}_{L/E}.$$

Let L/K be a finite extension of K of degree d, and let use the notations R_L and $\mathbf{k}_L$ be denoted the discrete valuation ring and the residue field of the field L, respectively. However, in order to do those notations, we first need to claim whether L is a discrete valuation field as well as the completeness.

Lemma 1.20. ***(Hensel's lemma)*** *Let R be the discrete valuation ring of K, and let $f(X)$ be a polynomial of order n with coefficients in R such that: there are polynomials $g_1(X)$ and $h_1(X)$, that are relatively prime, $g_1(X)$ is monic and $f(X) \equiv g_1(X)h_1(X) \mod (\pi)$. Then there exist polynomials $g(X)$ and $h(X)$ with coefficients in R such that*

- *$g(X)$ is monic, $g(X)$ and $h(X)$ are relatively prime,*
- *$g(X) \equiv g_1(X) \mod (\pi)$ and $h(X) \equiv h_1(X) \mod (\pi)$,*

- $f(X) = g(X)h(X)$.

Proof. Induction is an effective strategy to solve this problem, starting with $g_1(X)$ and $h_1(X)$, we assume that the degree of $g_1(X)$ is $m \neq n$, then the degree of $h_1(X)$ does not exceed $n - m$. We construct two sequences of polynomials $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ with coefficients in R by strong induction such that. Assuming that we have successfully defined $g_n(X)$ and $h_n(X)$ with

- $g_n(X)$ is a monic polynomial of degree m,
- $g_n(X)$ and $h_n(X)$ are relatively prime,
- $g_{n+1}(X) \equiv g_n(X) \mod (\pi^n)$,$h_{n+1}(X) \equiv h_n(X) \mod (\pi^n)$, and
- $f(X) \equiv g_n(X)h_n(X) \mod (\pi^n)$.

It is clear that $g_1(X)$ and $h_1(X)$ are satisfied. We inductively assume the first n polynomials have been successfully defined, and we construct the next terms in the manner

$$g_{n+1}(X) = g_n(X) + \pi^n u_n(X) \text{ and } h_{n+1}(X) = h_n(X) + \pi^n v_n(X)$$

in which we need to find the appropriate polynomials $u(X)$ and $v(X)$. Note that

$$g_{n+1}(X)h_{n+1}(X) = g_n(X)h_n(X) + \pi^n(v(X)g_n(X) + u(X)h_n(X)) + \pi^{2n}u(X)v(X)$$

and the ideal $\langle g_n(X), h_n(X)\rangle$ is a unity ideal of the polynomial ring $R[X]$, then there exist such $u(X)$ and $v(x)$ that

$$\frac{f(X) - g_n(X)h_n(X)}{\pi^n} \equiv v(X)g_n(X) + u(X)h_n(X) \mod (\pi)$$

In addition, changing $u(X)$ and $v(X)$ by $u(X) - t(X)g(X)$ and $v(X + t(X)h(X))$ does not change the congruence condition, so that without loss of generality, we can choose $u(X)$ such that degree of g_{n+1} remains as the degree of $g_n(X)$, moreover, g_{n+1} is monic.

Hence, by theorem 1.15, the sequences $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ component-wise converge to $g(X)$ and $h(X)$, respectively, with the properties are inherited. □

In the proof, the completeness of K plays a key role. Apparently, it raises the question that if the Hensel's lemma is true over a field that is not discrete valuation. The answer is "yes", and the terminology "Henselian ring" is used to name such rings, over the fraction field of which the Hensel's lemma works.

Corollary 1.21. *Let $f(X)$ be a monic irreducible polynomial in $K[X]$, then $f(0) \in R \Leftrightarrow f(X) \in R[X]$.*

Proof. It is obvious that the first statement is a corollary of the latter.

In order to prove the latter from the first one, let m be the smallest non-negative integer such that $\pi^m f(X) \in R[X]$. If $m = 0$ then there is nothing to discuss. We assume that $\pi^m f(X)$ is a polynomial with the least valuation of coefficients is 0, i.e, a unit in R. Then there exist a monic polynomial $g(X)$ with degree less than $f(X)$, such that $f(X) \equiv g(X) \cdot 1 \mod (\pi)$, to which the Hensel'lemma is applied to obtain a factorization of f by two polynomials with coefficients in R and degrees less than the degree of $f(X)$. That is a contradiction to the irreduciblity of $f(X)$. $\square$

Theorem 1.22. *L is a complete discrete valuation field. In addition, there is exactly one discrete valuation ν_L on L satisfying the restriction of $\hat{\nu_L}$ over K is a constant multiple of the discrete valuation on K.*

Proof. We construct a discrete valuation on L as following. Let $\hat{\nu_L}$ be a composition function $\nu \circ \mathbf{N}_{L/K}$ form $L^\times$ to $\mathbb{Z}$. It is straightforward to check the conditions of the definition 1.1 on $\hat{\nu_L}$.

According to (**i**) of the definition 1.16, the norm mapping is multiplicative, that instantly satisfies the condition (**ii**) of the definition 1.1. The condition (**iii**) is archived as follow.

- If $\hat{\nu_L}(x)$ is non-negative for an element $x \in L$, then let $f(X)$ be a minimal polynomial of x over K. The definition 1.16 shows that

  $$f(0)^{[L:K(x)]} = (-1)^d \mathbf{N}_{L/K}(x)$$

  then $\hat{\nu_L}(x)$ implies $f(X) \in R[X]$ as a result of the corollary 1.21. Thus, $f(X - 1) \in R[X]$ is the minimal polynomial of $x + 1$, and $\nu(f(-1)) \geq 0$ means that $\hat{\nu_L}(x + 1) \geq 0$.

- Let $x \in L$ with $\nu(x) \leq 0$. Then $\nu(x^{-1} + 1) \geq 0$ by the last argument, combining with the multiplicity of the norm mapping, we result

  $$\hat{\nu_L}(x + 1) = \hat{\nu_L}(x(1 + x^{-1})) = \hat{\nu_L}(x) + \hat{\nu_L}(1 + x^{-1}) \geq \min\{0, \hat{\nu_L}(x)\}$$

That proves the condition (**iii**) of the definition 1.1 is true in the case one of elements is unit, and another is of non-negative valuation. In addition, the multiplicity of the norm mapping is conveyed to the function $\hat{\nu_L}$, then in every case, we can convert it into the simple case as above.

For any $x \in K$, $\mathbf{N}_{L/K}(x) = (-x)^d$, then $\hat{\nu_L}|_{K^\times} = d\nu$. Furthermore, because degrees of minimal polynomials are divisors of d, then there are $c \mid d$ such that $\mathrm{Im}(\hat{\nu_L}) = c\mathbb{Z}$, then we define

$$\nu_L = \frac{1}{c}\hat{\nu_L} : L \to \mathbb{Z}$$

This function is surjective, then it is a discrete valuation on L, and $\nu_L|_{K^\times} = \frac{d}{c}\nu$. For the completeness, set $\mathbf{e}_{L/K} = \dfrac{d}{c}$, and let consider L as a finite dimensional K-vector space, with the absolute value

$$\|x\|_L = e^{\frac{-\nu_L(x)}{\mathbf{e}_{L/K}}}$$

Note that the restriction of this norm on K is the norm associated to the discrete valuation ν. Besides, let define another norm as a "max" column vector in the usual way, which is

$$\sum_{i=1}^{d} a_i e_i = \max\{|a_i|;\ i \in \{1, \ldots, d\}\}$$

where $\{e_1, \ldots, e_d\}$ is the standard basis. We apply two fundamental lemmas in Functional Analysis to this setting, see [5].

Lemma 1.23. *All norms over a certain vector space of finite dimension are equivalent.*

Lemma 1.24. *Every finite dimensional normed space over a complete field is a Banach space.*

As a result, K is complete then L is complete with respect to both defined absolute value. $\square$

Corollary 1.25. *The same method constructs a valuation on the algebraic closure $\overline{K}$ and the complement of $\overline{K}$ as well. In particular, the image of such a valuation is $\mathbb{Q}$.*

Definition 1.26. We divide the degree of an extension field as below:

- The number $\mathbf{e}_{L/K} \mid [L : K]$ is called Ramification Index of L/K. A finite extension is called Totally Ramifield if $\mathbf{e}_{L/K} = [L : K]$. In reverse, if $\mathbf{e}_{L/K} = 1$ then the extension field is called Unramifield.
- The number $\dfrac{[L : K]}{\mathbf{e}_{L/K}}$ is called Residue Degree, denoted by $\mathbf{f}_{L/K}$.

Theorem 1.27. *For any extension fields $M/L/K$, the followings hold*

(i) $[\mathbf{k}_L : \mathbf{k}] = \mathbf{f}_{L/K}$

(ii) $\mathbf{e}_{M/K} = \mathbf{e}_{M/L} \cdot \mathbf{e}_{L/K}$ *and* $\mathbf{f}_{M/K} = \mathbf{f}_{M/L} \cdot \mathbf{f}_{L/K}$.

Proof. It is trivial that $\mathbf{k} \subset \mathbf{k}_L$. In particular, let $x_1, \ldots, x_n$ be a $\mathbf{k}$-linear independent system of $\mathbf{k}_L$, then every set with $\{y_i \in L;\ y_i \in [x_i]\ \forall i \in \{1, \ldots, n\}\}$ is a K-linear independent in L. To prove it, let indirectly assume that there exists a such K-linear dependent set $\{y_i\}$, then there are $\{a_i;\ i \in \{1, \ldots, n\}\} \subset K$ such that at least one of them is nonzero, and

$$\sum_{i=1}^{n} a_i y_i = 0$$

Without loss of generality, we can assume $\{a_i;\ i \in \{1, \dots, n\}\} \in R$, and at least one of them is a unit. Then we observe both side in $\mod (\pi_L)$, then we obtain

$$\sum_{i=1}^{n} a'_i x_i = 0$$

where at least one a_i is nonzero. In other words, every $\mathbf{k}$-linear independent system of $\mathbf{k}_L$ induces a K-linear independent in L. That means $[\mathbf{k}_L : \mathbf{k}] = f \leq [L : K]$.

Let $x_1, \dots, x_f$ is a system of representatives of $\mathbf{k}_L$, and let consider the set $\{\pi_L^i;\ i \in \{1, \dots, \mathbf{e}_{L/K}\}\}$, then by evaluating the valuation ν_L, we can claim that the set is K-linear independent. Consequently, we achieve a K-linear independent set

$$\{x_i \pi_L^j;\ i \in 1, \dots, f; j \in \{1, \dots, \mathbf{e}_{L/K}\}\}$$

Thus, $\mathbf{e}_{L/K} \cdot f \leq d$. Meanwhile, theorem 1.15 show that for every $x \in L$ accepts a form

$$x = \sum_{m=n_0}^{\infty} b_m \pi_m$$

In particular, we choose $\pi_m = \pi^j \cdot \pi_L^i \forall m = j\, \mathbf{e}_{L/K} + i$, every unit b_m is rewrite in the $\mathbf{k}_L$-basis $\{x_1, \dots, x_f\}$, the result is of the form

$$x = \sum_{i=1}^{\mathbf{e}_{L/K}} \sum_{j=1}^{f} \pi_L^i x_j c_{ij}$$

with $c_i j$ in K, which is straightforward to $\{x_i \pi_L^j;\ i \in 1, \dots, f; j \in \{1, \dots, \mathbf{e}_{L/K}\}\}$ is a generating system, thus, $\mathbf{e}_{L/K} \cdot f \geq d$, combing with the above, we attain that $f = \mathbf{f}_{L/K}$. In addition, because every element of the D.V.R of L is the series starting at at least 0, then the D.V.R of L is a R-free module of degree d as well.

The statement (**i**) is a direct corollary of the definition of $\mathbf{e}_{L/K}$ and $\mathbf{f}_{L/K}$. □

Theorem 1.28. *Let $\overline{K}$ be the algebraically closure of K. In terms of fields and rings, the followings hold.*

- $\overline{R} = \overline{B}(0, 1) = \{x \in \overline{K};\ \nu(x) \geq 0\}$ *is a (local) valuation ring with the valuation nu satisfying all the laws of the definition* 1.1 *when replacing the range* $\mathbb{Z}$ *by* $\mathbb{Q}_{\geq \nvdash}$.
- *The maximal ideal of* $\overline{R}$ *is* $\mathcal{O}_{\overline{K}} = B(0, 1) = \{x \in \overline{K};\ \nu(x) > 0\}$.
- *The residue field of* $\overline{K}$ *is* $\overline{\mathbf{k}} = \overline{R}/\mathcal{O}_{\overline{K}}$, *which is an algebraic closure of* $\mathbf{k}$.

Proof. Most of those are corollaries of extending the field K. The difficult one is proving the algebraic closure of $\mathbf{k}$ is $\overline{R}/\mathcal{O}_{\overline{K}}$. For any element $x \in \overline{R}/\mathcal{O}_{\overline{K}}$, there is an finite extension L of K such that $x \in \mathbf{k}_L$, then $\overline{R}/\mathcal{O}_{\overline{K}}$ is a sub-field of the algebraic closure of

$\mathbf{k}$. Moreover, it is easy to see that all algebraic of $\mathbf{k}$ is in $\overline{R}/\mathcal{O}_{\overline{K}}$. That provides what we need. $\square$

2 Laurent series on discrete valuation fields

The Laurent series is a well-known objective to study in terms of real or complex analysis. Once we define an absolute value, we can examine the convergence of sequence or series on a complete valued field. The goal of this section is to learn the behavior of Laurent series on a field equipped with an absolute value associated with a discrete valuation, see also [6] and [7].

2.1 Laurent series on closed annuli and Gauss norms

Unless explicitly stated, we use the notation $K, R, \nu, |\cdot|$, etc as above.

Definition 2.1. A Laurent series with coefficient in K is a formal series

$$F(X) = \sum_{n\in\mathbb{Z}} a_n X^n.$$

The Laurent series is said to be convergent at the point x_0, if the sequence of partial sum $S_N = \sum_{n=-N}^{N} a_n x_0^n$ is a Cauchy sequence, and the limit point is the value of $F(x_0)$. The set of all Laurent series is denoted by $K\left[[X]\right]$.

Theorem 2.2. ***(Radii of convergence)*** *Let $F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in K\left[[X]\right]$. Then the radii of convergence r_1 and r_2 are given by formula*

$$r_1 = \limsup_{n\to+\infty} |a_{-n}|^{1/n} \quad \textit{and} \quad r_2 = \frac{1}{\limsup_{n\to+\infty} |a_n|^{1/n}}$$

The Laurent series $F(X)$ converges at every element of the annulus

$$\mathcal{A}_{(r_1,r_2)} = \{x \in \overline{K};\ r_1 < |x| < r_2\}.$$

In addition, for every points not in the closed annulus $\mathcal{A}_{[r_1,r_2]} = \{x \in \overline{K};\ r_1 \le |x| \le r_2\}$, the Laurent series $F(X)$ diverges.

Proof. For any $x \in \mathcal{A}_{(r_1,r_2)}$, then there exist a positive integer N_0, for all $n \ge N_0$

$$|a_n x^n| \le \left|\frac{a_n}{\sup_{n\ge N_0}\{|a_n|\}}\right| < 1 \text{ and } \left|\frac{a_{-n}}{x^n}\right| \le \left|\frac{a_{-n}}{\sup_{n\ge N_0}\{|a_{-n}|\}}\right| < 1$$

Combing with the non-archimedean property of the absolute value associated to a discrete valuation, we obtain that the two power series $\sum_{n=0}^{+\infty} a_n x^n$ and $\sum_{n=1}^{+\infty} a_{-n} x^{-n}$ converge, so that the Laurent series $F(X)$ converges at the point $x \in \mathcal{A}_{(r_1,r_2)}$.

For any $x \in \mathcal{A}^c_{[r_1,r_2]}$ and an arbitrary positive integer N, there exist another positive integer N_1 such that $N_1 > N$ and either $|a_N x_N| > 1$ or $a_{-N}x^{-N} > 1$, so that the Laurent series does not converge at the point x in the positive or negative direction. □

Remark 2.3. Note that the non-archimedean property also converts the convergence condition in the sense: a series $\sum_{i=0}^{n}$ converges if and only if

$$\lim_{n\to+\infty}|a_n| = 0.$$

In terms of valuation, it is equivalent to

$$\lim_{n\to+\infty}\nu(a_n) = +\infty.$$

In addition, we can transform the radii of convergence as well.

$$|x| > \limsup_{n\to+\infty}|a_{-n}|^{1/n} \Leftrightarrow \nu(x) < \liminf_{n\to+\infty}\frac{\nu(a_{-n})}{n}$$

$$|x| < \frac{1}{\limsup_{n\to+\infty}|a_n|^{1/n}} \Leftrightarrow \nu(x) > -\liminf_{n\to+\infty}\frac{\nu(a_n)}{n}.$$

Given real numbers $0 < \delta < \varepsilon$, let denote the set of all $[\delta,\varepsilon]$-convergent Laurent series with coefficients in K by $\mathcal{A}_{[\delta,\varepsilon]}$, in which we equip component-wise addition and multiplication as follow

$$\left(\sum_{n\in\mathbb{Z}} a_n X^n\right)\left(\sum_{n\in\mathbb{Z}} b_n X^n\right) = \sum_{n\in\mathbb{Z}}\left(\sum_{k+l=n} a_k b_l\right) X^n$$

Lemma 2.4. *$\mathcal{A}_{[\delta,\varepsilon]}$ is an integral domain with respect to the addition and multiplication.*

Proof. The component-wise sum of any two elements in $\mathcal{A}_{[\delta,\varepsilon]}$ is convergent in $[\delta,\varepsilon]$ because of radii of convergence and non-archimedean property. Thereby, $\mathcal{A}_{[\delta,\varepsilon]}$ is a commutative group with respect to the addition.

Let $A(X) = \sum_{n\in\mathbb{Z}} a_n X^n$ and $B(X) = \sum_{n\in\mathbb{Z}} b_n X^n$ be two arbitrary Laurent series converging in $[\delta,\varepsilon]$. The multiplication defined as above prominently satisfies the law of associativity, two-side distribution and the unity elements $1 \in K$. The rest is to prove the series $\sum_{n\in\mathbb{Z}}\left(\sum_{k+l=n} a_k b_l\right) X^n$ converges in $[\delta,\varepsilon]$. Let $x \in K$ such that $|x| \in [\delta,\varepsilon]$, then $A(x)$ and $B(x)$ converge, which is equivalent to

$$\lim_{n\to+\infty}|a_n x^n| = \lim_{n\to+\infty}|b_n x^n| = \lim_{n\to+\infty}|a_{-n}x^{-n}| = \lim_{n\to+\infty}|b_{-n}x^{-n}| = 0$$

Hence, the set $\{|a_n x^n|, |b_n x^n|;\ n \in \mathbb{Z}\}$ is bounded. Therefrom, we obtain that

$$\lim_{n\to+\infty}|a_k b_l x^{n=k+l}| = \lim_{n\to-\infty}|a_k b_l x^{n=k+l}| = 0$$

That provide the convergence of the product series. $\square$

Take $\delta \leq \delta' \leq \varepsilon' \leq \varepsilon$, then

$$\mathcal{A}_{[\delta,\varepsilon]} \subset \mathcal{A}_{[\delta',\varepsilon']}$$

In particular,

$$\mathcal{A}_{[\delta,\varepsilon]} = \bigcap_{\rho\in[\delta,\varepsilon]} \mathcal{A}_{[\rho,\rho]}$$

For simplicity, we use $\mathcal{A}_\rho$ instead of $\mathcal{A}_{[\rho,\rho]}$. Moreover, for any $\rho \in [\delta, \varepsilon]$, $\mathcal{A}_{[\delta,\varepsilon]}$ can be viewed as a sub-space or sub-ring of the K-vector space as well as ring $\mathcal{A}_\rho$.

Definition 2.5. We define a function $\|\cdot\|_\rho : \mathcal{A}_\rho \to \mathbb{R}_{\geq 0}$ by

$$|\sum_{n\in\mathbb{Z}} a_n X^n\|_\rho = \max_{n\in\mathbb{Z}} |a_n|\rho^n.$$

This function is called a ρ-Gauss norm.

Proposition 2.6. *The ρ-Gauss norm is a well-defined norm of the vector space $\mathcal{A}_\rho$. The norm defines a complete metric on $\mathcal{A}_\rho$.*

Proof. It is clear the reflexive property and multiplication by scalar are satisfied. The triangle inequality is an instant result of the non-archimedean property. Moreover, the metric space induced by the ρ-Gauss norm on $\mathcal{A}_\rho$ is an ultra-metric space as well.

For the multiplicity, let $k, l \in \mathbb{Z}$ be the smallest indices such that $\|\sum_{n\in\mathbb{Z}} a_n X^n\|_\rho = |a_k|\rho^k$ and $\|\sum_{n\in\mathbb{Z}} b_n X^n\|_\rho = |b_l|\rho^l$. Thus,

$$\nu(a_k b_l) + (k+l)\rho = \min\{\nu(a_m b_n) + (m+n)\rho\}$$

In addition, for all $m, n \in \mathbb{Z}$ such that $m + n = k + l$, it happens that either $m < k$ or $n < l$, which implies that $\nu(a_m + b_n) > \nu(a_k b_l)$, according to property 1.3, the absolute value of $\sum_{n+m=k+l} a_m b_n$ is equal to the absolute value of $a_k b_l$. Therefore, we obtain that

$$\Big\|\sum_{n\in\mathbb{Z}} a_n X^n\Big\|_\rho \Big\|\sum_{n\in\mathbb{Z}} b_n X^n\Big\|_\rho = |a_k b_l|\rho^{k+l} = \left\|\left(\sum_{n\in\mathbb{Z}} a_n X^n\right)\left(\sum_{n\in\mathbb{Z}} b_n X^n\right)\right\|_\rho.$$

For the completion, let $\{F_n(X)\}_{n\in\mathbb{N}}$ be a Cauchey sequence in $\mathcal{A}_\rho$. We write the coefficients of each F_n

$$F_n(X) = \sum_{i\in\mathbb{Z}} a_i^{(n)} X^i$$

Thus, for any $n \in \mathbb{N}$, $|a_i^{(n)} - a_i^{(m)}| \leq \|F_n - F_m\|$, which implies all the sequences $(a_i^{(n)})_{n\in\mathbb{N}}$ are Cauchy sequences. Whence $F_n \to \sum_{i\in\mathbb{Z}} \lim_{n\to+\infty} a_i^{(n)} X^i \in \mathcal{A}_\rho$ because of the completeness of K. $\square$

Theorem 2.7. ***(The maximal principal)*** *Let $0 \neq F(X) \in \mathcal{A}_\rho$, then*

$$\|F(X)\|_\rho = \max\{\|F(X)\|;\ x \in \overline{K};\ |x| = \rho\}$$

Proof. Let $\|F(X)\|_\rho = a \neq 0$, and let y be an arbitrary element of $\overline{K}$ such that $|y| = \rho$. Fix $F(X) = \sum_{n\in\mathbb{Z}} a_n X^n$, then we consider a new Laurent series defined by

$$G(X) = \sum_{n\in\mathbb{Z}} a^{-1} a_n y^n X^n$$

That has properties: $G(X) \in \mathcal{A}_1$ and $\|G(X)\|_1 = 1$. Thereby, it suffices to prove that $\max\{G(x);\ x \in \overline{K};\ |x| = 1\} = 1$. In particular, $\|G(X)\|_1 = 1$ means there is at least one coefficient is a unit, by embedding the Laurent series to the residue field $\overline{\mathbf{k}}$, we obtain a Laurent polynomial of finite degree. Let

$$G(X) \equiv P(X) \mod \langle \mathcal{O}_{\overline{K}} \rangle$$

Hence, take x is not a root of $P(X)$, then $G(X)$ is a unit, then $\max\{G(x);\ x \in \overline{K};\ |x| = 1\} \geq 1$. In addition, $\|G(X)\|_1 = 1$, that means for any unit x, $|a_n x^n| \leq 1$, then by the non-archimedean property, $\max\{G(x);\ x \in \overline{K};\ |x| = 1\} = 1$. □

2.2 Newton polygons of Laurent series

One is familiar with Newton polygons of polynomials, which provides a geometric interpretation of a polynomial in terms of its indices and valuation. In general, applying Newton polygon method on Laurent series also results surprising properties similar to polynomials, see details in [8], [9] and [10]. Our goal is to extend it to Laurent series, in fact, there is a strong bond between the Gauss norms and Newton polygons, which is a generalization of polynomials as well as power series. Therefore, in this section, we shall state Newton polygons in the language of Gauss norms.

Definition 2.8. The Newton polygon of a Laurent series $\sum_{n\in\mathbb{Z}} a_n X^n$ is the boundary of the lower convex hull of the set

$$\{(n, \nu(a_n))\,;\ n \in \mathbb{Z};\ a_n \neq 0\}.$$

That consists of at most infinitely many segments of distinct slopes, the joint of two consecutive segments is called a break. The part of segments with slopes retaining in $[\log(\delta), \log(\varepsilon)]$ is called the Newton polygon with respect to the segment $[\delta, \varepsilon]$.

Definition 2.9. Let $F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{[\delta,\varepsilon]}$, and the real number $\rho \in [\delta, \varepsilon]$ is called a critical radius of $F(X)$ if there are two distinct indices $n_1 < n_2$ such that

$$\|F(X)\|_\rho = |a_{n_1}|\rho^{n_1} = |a_{n_2}|\rho^{n_2}.$$

Lemma 2.10. *F has at most finitely many critical radii in $[\delta, \varepsilon]$*

Proof. Let k and l be indices such that

$$|a_{-k}|\delta^{-k} = \max_{n\leq 0}|a_n|\delta^n \quad \text{and} \quad |a_l|\varepsilon^l = \max_{n\geq 0}|a_n|\varepsilon^n$$

For any $\rho\,(\delta, \varepsilon)$ and $n > l$, then $|a_n|\varepsilon^n \leq |a_l|\varepsilon^l \Rightarrow |a_n|\rho^n < |a_l|\rho^l$. Similarly, we obtain $|a_n|\rho^n < |a_{-k}|\rho^{-k}\ \forall n \leq -k$. That implies that for any critical radius $\rho \in (\delta, \varepsilon)$, there is a pair of indices $-k \leq n_1 < n_2 \leq l$ such that, $\rho = \left(\frac{|a_{n_1}|}{|a_{n_2}|}\right)^{\frac{1}{n_2 - n_1}}$. Hence, the number of critical radii is bounded. □

Fix $F(X)$ and let $\delta = \rho_0 < \rho_1 < \ldots < \rho_m < \rho_{k+1} = \varepsilon$ be the sequence of all critical radii of $F(X)$, and let k and l be the indices defined as in the previous proof.

Lemma 2.11. *For any $i \in \{0, \ldots, k\}$, there is exactly an index n_i such that*

$$\|F(X)\|_\rho = |a_i|\rho^i\ \forall \rho \in (\rho_i, \rho_{i+1})\,.$$

Proof. Let $P_N = \{\rho \in (\rho_i, \rho_{i+1})\};\ |a_N|\rho^N = \max_{n\in\mathbb{Z}}\{|a_n|\rho^n\}$. Because there exist such indices k and l that

$$|a_n|\rho_n < \max_{-k\leq m\leq l}\{|a_m|\rho^m\}\ \forall n \notin \{-k, \ldots, l\}$$

Thus, for all $n < -k$ or $n > l$, $P_n = \emptyset$. In addition, for any n such that $P_n \neq \emptyset$

$$P_n = \{\rho \in (\rho_i, \rho_{i+1})\};\ |a_N|\rho^N = \max_{-k\leq m\leq l}\{|a_n|\rho^n\}$$

Therefore, P_n's are open and distinct because ρ is not a critical radius. The fact that the union of all the sets P_n's is the open interval (ρ_i, ρ_{i+1}) shows that there is only an index $n_i \in \{-k, \ldots, l\}$ such that $P_{n_i} = (\rho_i, \rho_{i+1})\,.$ □

Let observe the function $\|F\|_\rho$ with ρ run all over the closed interval $[\delta, \varepsilon]$. In terms of valuation, we can convert it to the form

$$\begin{aligned}\mathbf{W}_F : [\log(\delta), \log(\varepsilon)] &\to \mathbb{R}\\ t &\mapsto \log\left(\|F\|_{e^t}\right) = \max_{n\in\mathbb{Z}}\{nt - \nu(a_n)\}\end{aligned}$$

That is a convex and piece-wise affine-linear functions (i.e, covered by finitely many affine function of the form $ax + b$). In other hand, let $(i, \nu(a_i))$ and $(j, \nu(a_j))$ be two consecutive break of the Newton polygons, that means for all $u \neq i$,

$$\nu(a_u) \geq \nu(a_i) + (u - i)\frac{\nu(a_j) - \nu(a_i)}{j - i} \Leftrightarrow i\frac{\nu(a_j) - \nu(a_i)}{j - i} - \nu(a_i) \geq u\frac{\nu(a_j) - \nu(a_i)}{j - i} - \nu(a_u)$$

Then $(i, \nu(a_i))$ is a break if and only if there exists a $t \in [\log(\delta), \log(\varepsilon)]$ such that $\mathbf{W}_F(t) = ni - \nu(a_i)$. That means the Newton polygon with respect to the interval $[\log(\delta), \log(\varepsilon)]$ has slopes is the breaks of $\mathbf{W}_F(t)$, those are $\{\log(\rho_i);\ i \in \{1, \dots, k\}$, and we define

$$\mathbf{n}(F, \rho) = \min\{n; \|F\|_\rho = |a_n|\rho^n\}$$

$$\mathbf{N}(F, \rho) = \max\{n; \|F\|_\rho = |a_n|\rho^n\}$$

Then the breaks of the Newton polygon is $\{\mathbf{n}(F, \rho); \rho \text{ is a critical radius}\}$, and we have $\mathbf{N}(F, \rho_i) = \mathbf{n}(F, \rho_{i+1})$. In conclusion, we have the following theorem about the Newton polygon.

Theorem 2.12. *Let $F(X) \in \mathcal{A}_{[\delta,\varepsilon]}$, then consider its Newton Polygon with respect to the interval $[\delta, \varepsilon]$. Let $\{\rho_1, \dots, \rho_k\}$ be the set of all critical radii, then there are exactly k interval with slopes in the interval $(\log(\delta), \log(\varepsilon))$ at the breaks $\mathbf{n}(F, \rho_i)$. In particular, those slopes are $\log(\rho_i) = \dfrac{\nu(a_{\mathbf{N}(F,\rho_i)}) - \nu(a_{\mathbf{n}(F,\rho_i)})}{\mathbf{N}(F, \rho_i) - \mathbf{n}(F, \rho_i)}$'s.*

Theorem 2.13. *Let $F(X)$ and $G(X) \in \mathcal{A}_{[\delta,\varepsilon]}$, then the Newton polygon of FG has the set of slopes consists all slopes of F and G.*

Proof. It suffices to prove that $\mathbf{n}(FG, \rho) = \mathbf{n}(F, \rho) + \mathbf{n}(G, \rho)$ and $\mathbf{N}(FG, \rho) = \mathbf{N}(F, \rho) + \mathbf{N}(G, \rho)$, that can be easily check by the non-archimedean property. $\square$

Lemma 2.14. *$0 \neq F(X) \in \mathcal{A}_{[\delta,\varepsilon]}$ is a unit if and only of $\mathbf{n}(F, \delta) = \mathbf{N}(F, \varepsilon)$.*

Proof. If $F(X)$ is a unit then

$$\mathbf{n}(F, \delta) + \mathbf{n}(F^{-1}, \delta) = \mathbf{n}(1, \delta) = 0 = \mathbf{N}(1, \varepsilon) = \mathbf{N}(F, \varepsilon) + \mathbf{N}(F^{-1}, \varepsilon)$$

In other hand, $\mathbf{n}(F, \delta) < \mathbf{N}(F, \varepsilon)$ because of the convexity of $\mathbf{W}_F(t)$. Hence, $\mathbf{n}(F, \delta) = \mathbf{N}(F, \varepsilon)$.

In converse, let $m = \mathbf{n}(F, \delta) = \mathbf{N}(F, \varepsilon)$, then $F(X) = aX^m(1_G(X))$, where $a \neq 0$ and in $K^\times$ and $F(X)$ has no critical radius in $[\delta, \varepsilon]$, then $\max\{\|G\|_\rho;\ \rho \in [\delta, \varepsilon]\} = b < 1$, hence, the series

$$\sum_{i=0}^{+\infty} G(X)^i$$

converges, which is inverse of $1 - G(X)$. Therefore, $F(X)$ is invertible. $\square$

2.3 Factorization of Laurent series

The Weierstrass preparation theorem is an important tool to determine the valuation of roots of a certain power series with coefficients in a discrete valuation ring, see [8] and [10]. Thereby, we divide a Laurent series into 'positive' part, which is of all factors with positive indices, and 'negative' part, which is the rest to apply the Weirestrass preparation theorem. Besides, we construct the division with remainder over the integral domain $\mathcal{A}_{[\delta,\varepsilon]}$ under special condition.

Definition 2.15. A polynomial $P(X) = a_0 + \ldots + a_n X^n \in K[X]$ is called pure if its Newton polygon has only one slope, equivalently, $\mathbf{n}(P,\rho) = 0$ and $\mathbf{N}(P,\rho) = \deg(P)$ for some ρ.

Definition 2.16. A polynomial $P(X) = a_0 + \ldots + a_n X^n \in K[X]$ is ρ-dominant or ρ-extreme if $\mathbf{N}(P,\rho) = n$ or $\mathbf{N}(P,\rho) = n$ and $\mathbf{n}(P,\rho) = 0$.

Lemma 2.17. *$P(X) = a_0 + \ldots + a_n X^n \in K[X]$ is ρ-dominant or ρ-extreme if and only if the every root of $F(X)$ has absolute value at most ρ or exactly ρ, respectively.*

Proof. We factorize the polynomial $P(X) = a_n(X - \alpha_1)\ldots(X - \alpha_n)$, where $\alpha_i \in \overline{K}$. Then $\mathbf{N}(P,\rho) = \sum_{i=1}^{n} \mathbf{N}(X - \alpha_i, \rho)$. Therefore,

$$\mathbf{N}(F,\rho) = n \Leftrightarrow \mathbf{N}(X - \alpha_i, \rho) = 1 \ \forall i \in \{1, \ldots, n\}$$

That imples all $|\alpha_i| \leq \rho$. □

We present the two theorems to provide a perception about the division with remainder over $\mathcal{A}_{[\delta,\varepsilon]}$. It can be considered as a generalization of Washington's theorem, see [9].

Theorem 2.18. ***(Dividing a power series by a polynomial)*** *Let $F(X) = \sum_{n=0}^{+\infty} a_n X^n \in \mathcal{A}_\rho$ and $P(X) = b_0 + \ldots + b_n X^n \in K[X]$ is a ρ-dominant. Then there exist uniquely a power series $G(X) \in \mathcal{A}_\rho$ and a polynomial $R(X) \in K[X]$ of degree at most $n-1$ such that*

$$F(X) = P(X)G(X) + R(X)$$

In addition

$$\|F\|_\rho = \max\{\|PG\|_\rho, \|R\|_\rho\}$$

Proof. For uniqueness, let assume there are such power series and polynomials $Q_1 \neq Q_2$ and $R_1 \neq R_2$ with $Q_1 P + R_1 = Q_2 P + R_2 \Leftrightarrow p(Q_1 - Q_2) = R_1 - R_2$. Therefrom, we evaluate the largest indices with the largest absolute value of both side.

$$n - 1 \geq \mathbf{N}(R_1 - R_2, \rho) = \mathbf{N}(P(Q_1 - Q_2), \rho) = \mathbf{N}(P,\rho) + \mathbf{N}(Q_1 - Q_2, \rho) \geq n$$

That leads to a contradiction.

Once we have the form $F = PQ + R$, if $\|R\|_\rho \geq \|PQ\|_\rho$ then

$$\|F\|_\rho = \|PG\|_\rho = \max\{\|PG\|_\rho, \|R\|_\rho\}$$

because $\mathbf{N}(PQ, \rho) = \mathbf{N}(P, \rho) + \mathbf{N}(Q, \rho) \geq n > \deg(R)$, and by the non-archimedean property shows that for all $d < n$, the coefficient of degree d of both PQ and R has absolute value at most $\|PQ\|_\rho$. If $\|R\|_\rho > \|PQ\|_\rho$, then according to property 1.3, we achieve what we need.

The remaining part is existence, which can be prove by dividing each term $a_m X^m$ by the polynomial P to obtain the quotient polynomials Q_m and the remainder polynomials R_m of degree at most $n-1$. In particular, $|a_m|\rho^m = \max\{\|PQ_m\|_\rho, \|R_m\|_\rho\}$ holds for all m. The completeness of $\mathcal{A}_\rho$ with respect to $\|\cdot\|_\rho$ ensures the existence of

$$Q = \sum_{m=0}^{+\infty} Q_m \quad \text{and} \quad R = \sum_{m=0}^{+\infty} R_m.$$

$\square$

Theorem 2.19. ***(Dividing a Laurent series by a polynomial)*** *Let $F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{[\delta,\varepsilon]}$, $\rho \in [\delta, \varepsilon]$ and $P(X) = b_0 + \ldots + b_n X^n \in K[X]$ is a ρ-extreme. Then there exist uniquely a Laurent series $G(X) \in \mathcal{A}_{[\delta,\varepsilon]}$ and a polynomial $R(X) \in K[X]$ of degree at most $n-1$ such that*

$$F(X) = P(X)G(X) + R(X)$$

In addition

$$\|F\|_\rho = \max\{\|PG\|_\rho, \|R\|_\rho\}$$

Proof. The uniqueness is proved in the same way above using the inequality

$$\deg(R_1 - R_2) \geq \mathbf{N}(R_1 - R_2, \rho) - \mathbf{n}(R_1 - R_2, \rho)$$

The property $\|F\|_\rho = \max\{\|PG\|_\rho, \|R\|_\rho\}$ is obtained by proving $\mathbf{N}(PQ, \rho) - \mathbf{n}(PQ, \rho) \geq n$, which can be easily checked.

We divide $F(X) = F^+(X) + F^-(X)$, where $F^+(X) = \sum_{i\geq 0} a_i X^i$ and $F^-(X) = \sum_{i<0} a_i X^i$. For every $\mu \in [\rho, \varepsilon]$, we apply the theorem 2.18 to $F^+(X)$ with $F^+(X) \in \mathcal{A}_{[\delta,\varepsilon]} \subset \mathcal{A}_\mu$ and the μ-dominant polynomial $P(X)$ to obtain

$$F^+(X) = P(X)Q_1(X) + R_1(X)$$

where $Q_1(X)$ is a power series in $\mathcal{A}_{[\rho,\varepsilon]}$. In addition, $F^=, R_1, P \in \mathcal{A}_{[\delta,\varepsilon]}$ implies that

$Q_1 \in \mathcal{A}_{[\delta,\varepsilon]}$ as well.

Let define

$$F_0^-(X) = X^{n-1}F^-(X^-1) \text{ be a power series in } \mathcal{A}_{[\varepsilon^{-1},\delta^{-1}]}$$

and

$$P'(X) = X^n P(X^{-1}) \text{ be a } \rho^{-1}\text{-extreme of degree } n$$

We similarly show that $F_0^-(X) = P'(X)Q'(X) + R_2'(X)$, where $Q'(X)$ is a power series in $\mathcal{A}_{[\varepsilon^{-1},\delta^{-1}]}$, and $R_2'(X)$ is a polynomial of degree at most $n-1$. Hence

$$X^{n-1}F^-(X^-1) = X^n P(X^{-1})Q_2'(X) + R_2'(X)$$

Substituting X by X^{-1}, we obtain

$$F^-(X) = X^{-1}Q_2'(X^-1) + X^{n-1}R_2'(X^{-1})$$

Equivalently, there are a Laurent series $Q_2(X)$ with all non-negative coefficients omitted a polynomial $R_2(X)$ of degree at most $n-1$ such that

$$F^-(X) = P(X)Q_2(X) + R_2(X)$$

Putting two parts of $F(X)$ provides as we need. □

Theorem 2.20. ***(A generalization of Weierstrass theorem)*** *Let* $0 \neq F(X) = \sum_{i\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{[\delta,\varepsilon]}$ *with* $0 < \delta\varepsilon$*, then there are uniquely determined polynomial* $P(X) \in K(X)$ *of degree* $\mathbf{N}(F,\varepsilon) - \mathbf{n}(F,\delta)$ *and a unit Lauren series* $U(X)\mathcal{A}_{[\delta,\varepsilon]}^{\times}$ *such that* $F(X) = P(X)U(X)$*. In addiction, all roots of* $P(X)$ *has absolute value in the interval* $[\delta,\varepsilon]$*.*

Proof. For the uniqueness, we indirectly assume that there are polynomials $P(X)$ and $P'(X)$, units $Q(X)$ and $Q'(X)$ satisfying $P(X)Q(X) = P'(X)Q'(X)$, then it is easily claimed, the polynomials $P(X)$ and $P'(X)$ has the same degree and roots over $\overline{K}$. That proves as we need. If $F(X)$ is a unit, then theorem is claimed because of lemma 2.14. Let ρ is a critical radius and $1 \leq d = \mathbf{N}(F,\rho) - \mathbf{n}(F,\rho)$ and define

$$P_1(X) = X^{-\mathbf{n}(F,\rho)} \sum_{\mathbf{n}(F,\rho)}^{\mathbf{N}(F,\rho)} a_n X^n$$

This is a ρ-extreme polynomial of degree d and $\|F - P_1\|_\rho < \|F\|_\rho$. Making use of theorem 2.18, there exist a uniquely determined polynomial $R_1(X)$ of degree at most $d-1$ and a Laurent series $Q_2(X) \in \mathcal{A}_{[\delta,\varepsilon]}$, such that

$$F(X) = P_1(X)Q_1(X) + R_1(X)$$

Therefrom, we recursively construct sequences of polynomials and Laurent series in the manner

$$P_{m+1}(X) = P_m(x) + R_m(X) \text{ and } F(X) = P_{m+1}(X)Q_{m+1}(X) + R_{m+1}(X).$$

In order to make this process works, we need to prove that $P_m(X)$ is a ρ-extreme polynomial for all m. We inductively prove it by

$$\|F - P_m\|_\rho = \max\{\|P_m\|_\rho\|Q_m - 1\|_\rho, \|R_m\|_\rho\}$$

and

$$\|F - P_{m+1}\|_\rho \leq \max\{\|F - P_m\|_\rho, \|R_m\|_\rho\} = \|F - P_m\|_\rho$$

Hence, all the polynomial P_m's are ρ-extreme. Moreover, $\|F - P_m\|_\rho < \|F\|_\rho \; \forall m$, which leads to

$$\|Q_m - 1\|_\rho\|F\|_\rho = \|Q_m - 1\|_\rho\|P_m\|_\rho \leq \|F - P_m\|_\rho$$

Then for all m, the inequality $\|Q_m - 1\|_\rho \leq \dfrac{\|F - P_m\|_\rho}{\|F\|_\rho} = c < 1$ holds. We apply the inequality of theorem 2.18

$$\begin{aligned}
F(X) &= P_m(X)Q_m(X) + R_m(X) = P_{m+1}(X)Q_{m+1}(X) + R_{m+1}(X) \\
&\Rightarrow R_m(X)(Q_{m+1}(X) - 1) = -P_m(X)(Q_{m+1}(X) - Q_m(X)) - R_{m+1}(X) \\
&\Rightarrow \|R_m\|_\rho \cdot c \geq \|P_m(Q_{m+1} - Q_m) + R_{m+1}\|_\rho \\
&\Rightarrow \|R_m\|_\rho \cdot c \geq \max\{\|P_m(Q_{m+1} - Q_m)\|_\rho, \|R_{m+1}\|_\rho\} \\
&\Rightarrow \|R_m\|_\rho \cdot c \geq \|R_{m+1}\|_\rho
\end{aligned}$$

The last inequality shows that $R_m \to 0$ as $m \to 0$. Consequently, the non-archimedean property implies the convergence of $\{P_m\}_{m\geq 0}$, let the limit polynomial is $P^*(X)$, which is inherited the properties of P_m's, namely, $P(X)$ is a ρ-extreme polynomial of degree n with coefficients in K and there exists uniquely determined Laurent series $Q^*(X) \in \mathcal{A}_{[\delta,\varepsilon]}$, such that $F(X) = P^*(X)Q^*(X)$.

We repeat the process on all finitely many critical radii of $F(X)$ to achieve a polynomial $P(X)$ of degree $\mathbf{N}(F, \varepsilon) - \mathbf{n}(F, \delta)$ and a Laurent series $Q(X) \in \mathcal{A}_{[\delta,\varepsilon]}$ satisfying

$$F(X) = P(X)Q(X)$$

In addition, $\mathbf{N}(Q, \varepsilon) = \mathbf{n}(Q, \delta)$ is obtained means that $Q(X)$ is a unit. □

The theorem leads us directly to essential corollaries as follows.

Corollary 2.21. *Every root of a Laurent series is algebraic.*

Corollary 2.22. *If $\delta = 0$, we substitute Laurent series by power series, the same result is acquired.*

Corollary 2.23. *When we substitute K by its discrete valuation ring R, theorem 2.20 and the corollary 2.22 remain true.*

Remark 2.24. We can express the set of all zeros of a Laurent series in terms of Newton polygon. For a given Laurent series over a closed interval, all zeros can be divided into sets of zeros with the same valuation. In particular, a line segment of Newton polygon has its width projecting to x-axis representing the number of zeros with the valuation of inverse slope of the line segment.

Definition 2.25. The polynomial $P(X)$ of $F(X)$ as above is called the distinguished polynomial of the Laurent series, denoted by $\mathscr{C}(F)$.

Corollary 2.26. ***(Division of two Laurent series)*** *Let $F(X)$ and $G(X)$ be two power series in $\mathcal{A}_{[\delta,\varepsilon]}$, then there are a uniquely determined polynomial $R(X)$ of degree less than* $\deg(\mathscr{C}(G))$ *and a Laurent series $Q(X) \in \mathcal{A}_{[\delta,\varepsilon]}$ such that*

$$F(X) = G(X)Q(X) + R(X).$$

Theorem 2.27. *The integral domain $\mathcal{A}_{[\delta,\varepsilon]}$ is a principal ideal domain (P.I.D).*

Proof. Notice that in $\mathcal{A}_{[\delta,\varepsilon]}$, the ideal generated by a Laurent series is generated by its distinguished polynomial, which induces a bijection

$$\begin{aligned} \text{The set of all ideals in } \mathcal{A}_{[\delta,\varepsilon]} &\to \text{The set of all ideals in } K[X] \\ I &\mapsto I \cap K[X] \end{aligned}$$

This bijection preserves inclusion and union, and since $K[X]$ is a P.I.D, then so is $\mathcal{A}_{[\delta,\varepsilon]}$. In details,

$$\mathcal{A}_{[\delta,\varepsilon]} = K[X] \cdot \mathcal{A}^{\times}_{[\delta,\varepsilon]}.$$

□

Definition 2.28. We defined the Gauss-norm on $\mathcal{A}_{[\delta,\varepsilon]}$

$$\begin{aligned} \|\cdot\|_{\delta,\varepsilon} : &\mathcal{A}_{[\delta,\varepsilon]} \to \mathbb{R}_{\geq 0} \\ &F(X) \mapsto \max\{\|F\|_\rho;\ \delta \leq \rho \leq \varepsilon\} \end{aligned}$$

Proposition 2.29. *The ring $\mathcal{A}_{[\delta,\varepsilon]}$ is a complete K-algebra with respect to the Gauss norm.*

Proof. The Gauss norm is a well-defined norm because it satisfies all the laws of a norm over a vector space conveying from ρ-Gauss norm. The rest to to prove the inequality

$$\|FG\|_{\delta,\varepsilon} \leq \|F\|_{\delta,\varepsilon}\|G\|_{\delta,\varepsilon} \;\forall F(X), G(X) \in \mathcal{A}_{[\delta,\varepsilon]}.$$

Notice that

$$\begin{aligned}
&\max\{\|F\|_\rho;\; \delta \leq \rho \leq \varepsilon\} = \max\{\|F\|_\delta, \|F\|_\delta\|F\|_\varepsilon\} \\
&\quad \Rightarrow \|FG\|_{\delta,\varepsilon} = \max\{\|FG\|_\delta, \|FG\|_\delta\|F\|_\varepsilon\} \leq \max\{\|F\|_\delta, \|F\|_\delta\|F\|_\varepsilon\}\max\{\|G\|_\delta, \|F\|_\delta\|G\|_\varepsilon\} \\
&\quad \Rightarrow \|FG\|_{\delta,\varepsilon} \leq \|F\|_{\delta,\varepsilon}\|G\|_{\delta,\varepsilon}
\end{aligned}$$

□

2.4 The Laurent series on semi-open and open annuli

On a closed interval, we learn the properties of a Laurent series by studying the endpoints of a interval. Let use the notation $\mathcal{A}_{[\delta,\varepsilon)}$ and $\mathcal{A}_{(\delta,\varepsilon)}$ to denote the sets of all Laurent series converging on $[\delta,\varepsilon)$ and (δ,ε) respectively. In terms of inclusion, we describe the rings

$$\mathcal{A}_{[\delta,\varepsilon)} = \bigcap_{\delta<\varepsilon'<\varepsilon} \mathcal{A}_{[\delta,\varepsilon']} = \bigcap_{i=1}^{+\infty} \mathcal{A}_{[\delta,\varepsilon_i]}$$

and

$$\mathcal{A}_{(\delta,\varepsilon)} = \bigcap_{\delta<\delta'<\varepsilon} \mathcal{A}_{[\delta',\varepsilon)} = \bigcap_{i=1}^{+\infty} \mathcal{A}_{[\delta_i,\varepsilon)}$$

Where $(\varepsilon_i)_{i\in\mathbb{N}} \subset (\delta,\varepsilon)$ converges to ε, and $(\delta_i)_{i\in\mathbb{N}} \subset (\delta,\varepsilon)$ converges to δ. For simplicity, let fix those two sequences.

Lemma 2.30. ***(Unit elements)*** *Let $F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{[\delta,\varepsilon)}$, then $F(X)$ is a unit if and only if there exists a index n_0 with the following properties*

$$|a_{n_0}|\delta^{n_0} < |a_n|\delta^n \quad \textit{and} \quad |a_{n_0}|\varepsilon^{n_0} \leq |a_n|\varepsilon^n \;\forall n \neq n_0$$

Proof. Notice that

$$F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{[\delta,\varepsilon)} \subset \mathcal{A}_{[\delta,\varepsilon']} \;\forall \delta < \varepsilon' < \varepsilon$$

then $F(X)$ is a unit of $\mathcal{A}_{[\delta,\varepsilon)}$ if and only if it is a unit of $\mathcal{A}_{[\delta,\varepsilon']}$ $\forall \delta < \varepsilon' < \varepsilon$. By lemma 2.14, that implies

$$n_0 = \mathbf{n}(F,\delta) = \mathbf{N}(F,\varepsilon') \;\forall \delta < \varepsilon' < \varepsilon$$

Let ε' tend to ε then we obtain what we need. □

Remark 2.31. Note that we have the norm $\|\cdot\|_{\delta,\varepsilon_n}$ on the ring $\mathcal{A}_{[\delta,\varepsilon_n]}$, then we define a topology on $\mathcal{A}_{[\delta,\varepsilon)}$ by taking the intersection of all topologies generated by the norms $\|\cdot\|_{\delta,\varepsilon_n}$'s. Consequently, the completeness is conveyed as well. In this section, we use this topology when referring to open and closed sets.

Corollary 2.32. *Let $F(X)=\sum_{n\in\mathbb{Z}} a_nX^n \in \mathcal{A}_{(\delta,\varepsilon]}$, then $F(X)$ is a unit if and only if there exists a index n_0 with the following properties*

$$|a_{n_0}|\delta^{n_0} \leq |a_n|\delta^n \quad \textit{and} \quad |a_{n_0}|\varepsilon^{n_0} < |a_n|\varepsilon^n \ \forall n \neq n_0$$

Proposition 2.33. *Let $\rho \in [\delta,\varepsilon)$ and equip the ρ-Gauss norm on $\mathcal{A}_{[\delta,\varepsilon)}$, then the followings are achieved.*

(**i**) *The convergence with respect to the ρ-Gauss norm implies the coefficient-wise convergence as well. The metric space induced by the ρ-Gauss norm is complete.*

(**ii**) *In the ring $\mathcal{A}_{[\delta,\varepsilon]}$, every ideal is closed with respect to the ρ-Gauss norm.*

(**iii**) *In the ring $\mathcal{A}_{[\delta,\varepsilon)}$, every principal ideal is closed with respect to the ρ-Gauss norm.*

Proof. The statement (**i**) is an instant corollary of the ρ-Gauss norm on $\mathcal{A}_{[\delta,\varepsilon]}$. For the latter, let I be an ideal generated by an element $P(X)$, then for any Cauchy sequence $\{PF_n\}_{n\in\mathbb{N}}$, then by the multiplicative property of the ρ-Gauss norm, the sequence $\{F_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence as well, let the limit Laurent series is F, then $PF \in I$ is the limit of $\{PF_n\}_{n\in\mathbb{N}}$. □

Lemma 2.34. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials in $K[X]$ with the properties*

- *$\{f_n\}_{n\in\mathbb{N}}$ is pair-wise relatively prime,*
- *for any $n \in \mathbb{N}$, all roots of f_n is of absolute value at most ε_n, where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a strictly increasing to ε.*

Then there exist a power series $F(X) \in \mathcal{A}_{[0,\varepsilon)}$ whose the set of all zeros consists of all roots of f_n.

Proof. Without loss of generality, we can assume that f_n's are ε_n-extreme with its constant coefficient 1. The main idea is to convert the sequence into another convergent sequence, in which we show several special properties to easily determine all zeros of the limit power series. The construction follows in the manner.

- Notice that $f_n(X)$ is a $\varepsilon_n < \varepsilon$-extreme polynomial, hence, there exists a power series $f_n^{-1}(X) = \sum_{i=0}^{+\infty} c_{n,i}X^i \in \mathcal{A}^{\times}_{[0,\varepsilon_n)}$ such that $f_n(X)f_n^{-1}(X) = 1$, then the constant coefficient of $f_n^{-1}(X)$ is 1 as well. In particular, $f_n^{-1}(X)$ has its convergence of radii are 0 and ε_n, that implies it diverges at all points of absolute value more than ε_n, thus, there exists an index m_n such that

$$m_n = \min\{k;\ |c_{n,k}|\varepsilon^k > 1\}.$$

then $\sum_{i=0}^{m_n-1} c_{n,i}X^i \in \mathcal{A}^{\times}_{[0,\varepsilon)}$, and we define

$$f'_n(X) = f_n(X) \cdot \sum_{i=0}^{m_n-1} c_{n,i}X^i.$$

then $f'_n(X)$ has the same roots as $f_n(X)$ over the interval $[0,\varepsilon)$.

- We bound the ρ-Gauss norm of $f'_n(X) - 1$ for $\rho \in [\delta, \varepsilon)$.

$$\|f'_n(X) - 1\|_\rho = \left\| f_n(X) \cdot \sum_{i=0}^{m_n-1} c_{n,i}X^i - f_n(X) \cdot \sum_{i=0}^{+\infty} c_{n,i}X^i \right\|_\rho = \|f_n(X)\|_\rho \left\| \sum_{m_n}^{+\infty} c_{n,i}X^i \right\|_\rho$$

Since it is clear that $\|f_n^{-1}\|_{\varepsilon_n} = 1$, then

$$\|f'_n(X) - 1\|_\rho \leq \left(\frac{\rho}{\varepsilon_n}\right)^{m_n}$$

for all $0 < \rho < \varepsilon_n$ since $\|f_n(X)\|_\rho = \|f_n(0)\|_\rho = 1$.

- Consider the sequence $(m_n)_{n\in\mathbb{N}}$, $\|f_n^{-1}\|_\rho = 1\ \forall 0 < \rho\varepsilon_n$ shows that $\nu(c_{n,k}) \geq k\log(\rho)\ \forall 0 < \rho < \varepsilon_n$, equivalently

$$\nu(c_{n,k}) \geq k\log(\varepsilon_n)\ \forall n$$

Let N be an arbitrary positive integer, then there exists a positive integer N_0 such that $\lfloor n\log(\varepsilon_n)\rfloor = \lfloor n\log(\varepsilon)\rfloor\ \forall n > N_0$, that means for all $k \leq N$,

$$\nu(c_{n,k}) \geq k\log(\varepsilon_n) \Rightarrow \nu(c_{n,k}) \geq k\log(\varepsilon) \Leftrightarrow |c_{n,k}|\varepsilon^k \leq 1$$

Therefore, $m_n > N$ for all $n > N_0$, equivalently, $\lim_{m\to\infty} m_n = +\infty$.

- Let define

$$g_n(X) = \prod_{i=1}^{n} f'_n(X)$$

whose roots in the interval $[0,\varepsilon]$ are the roots of $f_1(X), \ldots, f_n(X)$. Fix $\rho \in [\delta, \varepsilon)$, we have

$$\|g_{n+1} - g_n\|_\rho \leq \left(\frac{\rho}{\varepsilon_n}\right)^{m_n} \text{ and } \lim_{m\to\infty} m_n = +\infty$$

As a result, the $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the ρ-Gauss norm for all

$\rho \in [\delta, \varepsilon)$, which implies the coefficient-wise convergence as well. Let $F(X)$ be the limit power series of the sequence of polynomial g_n's, then g_n tends to $F(X)$ with respect to all $\|\cdot\|_{\delta,\varepsilon_n}$. Whence, the completeness of $\mathcal{A}_{[\delta,\varepsilon_n]}$ implies that $F(X) \in \bigcap_{n\in\mathbb{N}} \mathcal{A}_{[\delta,\varepsilon_n]} = \mathcal{A}_{[\delta,\varepsilon)}$. For any $n \in \mathbb{N}$, $g_n(X) \mid F(X)$. In particular, $\prod_{i=1}^{n} f_i(X)$ is the distinguished polynomial of power series $F(X)$ with respect to the closed interval $[\delta, \varepsilon_n]$. That proves $F(X)$ is the desired power series. $\square$

Corollary 2.35. *Let $(x_n)_{n\in\mathbb{N}}$ be the sequence of elements in $\overline{K}$, and let its valuation form a sequence $(\alpha_n)_{n\in\mathbb{N}}$, which is a strictly decreasing sequence of limit α. Then there exists a uniquely determined power series $F(X) \in K[[X]]$, that take all x_n's and its Galois conjugates as zeros. Moreover, every Laurent series with coefficients in K vanishing at all points x_n's is divisible by $F(X)$.*

Proof. Let f_n be the minimal polynomial of x_n over K and apply the above lemma. $\square$

Theorem 2.36. ***(Weierstrass preparation theorem)***
Let $F(X)$ be a Laurent series in $\mathcal{A}_{[\delta,\varepsilon)}$, then there exist a uniquely determined power series $P(X)$ with $\|P\|_\delta = \|P\|_\varepsilon = 1$ and a unit $U(X) \in \mathcal{A}_{[\delta,\varepsilon)}$ such that $F(X) = P(X)U(X)$. In terms of Newton polygon, all properties remain true except, the number of line segments is possibly infinite, and the limit of slopes is ε in that case.

Proof. The theorem is a direct corollary of what we have presented. $\square$

Theorem 2.37. *Every closed ideal in $\mathcal{A}_{[\delta,\varepsilon)}$ is a principal ideal.*

Proof. Let I be an arbitrary closed ideal in $\mathcal{A}_{[\delta,\varepsilon)}$. We denote the ideal generated by I in the ring $\mathcal{A}_{[\delta,\varepsilon_n]}$ by I_n. Then, we obtain

$$I \subset \ldots \subset I_n \subset \ldots \subset I_1$$

Because $\mathcal{A}_{[\delta,\varepsilon_n]}$ is a P.I.D, let $\{f_n\}_{n\in\mathbb{N}}$ be the set of generating polynomials of $\{I_n\}_{n\in\mathbb{N}}$ respectively. Whence, we obtain $f_{n-1} \mid f_n \ \forall n \in \mathbb{N}$. Making use of lemma 2.34 show that there exist a power series $F(X) \in \mathcal{A}_{[\delta,\varepsilon)}$, such that $F(X) = f_n(X)U_n(X)$, where $U_n(X)$ is a unit of $\mathcal{A}_{[\delta,\varepsilon_n]}$. The uniqueness of $F(X)$ also implies that every element of $\bigcap_{n\in\mathbb{N}} I_n \subset \bigcap_{n\in\mathbb{N}} \mathcal{A}_{[\delta,\varepsilon_n]} = \mathcal{A}_{[\delta,\varepsilon)}$ takes $F(X)$ as a divisor. That claims $\bigcap_{n\in\mathbb{N}} I_n$ is a principal ideal of $\mathcal{A}_{[\delta,\varepsilon)}$ generated by $F(X)$.

The above claim ensures that it suffices to prove $I = \bigcap_{n\in\mathbb{N}} I_n$, in which we already have $I \subset \bigcap_{n\in\mathbb{N}} I_n$ by definition. For any element $G(X) \in \bigcap_{n\in\mathbb{N}} I_n$. Since I generates I_n, then assume $G(X) = \sum_{i=1}^{k} T_i(X)H_i(X)$, where $T_i(X) \in I$ and $H_i(X) \in \mathcal{A}_{[\delta,\varepsilon_n]}$. For any $i \in \{1, \ldots, k\}$, we take an approximate partial sum of $H_i(X)$ in the manner

$$H_i(X) = \sum_{i\in\mathbb{Z}} c_n X^n$$

then there exists a positive integer M such that

$$\left\| T_i(X)H_i(X) - T_i(X) \sum_{i=-M}^{M} c_n X^n \right\|_{\delta,\varepsilon_n} < \frac{1}{n}$$

Furthermore, we also have $\sum_{i=-M}^{M} c_n X^n \in \mathcal{A}_{[\delta,\varepsilon)}$, thus, $T_i(X) \sum_{i=-M}^{M} c_n X^n \in I$. By the non-archimedean property, we can construct a Lauren series $G_n(X) \in I$ such that

$$\|G - G_n\|_{\delta,\varepsilon_n} < \frac{1}{n} \ \forall n \in \mathbb{N}$$

As a result of closeness of I, the ideal includes every element of $\bigcap_{n\in\mathbb{N}} I_n$, which provides what we need. □

Remark 2.38. We know that $\mathcal{A}_{[\delta,\varepsilon]}$ is a P.I.D, so it is natural to raise the same question about $\mathcal{A}_{[\delta,\varepsilon)}$. Unfortunately, it is not. The only thing we know is that every closed ideal is a principal ideal and, reversely, every principal ideal is closed. The counter-example below shall explain why $\mathcal{A}_{[\delta,\varepsilon)}$ is not a P.I.D.

In the proof of lemma 2.34, we take $F_n = \dfrac{F}{\prod_{i=1}^{n} f_n}$ and the ideal ring $I = \langle F_1, \ldots, F_n, \ldots \rangle$, then every element of I take all roots of f_m as zeros for every m large enough, which shows that $1 \notin I$. However, the sequence $\{F_n\}_{n\in\mathbb{N}}$ converges to 1, thus, the ideal I is not a principal ideal.

Lemma 2.39. *Let $\{g_n(X)\}_{n\in\mathbb{N}}$ be a sequence of polynomial with coefficient in K with the following properties*

- *For any $n \in \mathbb{N}$, $g_n(X)$ is a α_n-extreme polynomial*
- *$(\alpha_n)_{n\in\mathbb{N}} \subset [\delta, \varepsilon)$, and $\lim_{n\to+\infty} \alpha_n = \varepsilon$.*

Let $\{A_n(X)\}_{n\in\mathbb{N}}$ be a sequence of Laurent series such that $A_n(X) \in \mathcal{A}_{[\delta,\alpha_n]}$ $\forall n \in \mathbb{N}$ and

$$g_n(X) \mid A_m(X) - A_n(X) \ \forall m > n.$$

Then there exists a power series $G(X) \in \mathcal{A}_{[\delta,\varepsilon)}$ satisfying $\prod_{i=1}^{n} g_i(X) \mid G(X) - A_n(X)$ $\forall n \in \mathbb{N}$.

Proof. Let $f_n(X) = \prod_{i=1}^{n} g_i(X) \in K[X]$, and without loss of generality, we can assume that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is strictly increasing to ε. Consequently, for any $m > n$, $g_m(X) \in \mathcal{A}^{\times}_{[\delta,\alpha_n]}$. Now, we modify the sequence $\{A_n(X)\}_{n\in\mathbb{N}}$ to make it convergence to our desired power series.

Notice that by adding to $A_n(X)$ a multiple of $f_n(X)$ then the initial property remains true. Making use of dividing a Laurent series by a polynomial replaces all $A_n(X)$'s by polynomials $B_n(X)$'s. In detail, $A_n(X) = f_n(X)Q(X) + B_n(X)$. Then fixed an index n, we have

$$B_n(X) - B_{n-1}(X) = f_{n-1}(X)U(X)$$

for some polynomial $U(X) \in K[X]$. Let add $f_n(X)T(X)$ to $B_n(X)$ to obtain

$$B_n(X) + f_n(X)T(X) - B_{n-1}(X) = f_{n-1}(X)\left(U(X) + g_n(X)T(X)\right)$$

Since, $g_n(X) \in \mathcal{A}^{\times}_{[\delta,\alpha_{n-1}]}$, then $-U(X)g_n^{-1}(X)$ is a power series in $\mathcal{A}_{[\delta,\alpha_{n-1}]}$, and clearly $K[X]$ is a dense in the set of power series in $\mathcal{A}_{[\delta,\alpha_{n-1}]}$ with respect to the norm $\|\cdot\|_{\delta,\alpha_{n-1}}$, then it is possible to choose a polynomial $T(X)$ such that

$$\|B_n(X) + f_n(X)T(X) - B_{n-1}(X)\|_{\delta,\alpha_{n-1}} < \frac{1}{n-1}$$

Inductively processing on all polynomial $B_n(X)$'s to obtain a new sequence of polynomials $\{C_n(X)\}_{n\in\mathbb{N}} \subset K[X]$ with $f_n(X) \mid B_n(X) - C_n(X)$ $\forall n \in \mathbb{N}$ and most importantly,

$$\|C_{n+1} - C_n(X)\|_{\delta,\alpha_n} < \frac{1}{n}$$

Therefore, there exists the limit power series $G(X) \in \mathcal{A}_{[\delta,\varepsilon)}$ such that $g_n(X) \mid G(X) - A_n(X)$ $\forall n \in \mathbb{N}$. □

Corollary 2.40. *If we take the sequence of $f_n(X)$ of distinguished polynomials of a Laurent series $F(X)$, then we obtain $G(X) - A_n(X) \in \langle F\rangle \subset \mathcal{A}_{[\delta,\alpha_n]}$ $\forall n \in \mathbb{N}$.*

Theorem 2.41. *Every finite generated ideal of $\mathcal{A}_{[\delta,\varepsilon)}$ is a principal ideal.*

Proof. It suffices to prove in the case there are exactly two generator $F(X)$ and $G(X)$ because once we have the results in that case, the induction shall complete the proofs. Without loss of generality, we can assume that $F(X)$ and $G(X)$ are both power series according to the Weierstrass preparation theorem. By the above theorem, the closure of $\langle F, G\rangle$ is $\langle H\rangle$ with $H(X)$ is a power series in $\mathcal{A}_{[\delta,\varepsilon)}$. Therefore, $\langle F, G\rangle = \langle H\rangle$ in all $\mathcal{A}_{[\delta,\varepsilon_n]}$. As a result, for any $n \in \mathbb{N}$, there exist $A_n(X)$ and $B_n(X) \in \mathcal{A}_{[\delta,\varepsilon_n]}$ such that

$$H(X) = F(X)A_n(X) + G(X)B_n(X)$$

Let $F(X) = H(X)F'(X)$ and $G(X) = H(X)G'(X)$, then

$$1 = F'(X)A_n(X) + G'(X)B_n(X)$$

Hence, $A_m(X) - A_n(X) \in G'\mathcal{A}_{[\delta,\varepsilon_n]}$ $\forall m > n$. By lemma 2.39, we obtain a power series

$A(X) \in \mathcal{A}_{[\delta,\varepsilon)}$ such that

$$A(X) - A_n(X) \in G'\mathcal{A}_{[\delta,\varepsilon_n]} \Rightarrow 1 - F'(X)A(X) \in G'\mathcal{A}_{[\delta,\varepsilon)}$$

Equivalently, there exists some $B(X) \in \mathcal{A}_{[\delta,\varepsilon)}$ satisfying

$$1 = F'(X)A(X) + G'(X)B(X) \Leftrightarrow \langle H \rangle = \langle F, G \rangle.$$

□

Corollary 2.42. *The ring $\mathcal{A}_{[\delta,\varepsilon)}$ is a Bézout ring, i.e every finitely generated ideal is principal. In particular, $\mathcal{A}_{[\delta,\varepsilon)}$ is a G.C.D domains, i.e every two elements $F(X)$ and $G(X)$ has the greatest common divisor, which is a power series of all common zeros in $[\delta, \varepsilon)$.*

Definition 2.43. Let define a sub-ring of $\mathcal{A}_{[\delta,\varepsilon)}$

$$\mathcal{A}^{bd}_{[\delta,\varepsilon)} = \left\{ F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{[\delta,\varepsilon)};\ \exists n_0, |a_{n_0}|\varepsilon^{n_0} = \max_{n\in\mathbb{Z}} |a_n|\varepsilon^n \right\}$$

Theorem 2.44. *$\mathcal{A}^{bd}_{[\delta,\varepsilon)}$ is a principal ideal domain.*

Proof. It is clear that $\mathcal{A}^{bd}_{[\delta,\varepsilon)}$ is a ring, and every element $F(X)$ has finitely many critical radii, which implies its distinguished polynomial is determined. In other words,

$$\mathcal{A}^{bd}_{[\delta,\varepsilon)} = K[X] \left(\mathcal{A}^{b}_{[\delta,\varepsilon)} \right)^{\times}$$

The polynomial ring is a P.I.D, so is $\mathcal{A}^{bd}_{[\delta,\varepsilon)}$. □

3 The Robba ring

3.1 The ring structure of the Robba ring

In this section, we will discuss the properties of the Robba ring, which is a union of infinitely many Bézout rings

$$\mathcal{R} = \bigcup_{0<\delta<1} \mathcal{A}_{[\delta,1)}$$

It naturally questions what type of the Robba ring is what properties it possesses.

Definition 3.1. We define two main sub-rings of the Robba ring:

- The bounded Robba ring is the set of all bounded-coefficient element, defined as

$$\mathcal{R}^{bd} = \bigcup_{0<\delta<1} \mathcal{A}^{bd}_{[\delta,1)} = \left\{ F(X) = \sum_{i\in\mathbb{Z}} a_n X^n \in \mathcal{R};\ \{|a_n|\}_{n\in\mathbb{Z}} \text{ is bounded} \right\}$$

- The ring consists of all elements with coefficients in $\mathbb{R}$, defined as

$$\mathcal{R}^{int} = \left\{ F(X) = \sum_{i\in\mathbb{Z}} a_n X^n \in \mathcal{R};\ a_n \in R\ \forall n \in \mathbb{Z} \right\}$$

Note that

$$\mathcal{R}^{int} \subset \mathcal{R}^{bd} \subset \mathcal{R}$$

and each of those ring has surprising properties, that are presented as following theorems.

Theorem 3.2. *The ring $\mathcal{R}$ is a Bézout ring, i.e every finitely generated ideal is principal.*

Proof. Let I be an ideal of $\mathcal{R}$ generated by finitely many elements $F_1, \ldots, F_m$, therefore, there exists some $\delta \in (0,1)$ such that $F_1, \ldots, F_m \in \mathcal{A}_{[\delta,1)}$. Because $\mathcal{A}_{[\delta,1)}$ is a Bézout ring, then let $F(X) \in \mathcal{A}_{[\delta,1)}$ be the unique generator of $\langle F_1, \ldots, F_m \rangle \subset \mathcal{A}_{[\delta,1)}$. Let $Q_1, \ldots, Q_m$ be elements in $\mathcal{R}$, then there is some $\delta' \in (0,1)$, such that $Q_1, \ldots, Q_m \in \mathcal{A}_{[\delta',1)}$.

If $\delta' \leq \delta \Rightarrow \mathcal{A}_{[\delta',1)} \subset \mathcal{A}_{[\delta,1)}$, then we obtain

$$\sum_{i=1}^{m} F_i Q_i \in \langle F \rangle \subset \mathcal{A}_{[\delta,1)}$$

If $\delta \leq \delta' \Rightarrow \mathcal{A}_{[\delta,1)} \subset \mathcal{A}_{[\delta',1)}$, we can take F_i''s the distinguished polynomial of power series in $[\delta',1)$, which is a Bézout ring. Whence, there exists $F' \in \mathcal{A}_{[\delta',1)}$, and

$$\sum_{i=1}^{m} F_i Q_i \in \langle F' \rangle \subset \mathcal{A}_{[\delta',1)}$$

However, $F' \mid F$, then F is still the generator of the ideal $\langle F_1, \ldots, F_m \rangle$ in $\mathcal{R}$. $\square$

Corollary 3.3. *Let $F_1, \ldots, F_n \in \mathcal{R}$, and for any $i \in \{1, \ldots, n\}$, let $\{x_{i,k}\}_{k\in\mathbb{N}}$ be the set of all zeros of $F_i(X)$ in the interval $(0,1)$, then its greatest common divisor is a power series or a polynomial with the set of all zeros is*

$$\bigcap_{i=1}^{n} \{x_{i,k}\}_{k\in\mathbb{N}}.$$

Proof. It is directly from corollary 2.42. $\square$

In order to study the bounded Robba ring, we can look at the rings $\mathcal{A}^{bd}_{[\delta,1)}$,where $\delta \in (0,1)$. It is obvious that the 1-Gauss norm is a norms on $\mathcal{A}^{bd}_{[\delta,1)}$, therefore it induce a topology on $\mathcal{R}^{bd}$ as well.

Lemma 3.4. *Fix $\delta \in (0,1)$, these followings are equivalent.*

(**i**) $F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \in \mathcal{A}_{[\delta,1)}$.

(**ii**) *The Newton polygon has finitely many slopes.*

(**iii**) *The Weierstrass works on $F(X)$, and there exists its distinguished polynomial, that captures all roots of $F(X)$ in the interval $[\delta, 1)$.*

(**iv**) *There exist $\lim_{\rho\to 1} \|F\|_\rho$ and $\lim_{t\to 0} \mathbf{W}_F(t)$.*

Proof. The connection between the roots of $F(X)$ and its Newton polygons instantly shows the equivalence of (**ii**) and (**iii**). Moreover, if (**ii**) holds, then F(X) accepts the form

$$F(X) = P(X)U(X)$$

Where $P(X)$ is a polynomial and $U(X)$ is a unit, therefore $\mathbf{n}(F,1) = \mathbf{n}(P,1) + \mathbf{n}(U,1) < +\infty$, that implies (**i**). Conversely, if $F(X) \in \mathcal{A}_{[\delta,1)}$, then $F(X)$ has finitely many critical radii, equivalently, (**iii**) holds. In addition, there exist n_0, such that $\|F\|_\rho = |a_{n_0}|\rho^{n_0}$ for all ρ near 1 enough. That provides (**iv**).

If (**iv**) holds, then there is a constant C such that

$$\sup_{\rho\in(0,1);n\in\mathbb{N}} \{|a_n|\rho^n\} < C \Rightarrow |a_n| < C \ \forall n$$

Therefore, the set $\{\nu(a_n)\}_{n\in\mathbb{N}}$ has lower bound, but the value of $\nu(a_n)$ is integer, hence, there exists n_0 such that $|a_{n_0}| \max_{n\in\mathbb{Z}}|a_n|$, then (**i**) holds. $\square$

Theorem 3.5. *$\mathcal{R}^{bd}$ is a field, and $\mathcal{R}^\times = \mathcal{R}^{bd}\backslash\{0\}$.*

Proof. It is obvious that $\mathcal{R}^{bd}$ is a ring with multiplicative identity element, so to prove it í a field, we need to show that every element has its inverse. Indeed, let $F(X)$ be an arbitrary element, and $\delta \in (0,1)$ such that $F(X) \in \mathcal{A}^{bd}_{[\delta,1)}$, whence lemma 3.4 shows that $F(X)$ has finitely many roots in $[\delta, 1)$ let $\varepsilon \in (0,1)$ such that the absolute values of all roots is less than ε, thus $F(X) \in \mathcal{A}^\times_{[\varepsilon,1]} \subset \mathcal{R}^{bd}$.

For the latter, let $G(X)$ be an invertible element of $\mathcal{R}$, then there is some $\delta' \in (0,1)$, such that $G(X) \in \mathcal{A}^\times_{[\delta',1]}$, therefore lemma 2.30 implies that $G(X) \in \mathcal{A}^{bd}_{[\delta',1]} \in \mathcal{R}^{bd}$. $\square$

Corollary 3.6. *$\mathcal{R}^{bd}$ is a discrete valuation ring with the valuation function define by*

$$\mathbf{W}\left(\sum_{n\in\mathbb{Z}} a_n X^n\right) = \min_{n\in\mathbb{N}} \nu(a_n)$$

In addition, its non-archimedean absolute value that is associated to the valuation $\mathbf{W}$ is the 1-Gauss norm and the discrete valuation ring of $\mathcal{R}^{bd}$ is $\mathcal{R}^{int}$, namely $\mathcal{R}^{bd} = \mathcal{R}^{int}\left[\frac{1}{\pi}\right]$.

Proof. It is obvious from what we have proved. $\square$

However, we notice that the discrete valuation ring is complete. For an instant, we look at the following example.

Example 3.7. Let define a sequence of Laurent series ring $\{f_n\}_{n\in\mathbb{N}}$ by

$$f_n(X) = \sum_{i=0}^{n} \pi^{\lfloor \log(n) \rfloor} X^{-n}$$

It is clear that $g_n(X) \in \mathcal{R}$ and the sequence converges to a Laurent series $f(X)$ with respect to the 1-Gauss norm.

$$f(X) = \sum_{i=0}^{+\infty} \pi^{\lfloor \log(n) \rfloor} X^{-n}$$

However, for any $\delta \in (0,1)$ and for any n large enough, $n\delta > \log(n)$. Whence, there is no such $\delta \in (0,1)$ such that $f(X) \in \mathcal{A}_{[\delta,1)}$, equivalently, not in $\mathcal{R}^{bd}$ as well. The existence of this example shows the incompleteness of the bounded Robba ring.

Notice that for every element $g(X) = \sum_{n\in\mathbb{Z}} a_n X^n$, its coefficients are bounded and $\nu(a_n)$ tends to infinity as $n \to -\infty$ as quick as $n\delta$ for some $\delta \in (0,1)$, which is not satisfied on $f(X)$ in the previous example. The following theorem shall solve this matter by defining all elements of completion of $\mathcal{R}^{bd}$.

Theorem 3.8. *Let $\mathcal{E}$ be the set off all Laurent series with coefficients in K satisfying the conditions: there is a upper bound with respect to the* 1*-Gauss norm, equivalently, a lower bound with respect to valuation, and as index tends to $-\infty$, the valuation tends to infinity. In addition, we use the notation $\mathcal{E}^{int}$ to denote the complement of $\mathcal{R}^{int}$.*

Proof. It is obvious that $\mathcal{R}^{bd} \subset \mathcal{E}$, and $\mathcal{E}$ is a field under addition and multiplication defined on Laurent series. Recalling the above argument shows that the field $\mathcal{E}$ contains $\mathcal{R}^{bd}$, and the bound of coefficients makes the discrete valuation $\mathbf{W}$ available on $\mathcal{E}$. Moreover, let $f(X) = \sum_{i\in\mathbb{Z}} a_n X^n$ be an arbitrary element of $\mathcal{E}$, then we set $\{f_n\}_{n\in\mathbb{N}}$ defined by

$$f_n(X) = \sum_{i=-n}^{+\infty} a_i X^i \in \mathcal{R}^{bd}$$

The definition of $\mathcal{E}$ guarantees that $f_n \to f$ with respect to the 1-Gauss norm. Therefore, $\mathcal{R}^{bd}$ is dense in $\mathcal{E}$. Hence, if $\mathcal{E}$ is complete, then theorem 1.12 shall finish the proof.

Let $\{g_n(X) = \sum_{k\in\mathbb{Z}} a_k^{(n)} X^k\}_{n\in\mathbb{Z}}$ be an arbitrary Cauchy sequence in $\mathcal{E}$. That means for every $\varepsilon > 0$, there exists an intermediate index N such that for all $m, n \geq N$

$$\|g_m - g_n\|_1 < \varepsilon \Rightarrow |a_k^{(m)} - a_k^{(n)}| < \varepsilon \ \forall k \in \mathbb{Z}$$

Thereby, we obtain infinitely many Cauchy sequences $\left(a_k^{(n)}\right)_{n\in\mathbb{N}}$'s, the completeness of the field K implies there is a Laurent series $g(X) \in K$, which is the component-wise limit of $\{g_n\}_{n\in\mathbb{N}}$. In addition,

$$\|g_n - g\|_1 = \max_{k\in\mathbb{Z}} |a_k^{(n)} - a_k| < \varepsilon \ \forall n > N$$

Where a_k's are limits of the Cauchy sequences $\left(a_k^{(n)}\right)_{n\in\mathbb{N}}$'s. Whence, the bound of $(a_k)_{k\in\mathbb{Z}}$ is inherited form the bound of coefficients of g_n's. Moreover, the non-archimedean shows that

$$|a_k| \leq \max\{|a_k^{(n)} - a_k|, |a_k^{(n)}|\} \ \forall n$$

Thus, for all $\varepsilon > 0$, there exists an index N_ε such that

$$|a_k| \leq \max\{\varepsilon, |a_k^{(n)}|\} \ \forall n \geq N_\varepsilon \Rightarrow \lim_{k\to-\infty} |a_k| \leq \max\{\varepsilon, \lim_{k\to-\infty} \{|a_k^{(N_\varepsilon)}|\}\} = \varepsilon$$

Let ε tends to zero, we obtain that $g(X) \in \mathcal{E}$, which shows the completeness of $\mathcal{E}$. □

Lemma 3.9. ***(Nagata's lemma)*** *Let L be a discrete valuation ring with maximal ideal $\mathfrak{m}$, then the four following conditions are equivalent:*

(**i**) *L is a Henselian ring.*

(**ii**) *Every integral ring extension of L is a local ring.*

(**iii**) *Every Nagata polynomial, which is of the form $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in L[X]$ satisfying that $a_0 \in \mathfrak{m}$ and $a_1 \notin \mathfrak{m}$ has a root in $\mathfrak{m}$.*

(**iv**) *Every monic polynomial $g(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in L[X]$ such that $a_{n-1} \notin \mathfrak{m}$ and $a_i \in \mathfrak{m}\ \forall i = 0, 1 \ldots, n-2$ has a root in $a_{n-1} + \mathfrak{m}$.*

Theorem 3.10. *$\mathcal{R}^{int}$ is a Henselian ring.*

Proof. Let $G(X) = T^n + b_{n-1}T^{n-1} + \ldots + b_0 \in \mathcal{R}^{int}[T]$ be a polynomial such that $b_{n-1} \not\equiv 0 \mod \pi$ and $b_{n-2} \equiv \ldots \equiv b_0 \equiv 0 \mod \pi$. According to Nagata's lemma, we need to show that there is a root in $\mathcal{R}^{int}$ congruent to $b_{n-1} \mod \pi$. Making use of the transformation $(-b_{n-1})^{-d} G(-b_{n-1}T)$, we can assume $b_{n-1} = -1$. Notice that for any $Z \equiv 1 \mod \pi$, we have $G(Z) \equiv 0 \mod \pi$ and

$$G'(Z) = nZ^{n-1} - (n-1)Z^{n-2} \equiv 1 \mod \pi$$

Whence, we can recursively construct a sequence $(x_n)_{n\in\mathbb{N}}$ given by

$$x_1 = 1 \text{ and } x_{i+1} = x_i - \frac{G(x_i)}{G'(x_i)}$$

We inductively prove that

$$G(x_i) \equiv 0 \mod \pi^i \text{ and } x_{i+1} \equiv x_i \mod \pi^i$$

Let assume it is true up to k, we have to prove for the index $k+1$. Indeed, we observe

$$G(x_{k+1}) = G(x_k - \frac{G(x_k)}{G'(x_k)}) = G(x_k - \pi^k \frac{G(x_k)}{\pi^k G'(x_k)}) \equiv G(x_i) - \pi^k \frac{G(x_k)}{\pi^k} \equiv 0 \mod \pi^{k+1}$$

Thus, $G'(x_{k+1}) \equiv 1 \mod \pi \Rightarrow x_{k+1} \equiv x_k \mid \pi^k$. Therefore, $x_i \to x \in \mathcal{E}^{int}$.
For any $i \in \{0, \ldots, n-2\}$, $\|b_i\|_1 < 1$, then by lemma 3.4, there exits $\delta_1 \in (0,1)$ such that $b_i \in \mathcal{A}^{bd}_{[\delta_1,1)}$ and $\|b_i\|_\rho < 1\ \forall \rho \in [\delta_1, 1)$. Whence, we can inductively construct a sequence $\delta_1 < \ldots < \delta < 1$ such that

$$x_i \in \mathcal{A}^{bd}_{[\delta_i,1)} \text{ and } \|x_i\_\rho \leq 1\ \forall \rho \in [\delta_i, 1)$$

in the manner: the first step is trivial, let assume that we already have the first i elements, so $G(x_i), G'(x_i) \in \mathcal{A}^{bd}_{[\delta_i,1)}$ and $\|G(x_i)\|_\rho \leq 1$ and $G'(x_i) \equiv 1 \mod \pi \Rightarrow \|G'(x_i)\|_\rho = 1 \forall \rho \in [\delta_i, 1)$. That show $G'(x_i)$ is a unit in $\mathcal{A}^{bd}_{[\delta_{i+1},1)}$ for some $\delta_{i+1} > \delta_i$, then the non-archimedean implies

$$\|x_{i+1}\|_\rho \leq \max\left\{\|x_{i+1}\|_\rho, \left\|\frac{G(x_i)}{G'(x_i)}\right\|_\rho\right\} \leq 1\ \forall \rho \in [\delta_{i+1}, 1)$$

The upper bound exist because $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. Therefore, $x_i \in \mathcal{A}^{bd}_{[\delta,1)}$ and $\|x_i\|_\rho \leq 1\ \forall \rho \in [\delta, 1)\,, i \in \mathbb{N}$. Moreover, fix $\rho \in [\delta, 1)$, and let $x_i = \sum_{j\in\mathbb{Z}} a_j^{(i)} X^j$, we obtain

$$1 \geq \|x_i\|_\delta \geq a_j^{(i)} \rho^j \left(\frac{\delta}{\rho}\right)^j$$

Let $x = \sum_{j\in\mathbb{Z}} a_j X^j$, then $\lim_{j\to-\infty} |a_j|\rho^j = 0$ by letting $i \to +\infty$. That means $x \in \mathcal{A}_\rho\ \forall \rho \in [\delta, 1) \Rightarrow x \in \mathcal{R}^{int}$. $\square$

3.2 Semi-linear maps

Definition 3.11. Let K be a field and V be a K-vector space of finite dimension, $d = \dim_K V$. Given a injective field homomorphism $\sigma : K \to K$. A mapping on V is called semi-linear (σ-semi-linear) if it satisfies the condition

$$f(av_1 + bv_2) = \sigma(a)f(v_1) + \sigma(b)f(v_2)\ \forall a, b \in K; v_1, v_2 \in V.$$

Proposition 3.12. *Fix a K-basis $\{v_1, \ldots, v_d\}$, with respect to which we denote the associate matrix of f by A_f. Then we have an injective mapping*

$$\textit{The set of all semi-linear} \to M_d(K)$$
$$f \mapsto A_f$$

Notice that the homomorphism σ induces the map $A \mapsto \sigma(A) = (\sigma(a_{ij}))$ on $M_d(K)$

with the following properties:

- $\sigma(A+B)=\sigma(A)+\sigma(B)$,
- $\sigma(AB)=\sigma(A)\sigma(B)$,
- $\sigma(I)=I$.

Proof. Given a matrix $A=(a_{ij})_{d\times d}\in M_d(K)$, the associated semi-linear f is uniquely determined

$$f(c_1v_1+\ldots+c_dv_d)=\sum_{i=1}^{d}\sum_{j=1}^{d}\sigma(c_j)a_{ij}v_i$$

□

Lemma 3.13. *Let B is a transition matrix of the basic $\{v_1',\ldots,v_d'\}$, in particular*

$$(v_1',\ldots,v_d')=(v_1,\ldots,v_d)\,B$$

The associated matrix of f with respect to $\{v_1',\ldots,v_d'\}$ is $B^{-1}A_f\sigma(B)$.

Proof. We observe

$$f(v_j')=f(\sum_{l=1}^{d}b_{lj}v_l)=\sum_{l=1}^{d}\sigma(b_{lj})\sum_{k=1}^{d}a_{kl}v_k=\sum_{l=1}^{d}\sum_{k=1}^{d}a_{kl}\sigma(b_{lj})v_k$$

Then we obtain

$$BA_f'=A\sigma(B)$$

□

Definition 3.14. Let K be a field with characteristic p, which is a prime number. A field homomorphism σ is called Frobenius endomorphim if it is of the form

$$\begin{aligned}\sigma:&K\to K\\ &a\mapsto a^q\end{aligned}$$

Where q is a power of p.

Proposition 3.15. *A Frobenius endomorphism is injective. If K is a finite field then the Frobenius endomorphism is an automorphism.*

Proof. We notice that $\ker(\sigma)=\{0\}$ since $a^q=0$ if and only if $a=0$. Therefore, in the case K is a finite field, then the bijectivity follows from the injectivity. □

3.3 Special endomorphism

Given a positive integer q, and a Laurent series $u(X) = \sum_{n\in\mathbb{Z}} c_n X^n \in \mathcal{R}^{int}$ such that

$$u \equiv X^q \mod \pi$$

Since $u \in \mathcal{R}^{int}$, then there exist some $\delta_0 \in (0,1)$ such that $u \in \mathcal{A}^{\times}_{[\delta_0,1)}$, whence $\|u\|_\rho = \rho^q \ \forall \rho \in [\delta_0, 1)$. We study the substitution map φ induced by u on various domains

$$\varphi : F(X) = \sum_{n\in\mathbb{Z}} a_n X^n \mapsto F(u) = \sum_{n\in\mathbb{Z}} a_n u^n$$

Lemma 3.16. *Let $F(X) \in \mathcal{A}_{[\delta,1)}$ with $\delta \in [\delta_0, 1)$, then $F(u) \in \mathcal{A}_{\left[\delta^{1/q},1\right)}$. In addition, for all $\rho \in \mathcal{A}_{\left[\delta^{1/q},1\right)}$. we have*

$$\|F(u)\|_\rho = \|F\|_{\rho^q}$$

Proof. For all $\rho \in \left[\delta^{1/q}, 1\right)$,

$$\|u\|_\rho = \rho^q \Rightarrow \lim_{n\to\pm\infty} |a_n| \|u\|_\rho^n = \lim_{n\to\pm\infty} |a_n| \rho^{qn} = 0$$

Therefore, $F(u) \in \mathcal{A}_{\left[\delta^{1/q},1\right)}$. On other hand, notice that

$$u \in \mathcal{A}^{\times}_{[\delta_0,1)} \Rightarrow \|u - c_q X^q\|_\rho < |c_q| \rho^q = \rho^q \Rightarrow \left\| \frac{u}{c_q X^q} - 1 \right\|_\rho < 1$$

Because c_q is a unit in R, and

$$\left(\frac{u}{c_q X^q}\right)^n - 1 = \left(\frac{u}{c_q X^q} - 1\right) \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^k \left(\frac{u}{c_q X^q}\right)^{n-k}\right)$$

We have the latter sum is the sum of finite elements with the ρ Gauss norm 1, therefore, the non-archemedean shows that the ρ-Gauss norm of the whole sum is at most 1. Hence, we repeat the argument for $\dfrac{c_q X^q}{u}$ to obtain

$$\left\| \left(\frac{u}{c_q X^q}\right)^n - 1 \right\|_\rho < 1 \Rightarrow \|u^n - (c_q X^q)^n\|_\rho < |c_q| \rho^{qn} \ \forall n \in \mathbb{Z}$$

Combining this with

$$F(u) = \sum_{n\in\mathbb{Z}} a_n u^n = \sum_{n\in\mathbb{Z}} a_n \left(u^n - (c_q X^q)^n\right) + \sum_{n\in\mathbb{Z}} a_n (c_q X^q)^n$$

We have

$$\|F(u)\|_\rho = \left\| \sum_{n\in\mathbb{Z}} a_n (c_q X^q)^n \right\|_\rho = \|F\|_{\rho^n}$$

□

Theorem 3.17. *Let define φ on different domains, namely $\mathcal{R}, \mathcal{R}^{bd}$ and $\mathcal{E}$.*

- *The mapping $\varphi : \mathcal{R} \to \mathcal{R}$ is an injective continuous ring homomorphism.*
- *The mapping $\varphi : \mathcal{R}^{bd} \to \mathcal{R}^{bd}$ is an injective continuous field homomorphism. It uniquely induces an endomorphism φ^* on $\mathcal{E}$, whose restriction on $\mathcal{R}^{bd}$ is φ. Both functions preserves the valuation $\mathbf{W}$.*

Proof. Lemma 3.16 shows a ring homomorphism from $\mathcal{A}^{\times}_{[\delta,1)} \to \mathcal{A}_{\left[\delta^{1/q},1\right)}$, tending $\delta \to 1$ provides a ring homomorphsim $\mathcal{R} \to \mathcal{R}$. The property of Gauss norms implies that $\ker(\phi) = \{0\}$, which shows the injectivity and continuous.

For the domain $\mathcal{R}^{bd}$, it suffices to prove that for any $\delta \in [\delta_0, 1)$, ϕ maps elements of $\mathcal{A}^{bd}_{[\delta,1)}$ into $\mathcal{A}^{bd}_{[\delta,1)}$ itself. Indeed, we tend $\delta \to 1$ in the equality

$$\|F(u)\|_\rho = \|F\|_{\rho^q}$$

and apply lemma 3.4, we obtain the 1-Gauss norm is available on $F(u)$ as well, which implies $F(u) \in \mathcal{A}^{bd}_{[\delta,1)} \in \mathcal{R}^{bd}$. The extension of ϕ is trivial because of its continuity. □

Definition 3.18. Let σ be a field endomorphism on K, that preserves the valuation $\mathbf{W}$. A ring homomorphism ϕ is called σ-special if

$$\phi : F(X) \mapsto \sigma(F)(u)$$

for some $u \equiv X^q \mod \pi$ and $\sigma\left(\sum_{n\in\mathbb{Z}} a_n X^n\right) = \sum_{n\in\mathbb{Z}} \sigma(a_n) X^n$.

Remark 3.19. Since σ is a valuation preserving endomorphism, which implies the injectivity, thus, ϕ is an injective endomorphism of all domains $\mathcal{R}, \mathcal{A}_{[\delta,1)}, \mathcal{R}^{bd}, \mathcal{A}^{bd}_{[\delta,1)}, \mathcal{R}^{int}, \mathcal{E}$ and $\mathcal{E}^{int}$.

Theorem 3.20. *Let n be a positive integer, and let A be a matrix of size $n \times n$ in $M_n(R^{int})$, then every σ-special ϕ with $q > 1$ satisfies*

$$\begin{aligned} \left(\mathcal{R}\backslash\mathcal{R}^{bd}\right)^n &\to \left(\mathcal{R}\backslash\mathcal{R}^{bd}\right)^n \\ v + \left(\mathcal{R}^{bd}\right)^n &\mapsto v - A\phi(v) + \left(\mathcal{R}^{bd}\right)^n \end{aligned}$$

is a group automorphism under addition.

Proof. We add several properties to the matrix A in the manner: Since $u \in \mathcal{R}^{int} \subset \mathcal{R}^{bd}$, there exists a δ_0 as defined above such that

$$u \in \mathcal{A}^{\times}_{[\delta_0,1)} \Rightarrow \mathbf{W}_u(t) = \alpha t \ \forall t \in [\log(\delta_0), 0 = \log(1)]$$

For any integer m, the mapping $F \mapsto X^m F$ is trivially a bijection on $\left(\mathcal{R}\backslash\mathcal{R}^{bd}\right)^n$, which induces a commutative diagram

$$\begin{array}{ccc} \left(\mathcal{R}\backslash\mathcal{R}^{bd}\right)^n & \xrightarrow{v-A\phi(v)} & \left(\mathcal{R}\backslash\mathcal{R}^{bd}\right)^n \\ X^{-m}\cdot\Big\downarrow & & \Big\downarrow X^{-m}\cdot \\ \left(\mathcal{R}\backslash\mathcal{R}^{bd}\right)^n & \xrightarrow{v-\hat{A}\phi(v)} & \left(\mathcal{R}\backslash\mathcal{R}^{bd}\right)^n \end{array}$$

Where the matrix $\hat{A}$ is determined by

$$X^{-m}\left(v - A\phi(v)\right) = X^{-m}v - \hat{A}\phi(X^{-m}v) \Rightarrow \hat{A} = X^{-m}u^m A$$

This follows because in $\mathcal{R}^{int} \subset \mathcal{R}^{bd}$, $u = \phi(X) = X^q + \left(\mathcal{R}^{bd}\right)$. Hence, it suffices to prove the theorem is true for $\hat{A}$. For an entry $F \in \mathcal{R}^{int}$ of A, there exists some $\delta \in [\delta_0, 1)$ such that $F \in (\mathcal{A}^b_{[\delta,1)})^\times$. That implies $\mathbf{W}_F(t) = \beta t + \gamma\ \forall t \in [\delta, 0]$ with $\gamma = \mathbf{W}_F(0) \geq 0$, therefore we obtain

$$\mathbf{W}_{X^{-m}u^m F}(t) = \left(m(\alpha-1) + \beta\right)t + \gamma$$

Choosing $m \geq \dfrac{\beta}{m(\alpha-1)}$ makes the ρ-Gauss norm of the corresponding entry of $\hat{A}$ at most 1 for all $\rho \in [\delta, 1]$. Let move δ closed to 1 and choose m in that sense so that we can obtain the same property for all entry of the matrix $\hat{A}$.

Injectivity: We shall show that the kernel is $\{0\}$, let $v = (v_1, \ldots, v_n)$ be a element of $\mathcal{R}$ such that $(w_1, \ldots, w_n) = v - \hat{A}\phi(v) \in \mathcal{R}^{bd}$.Then $\|w_i\|_\rho$'s are bounded as the variable ρ run over the interval $[\delta, 1)$. Similarly, $\|\phi(v_i)\|_\rho$'s are also bounded on the closed interval $\left[\delta, \delta^{1/q}\right]$, hence, let a constant C be an upper bound. Making use of the equality

$$(w_1, \ldots, w_n) = (v_1, \ldots, v_n) - \hat{A}\left(\phi(v_1), \ldots, \phi(v_n)\right)$$

, all entries of $\hat{A}$ of the ρ-Gauss norm at most 1 and lemma 3.16, we obtain

$$\max_{i\in\{1,\ldots,n\}} \|v_i\|_\rho \leq C\ \forall\rho \in \left[\delta, \delta^{1/q}\right] \Rightarrow \max_{i\in\{1,\ldots,n\}} \|v_i\|_\rho \leq C\ \forall\rho \in \left[\delta, \delta^{1/q^2}\right]$$

Notice that $\lim_{n\to+\infty} \delta^{1/q^n} = 1$, then induction provides that C is an upper bound of $\{\|v_i\|_\rho; \|v_i\|_\rho; \rho \in [\delta, 1\}$, which means $v \in \left(\mathcal{R}^{bd}\right)^n$.

Surjectivity: Let $v = (v_1, \ldots, v_n) \in (\mathcal{R})^n$, then there exists a $\delta \in (0, 1)$ such that $v_i \in \mathcal{A}_{[\delta,1)}\ \forall i \in \{1, \ldots, n\}$ as well as all entries of $\hat{A}$. We inductively construct a sequence $\left(v^{(k)}\right)_{k\in\mathbb{N}} \subset (\mathcal{R})^n$. For each component $v_i^{(k)}$ of $v^{(k)}$, we divide it into two parts $v_i^{(k)+} = \sum_{j=0}^{+\infty} a_{kij}X^j$ and $v_i^{(k)-} = \sum_{j<0} a_{kij}X^j$, then $v^{(k)} = v_i^{(k)+} + v_i^{(k)-} = \sum_{j\in\mathbb{Z}} a_{kij}X^j$, and $v^{(k+1)}$ is defined recursively

$$v^{(k+1)} = \hat{A}\phi\left(v^{(k)+}\right)$$

Since $v^{(k)+} = \left(v_1^{(k)+}, \ldots, v_n^{(k)+}\right) \in \left(\mathcal{A}_{[\delta,1)}\right)^n$, then $v^{(k+1)} \in \left(\mathcal{A}_{\left[\delta^{1/q},1\right)}\right)^n$, and

$$\max_{i\in\{1,\ldots,n\}} \|v_i^{(k+1)+}\|_\rho \leq \max_{i\in\{1,\ldots,n\}} \|v_i^{(k+1)}\|_\rho \leq \max_{i\in\{1,\ldots,n\}} \|v_i^{(k)+}\|_{\rho^q}$$

$$\leq \rho^q \max_{i\in\{1,\ldots,n\}} \|X^{-1}v_i^{(k)+}\|_{\rho^q} \leq \rho^q \max_{i\in\{1,\ldots,n\}} \|X^{-1}v_i^{(k)+}\|_\rho \leq \rho^{q-1} \max_{i\in\{1,\ldots,n\}} \|v_i^{(k)+}\|_\rho$$

The latter happens because $\rho^q < \rho$. Therefore, we obtain $\lim_{j\to+\infty} \|v_i^{(k)+}\|_\rho = 0$ for all $\rho \in [\delta, 1)$ and $i \in \{1, \ldots, n\}$. Then there exists $w = (w_1, \ldots, w_n) \in \mathcal{A}_{[0,1)}^n$, whose components are power series given by $w_i = \sum_{k=0}^{+\infty} v_i^{(k)+}$. Whence, it follows

$$v - w + \hat{A}\phi(w) = \sum_{k=0}^{+\infty} v_i^{(k)-}$$

$\sum_{k=0}^{+\infty} v_i^{(k)-}$ has zero coefficients for all terms of positive degree, and it converges because

$$\max_{i\in\{1,\ldots,n\}} \|v_i^{(k+1)-}\|_\rho \leq \max_{i\in\{1,\ldots,n\}} \|v_i^{(k+1)}\|_\rho \leq \rho^{q-1} \max_{i\in\{1,\ldots,n\}} \|v_i^{(k)+}\|_\rho \to 0$$

Hence, $v - w + \hat{A}\phi(w) \in \mathcal{A}^{bd}_{\left[\delta^{1/q},1\right)} \subset \mathcal{R}^{bd} \Rightarrow v = w + \hat{A}\phi(w) \mod \mathcal{R}^{bd}$. □

Remark 3.21. When the reside field $\mathbf{k}$ is a field with characteristic p, the Frobenius-special endomorphism is called power Frobenius lift, which also holds all the proved properties.

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