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Modeling of CDS Index Tranches

MSc Thesis

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Contents

Introduction

A significant part of the financial derivatives market is the credit derivatives nowadays and even earlier too. An important example is the Collateralized Debt Obligations. The CDOs are structured finance products whose basket includes loans and bonds. The first CDOs were created in the 1990s and their markets have grown rapidly because with these products the investment banks could issue more and more loans. Soon, in the late 1990s, appeared the first synthetic CDOs, whose basket includes other credit derivatives, like CDSs.

The Credit Default Swap is a credit derivative between two counterparties. The first CDSs in their current form have existed since the early 1990s and growth enormous market in the early 2000s. At the beginning of the 2007-2009 financial crisis the CDS market was the third biggest OTC derivatives market in the world. By the end of 2007 the outstanding CDS amount was $62.2 trillion which fell back to $26.3 trillion by the mid-year 2010 and stayed $5.5 trillion in early 2012.

After the financial crisis the regulations of these products were necessary. The most CDS are documented using standard forms drafted by the International Swaps and Derivatives Association (ISDA) [1].

In general, synthetic CDOs have lost in popularity since 2008 crisis, however e.g. standardized CDS index tranches are still an important example both in the US and Europe for credit correlation trading in the credit derivatives markets. There is $8 trillion notional value outstanding CDS as of June 2018 [2]. And for 2019 the ISDA estimates $10 trillion of gross notional outstanding on single name and index CDS.

In the first sections of this thesis I'm writing about these CDS and CDO products and how we are pricing them by implementing the base model. Writing about the base model and the extended ones with stochastic recovery rate function and different copula methods. We would like to fit the models to the market prices and to analyze one of the most representative piece of the calculations, the expected loss. Our results are presented in section 4 while the conclusion can be found in section 5.

1 Overview of Credit Default Swaps

In this first section, we present the Credit Default Swap contracts based on the paper of Nina, Anna and Or [2] and Dezhong [3] article.

CDS is a bilateral agreement between a protection buyer and a protection seller where the buyer agrees to pay fixed periodic payments to the seller in exchange for protection against the credit event of an underlying. The most common credit events are the bankruptcy and the failure to pay. The Credit Default Swaps are derivative contracts which could be classified as follows: single-name, basket or credit default index swaps.

In 2021, Credit Default Swaps were the most traded product of all credit derivatives, which means that it was almost 90% of the credit derivative market, with $3.4 trillion dollars from the U.S. Comptroller of the Currency's report.

1.1 Single-Name CDS contracts

We are talking about single-name Credit Default Swap if there is a CDS contracts where the underlying is a single reference entity for example a single corporation or a sovereign. The protection buyer pays coupon payments to the seller and when the reference entity defaults the protection seller pays an amount to compensate for the loss of the protection buyer. The ISDA Master Agreement defined the properties and conditions of contracts. The standard credit events include bankruptcy or insolvency of the reference entity, failure to pay an amount above a special threshold, obligation default and repudiation.

The method of single-name CDS contracts was changed in 2015. Before that new on-the-run single-name contract rolled each quarter on the 20^{th} of March, June, September and December, but after that amended the new contracts only rolled in March and September in order to align with the roll frequency of CDS Index contract and develop the liquidity. So for example, if there was a 5-year single-name CDS contract on June 20, 2015 with September 20, 2020 maturity under the old convention then it was considered as on-the-run for 3 months between June and September, 2015. If there is a 5-year single-name CDS contract under the new convention which is rolled on March 20, 2016 and maturing on June 20, 2021 then it is considered as on-the-run until September, 2016. So after 2015 the single-name contracts are considered as on-the-run for 6 months and then another series begin.

1.2 Basket Default Swap

Basket default swap is a credit derivative on a portfolio of reference entities. These are the first-to-default, second-to-default and n^{th}-to-default swaps. Obviously, the first-to-default swap provides insurance for only the first default, second-to-default swap provides insurance for only the second default, etc.

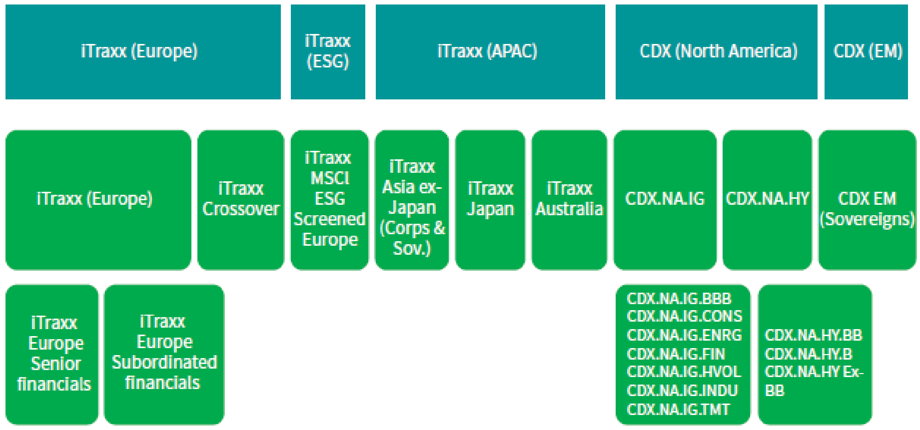
1.3 Credit Default Swap Index

We are talking about Credit Default Swap Index (denoted by CDX) if there is a CDS contract where the underlying is a portfolio of reference entities so the CDS Index is a portfolio of single-name CDS. Also known as the contract that provides protection against the credit risk of a standardized basket of reference entities. The mechanics of a CDX payments are a slightly different from the payment of single-name CDS. If there is default the coupon payment doesn't cease just decrease because less reference entities are being protected and the protection seller pays less recovery.

The most popular CDS Index families are Markit CDX indices, covering North American (NA), Emerging Markets (EM) and International Index Company (IIC) iTraxx indices including Europe, Australia, Japan and non-Japan Asia. The Markit CDX Indices family includes the North American Investment Grade CDX Index (CDX.NA.IG), the High-Yield CDX Index (CDX.NA.HY) and the Emerging Markets CDX Index (CDX.EM). The iTraxx Indices family includes the iTraxx Europe index and iTraxx Crossover index. These indices and their properties are presented in figure 1 from IHS Markit Paper [1].

As we see, the CDX.NA.HY index is a portfolio of 100 North American reference entities with 500 bps fixed coupon. These contracts are rolled semi-annually, on 27^{th} of March and September and the determined tenors are 3,5,7 and 10 years. The CDX.NA.IG index is a portfolio of 125 North American reference investment-grade-rated corporate firms with 100 bps fixed coupon and its standard maturities are 1, 2, 3, 5, 7, 10 years. iTraxx Europe index includes 125 equally weighted investment-grade European reference entities and this iTraxx Europe index family comprises 3 sub-indices sectors, which are the financial senior, the financial sub and the non-financial sectors. The iTraxx Crossover (iTraxx Xover) index is composed of the 75 sub-investment-grade European entities with 500 bps fixed coupon and its standard maturities are only 3,5,7,10 years. The indices could be traded either on spread or price. This convention depends on the cash market where the bonds trade on yields and other on price. These CDS Indices convention based on the underlying cash instruments. Therefore the quoting convention of CDX.NA.HY index is price, CDX.NA.IG and iTraxx indices are spread. In the section 4 I

IHS Markit CDS Index offering



Product descriptions

Index	# of Entities	Fixed Coupon (bps)	Currency	Quoting convention	Recovery Rates (%)	Roll Dates	Tenors
CDX.EM	18	100	USD	Price	40	20th March/Sept	5Y, 10Y
CDX.NA.HY	100	500	USD	Price	40	27th March/Sept	3Y, 5Y, 7Y, 10Y
CDX.NA.IG	125	100	USD	Spread	40	20th March/Sept	1Y, 2Y, 3Y, 5Y, 7Y, 10Y
iTraxx Asia ex-Japan	40	100	USD	Spread	35	20th March/Sept	5Y
iTraxx Australia	25	100	USD	Spread	40	20th March/Sept	5Y
iTraxx Crossover	75	500	EUR	Spread	40	20th March/Sept	3Y, 5Y, 7Y, 10Y
iTraxx Europe	125	100	EUR	Spread	40	20th March/Sept	1Y, 3Y, 5Y, 7Y, 10Y
iTraxx Europe Senior Financials	30	100	EUR	Spread	40	20th March/Sept	5Y, 10Y
iTraxx Europe Subordinated Financials	30	100	EUR	Spread	20	20th March/Sept	5Y, 10Y
iTraxx Japan	40	100	JPY	Spread	35	20th March/Sept	5Y
iTraxx MSCI ESG Screened Europe	Variable	100	EUR	Spread	40	20th March/Sept	5Y

Figure 1: Credit Default Swap Indices

will use the CDX.NA.IG and iTraxx Xover indices during the simulation which have got 40% recovery rate.

As mentioned in section 1.1, the contracts are rolled every six months, hence a new series of the CDS Index is created with updated constituents. As of writing this thesis, the latest indices for example are the CDX.NA.IG S38, where S38 marks this is the series 38, which is started trading on March 20th, 2022 and the iTraxx Xover S37, which is started trading on March 20th, 2022.

1.4 CDS Index Tranche

In this section we introduce the CDS Index Tranches based on [2] study and providing some interest inside from the IHS Markit article [1].

Some of the above mentioned CDS Indices are also available in a tranched format which allows investors to gain exposure on a particular portion of the index loss distribution. Tranches are defined by attachment and detachment points for the index loss distribution. It means that when default events occur the lower level tranches absorb the loss up to the detachment point, before moving to the next senior tranche. Once the total loss reaches the detachment point, that the tranche notional is fully written down. Therefore the attachment and detachment point refer to the loss amounts.

There is an important difference between indices and tranches that is while the CDS Indices rolled semi-annually, the CDS Tranches rolled only once a year, in September.

Figure 2 shows an example for the CDX.NA.IG tranche:

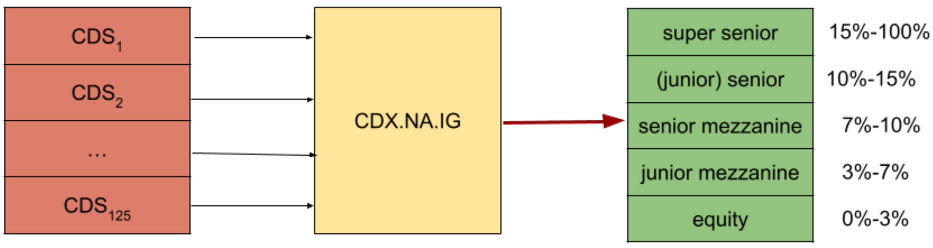


Figure 2: Tranche structure

where CDS_1, CDS_2, $\ldots$, CDS_{125} are equally weighted and the attachment point of the senior mezzanine tranche is 7%, and the corresponding detachment point is 10%. The CDX North American Investment Grade and High Yield, the iTraxx Europe and Crossover Index Tranches with their attachment and detachment points can be seen in the figure 3:

Index / Tranche	CDX.NA.IG		CDX.NA.HY		iTraxx Europe		iTraxx Crossover	
	Tranche (%)	Coupon (bps)	Tranche (%)	Coupon (bps)	Tranche (%)	Coupon (bps)	Tranche (%)	Coupon (bps)
Equity	0 - 3	100	0 - 15	500	0 - 3	100	0 - 10	500
Mezzanine	3 - 7	100	15 - 25	500	3 - 6	100	10 - 20	500
Senior	7 - 15	100	25 - 35	500	6 - 12	100	20 - 35	500
Super Senior	15 - 100	100	35 - 100	500	12 - 100	100	35 - 100	500

Figure 3: Index Tranches

CDS index tranches are liquid instruments that, among other use cases, facilitate the correlation trading. In this context, correlation refers to the probability of default of one reference entity in relation to others. Generally, the higher correlation implies higher joint default probability for the index constituents, and also implies an increasing value of the tranche. Inversely, the lower correlation implies a lower joint default probability for the index constituents, and thereby decreasing the value of the tranche.

We can think about index tranches as a layered protection technology which is developed for protecting the portfolio credit risk like in the Basket Default Swap 1.2. There, n layer (basket default swap) protects the n^{th} default in the portfolio. Now, these layers are specified by a range of percentage. The layer protection derivative products include CDO and CDS index tranches. A Collateralized Debt Obligation (CDO) is a security backed by a pool of one or more kinds of debt obligations such as loans, bonds, credit default swaps or other assets.

1.5 CDS Index Option

There are the CDS index options besides the CDS indices and index tranches which is significantly increased in the market. CDS Index Options or Swaptions are contracts that promise the holder of the swaption the right to enter into a CDS index position at option expiry at the specified strike level.

2 Pricing Methodology

In this second section we represent how CDSs and synthetic CDOs are priced, through the hazard rate and copula functions.

2.1 CDS Pricing

When we are pricing a Credit Default Swap, we have the following constructional parameters:

- N – the notional
- R – the recovery rate
- T – the maturity
- m – the payment dates (typically the number of quarters), so can be written:
  $0 = t_0,\ t_1,\ \ldots,\ t_m = T$
- s – the par spread in basis point (premium)

Before the 2007-2008 financial crisis the premium was equal with the spread, but after the standardization we use par spread and upfront payment. Moreover the other theoretical constructs are the follows:

- τ – Default time
- $p(t) = \mathbb{P}(\tau \leq t)$ – Default probability
- $D(0,t)$ – the discount factor of a risk-free bond maturiting at t

Assume, that $D(0,t)$ is a deterministic function, therefore independent from τ. Then the protection leg is the present value of the expected loss:

$$\text{ProtLeg} = \mathbb{E}\big[N(1-R)D(0,\tau)\mathbf{1}_{\{\tau \leq T\}}\big], \tag{1}$$

if the payment occurs in the same time as the default, at τ. If we assume that the payment will happen the next premium payment time after the default (we can assume that, because the discountfactors don't change significantly) then we can write the follows:

$$\text{ProtLeg} = \mathbb{E}\big[\sum_{j=1}^{m} N(1-R)D(0,T_j)\mathbf{1}_{\{T_{j-1} \leq \tau \leq T_j\}}\big] = N(1-R)\sum_{j=1}^{m} D(0,T_j)[p(T_{j-1}) - p(T_j)]. \tag{2}$$

The premium leg is the expected value of each premium payments in present value:

$$\text{PremLeg} = \mathbb{E}\left[\sum_{j=1}^{m} Ns\Delta_j D(0,t_j)\mathbf{1}_{\{t_j<\tau\}}\right] = Ns\sum_{j=1}^{m}\Delta_j D(0,t_j)p(t_j), \tag{3}$$

where $\Delta_j = t_j - t_{j-1}$.

Finally, the par spread is as follows:

$$s = \frac{\mathbb{E}\left[N(1-R)D(0,\tau)\mathbf{1}_{\{\tau\leq T\}}\right]}{N\sum_{j=1}^{m}\Delta_j D(0,t_j)p(t_j)} = \frac{(1-R)\sum_{j=1}^{m} D(0,T_j)[p(T_{j-1})-p(T_j)]}{\sum_{j=1}^{m}\Delta_j D(0,t_j)p(t_j)} \tag{4}$$

2.1.1 Hazard Rate

We briefly introduce the intensity based approach in CDO pricing models. Let us take a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ and assuming the absence of arbitrage to guarantee the existance of a unique risk-neutral probability measure $\mathbb{P}$. Under this framework, non-dividend paying assets (default-free) are martingales if discounted at the risk-free rate. The existence of such a risk neutral probability measure allows us to price our bonds or credit derivatives correctly, without the stronger market completeness assumption which would be required to hedge the securities. In addition, in order to simplify the exposition and focus on the credit risk modelling, we assume a flat interest rate structure, and at the beginning a fixed recovery rate and independence between default probabilities, interest rate curve and recovery rates. After this simple set up we later introduce stochastic recovery.

In general, pricing a CDO requires us to model both the risk neutral default probabilities for each name in the portfolio and the joint default distribution. The risk neutral default probabilities can be calculated following the popular practice of bootstrapping from CDS premiums.

Given the standard probability space defined above, we consider the $\{\mathcal{F}_t\}$-stopping time τ_i to model the default of the i^{th} obligor in a portfolio, the default probability distribution is given by $F_i(t) = \mathbb{P}\{\tau_i < t\}$ and the probability density distribution is $f_i(t)$. We define the "hazard rate" or "intensity" as follows:

$$\lambda_i(t) = \frac{f_i(t)}{1-F_i(t)}. \tag{5}$$

From that we can get the next ordinary differential equation:

$$\lambda_i(t) = -\frac{d\ln(1-F_i(t))}{dt}, \tag{6}$$

and solving the O.D.E., get an expression for the default probability distribution:

$$F_i(t) = 1 - \exp\Big(-\int_0^t \lambda_i(s)ds\Big). \tag{7}$$

We established the connection between the default distribution and the hazard rate, we can bootstrap the default probabilities based on the market observable CDS prices.

2.2 Synthetic CDO Pricing

What is Synthetic CDO? – A synthetic CDO is a type of collateralized debt obligation (CDO) that can offer extremely high yields to investors. They differ from traditional CDOs, which typically invest in regular debt products such as bonds, mortgages, and loans, in that they generate income by investing in noncash derivatives such as credit default swaps (CDSs), options, and other contracts. Synthetic CDOs are typically divided into credit tranches based on the level of credit risk assumed by the investor.

In contrast with the CDS pricing, the synthetic CDO pricing is a little bit more complex. Based on Andrew Lesniewski's [4] and Claudio Ferrarese's works [6] CDO tranches are defined by attachment a and detachment points d as, marking the lower and upper bound of a tranche. They are generally expressed as a percentage of the portfolio and determine the tranche size. Furthermore, we define the following constructional variables, some of them is same as in the CDS pricing:

- n – the number of reference entities (i.e. the number of CDS in a synthetic CDO)
- N_i – the nominal amount for the i^{th} ref.entity
- R_i – the recovery rate for the i^{th} ref.entity
- T – the maturity
- m – the payment dates (typically the number of quarters), so can be written:
  $0 = t_0,\ t_1,\ \ldots,\ t_m = T$
- s – the par spread in bps per period (premium)
- u – upfront premium, if s is fixed

And the other theoretical variables:

- τ_i – the default time for the i^{th} reference entity
- $p_i(t) = \mathbb{P}(\tau_i \leq t)$ – default probability of the i^{th} reference entity
- $B(0, t)$ – the discount factor of a risk-free bond maturiting at t

From these variables we get immediately that $N = \sum_{i=1}^{n} N_i$ the nominal amount of the portfolio. Moreover, we can write an equation for the risk-free discount bond, so $\mathbb{E}[D(0,t)] = e^{\int_0^t -r(s)ds}$ discount factor, where $r(s)$ is the risk-free interest rate. For every tranche, given the total portfolio loss in (8), and the cumulative tranche loss is given in (9):

$$L(t) = \sum_{i=1}^{n} N_i(1 - R_i)\mathbf{1}_{\{\tau_i \leq t\}} \tag{8}$$

$$L_{[a,d]}(t) = \max\{\min\{L(t), dN\} - aN, 0\}. \tag{9}$$

In the synthetic CDO contracts we are talking about *protection leg* and *premium leg* which two have to be equal. The protection leg covers the losses affecting the specific tranche, which given the following payment dates

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T$$

can be calculated by taking the expected value with respect to the risk neutral probability measure:

$$\text{ProtLeg}_{[a,d]} = \mathbb{E}\left[\sum_{j=1}^{m} D(0,t_j)\left(L_{[a,d]}(t_j) - L_{[a,d]}(t_{j-1})\right)\right]. \tag{10}$$

Similarly, assuming a continuous time payment, the protection leg can be written as:

$$\text{ProtLeg}_{[a,d]} = \mathbb{E}\left[\int_0^T D(0,s)dL_{[a,d]}(s)\right]. \tag{11}$$

On the other hand, the premium leg is generally paid quarterly in arrears to the protection seller. For this we have to know the notional of every tranche.

So the premium leg can be expressed as follows:

$$\text{PremLeg}_{[a,d]} = \mathbb{E}\left[\sum_{j=1}^{m} D(0,t_j)s\Delta_{t_j}\left[(d-a)N - L_{[a,d]}(t_j)\right]\right], \tag{12}$$

where $\Delta_{t_j} = t_j - t_{j-1}$. Because know that the premium and the protection legs have to be equal, but after the standardization of the CDS index tranche contracts the s premium is fixed, and we are pricing with the u upfront payment, which is the

$$\text{ProtLeg}_{[a,d]} = \text{PremLeg}_{[a,d]}, \tag{13}$$

the spread can be calculated by:

$$s = \frac{\mathbb{E}\left[\int_0^T D(0,s)dL_{[a,d]}(s)\right]}{\mathbb{E}\left[\sum_{j=1}^m D(0,t_j)\Delta_{t_j}\left[(d-a)N - L_{[a,d]}(t_j)\right]\right]}. \tag{14}$$

During the implementation we transform back equation (14) to the discrete time and change the order of expected value and sum:

$$\frac{\mathbb{E}\left[\sum_{j=1}^m D(0,t_j)\left(L_{[a,d]}(t_j) - L_{[a,d]}(t_{j-1})\right)\right]}{\mathbb{E}\left[\sum_{j=1}^m D(0,t_j)\Delta_{t_j}\left[(d-a)N - L_{[a,d]}(t_j)\right]\right]} = \frac{\sum_{j=1}^m D(0,t_j)\,\mathbb{E}\left[L_{[a,d]}(t_j) - L_{[a,d]}(t_{j-1})\right]}{\sum_{j=1}^m D(0,t_j)\Delta_{t_j}\left[(d-a)N - \mathbb{E}L_{[a,d](t_j)}\right]}. \tag{15}$$

As we wrote the premium and the protection legs have to be equal, but after the standardization of the CDS index tranche contracts the s premium is fixed, and we use the u upfront payment. In that case, the equation (13) can be rewrite as follows:

$$\text{ProtLeg}_{[a,d]} = \text{PremLeg}_{[a,d]} + Nu. \tag{16}$$

From that the value of u is

$$u = \sum_{j=1}^m D(0,t_j)\frac{1}{N}\mathbb{E}\left[\left(L_{[a,d]}(t_j) - L_{[a,d]}(t_{j-1})\right)\right] - \sum_{j=1}^m D(0,t_j)\frac{1}{N}s\Delta_{t_j}\left[(d-a)N - \mathbb{E}L_{[a,d](t_j)}\right] \tag{17}$$

To determine the given tranche price we have to predict the expected value of the given tranche loss at time t, $\mathbb{E}L_{[a,d]}(t)$. Therefore as the next step we would like to calculate the expected value of equation (8):

$$\mathbb{E}L(t) = \sum_{i=1}^n N_i(1-R_i)\mathbb{P}(\tau_i \leq t), \tag{18}$$

if we know the CDS index's spreads we can get the default probabilities. However, for this we have to familiarize ourselves with the loss distribution, because in equation (9) the functions are not linear.

During the approximation of the distribution we have to pay attention that the default of reference entities are not independent, i.e. there default intensities are correlated.

To investigate this, in the next subsection we examine the concepts of copulas and factor models.

2.2.1 Copulas

Using the market implicit approach we can construct the marginal distribution of survival time for each name in the portfolio. Assuming independence among the names is not realistic, considering that in real life, the default probability for a group of reference entities tends to be higher in a recession and lower when the economy is booming. This implies that there exists some form of positive dependence among the reference entities. To examine this correlation in the portfolio, we have to specify the joint distribution of survival/default times with given marginal distributions. For this, we introduce the copula function based on the article [10] written by Li.

Given $C : [0,1]^n \to [0,1]$ is a multidimensional function, $U_1, U_2, \ldots, U_n$ are uniformly distributed on the interval $[0,1]$. In that case, the **copula** function is defined as the joint cumulative distribution function of $(U_1, U_2, \ldots, U_n)$:

$$C(u_1, u_2, \ldots, u_n) = \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_m \leq u_n). \tag{19}$$

Sklar in 1959 [11] proved the following statement which is considered as the definition of the copula function. He showed if $F(x_1, x_2, \ldots, x_n)$ is a joint multivariate distribution function with univariate marginal distribution functions $F_1(x_1), F_2(x_2), \ldots, F_n(x_n)$, then there exists a $C(u_1, u_2, \ldots, u_n)$ copula function such that

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_m(x_n)). \tag{20}$$

In addition, if F_i is continuous for all i then C copula function is unique.

Several copula functions and factor models exist. These differ of what kind of distributions they assume for the common factors and the individual factors. Our model assume normal distribution and therefore Gaussian copula.

The Standard Gaussian Copula function is given by

$$C(u_1, u_2, \ldots, u_n) = \Phi_{n,\Sigma}\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n)\right), \tag{21}$$

where $\Phi_{n,\Sigma}$ is an n dimensional normal joint distribution function, Σ is a covariance matrix, and Φ is the standard normal distribution function.

The Standard Gaussian Copula is one of the most widely used copula in the financial applications, because the their easy numerical tractability. However we will explore other distributions to improve the model in the next sections, like Student's t and α-stable distribution as detailed in Section 3.3.

2.2.2 One Factor Gaussian Copula Model

As mentioned before, the idea behind the factor models is to assume that all names are influenced by one or more uncertainty from same sources. Moreover there is a positive correlation between the default events. The factor models are described as follows:

$$Z_i = \rho_i X + \xi_i \epsilon_i, \quad i = 1, \dots, n \tag{22}$$

where X is the common factor for the reference entities, $\epsilon_1, \dots, \epsilon_n$ are the individual factors which are assumed to be independent of X and each other and ρ_i are the correlation between the reference entities have to be from $[0, 1]$ interval. In this thesis we assume a single-factor gaussian copula model, i.e. there is only 1 common market factor, X as $X \sim N(0, 1)$ and $\epsilon_i \sim N(0, 1)$. We want that Z_i to be also standard normal distributed, so we have to choose the parameters as:

$$0 = \mathbb{E}(Z_i) = \rho_i \, \mathbb{E}(X) + \xi_i \, \mathbb{E}(\epsilon_i)$$
$$1 = \mathbb{D}^2(Z_i) = \rho_i^2 \mathbb{D}^2(M) + \xi_i^2 \mathbb{D}^2(\epsilon_i)$$

Therefore $\xi_i = \sqrt{1 - \rho_i^2}$ so our equation is amended as follows:

$$Z_i = \rho_i X + \sqrt{1 - \rho_i^2} \epsilon_i, \tag{23}$$

where the correlation between the names are

$$\text{corr}(Z_i, Z_j) = \frac{\text{cov}(Z_i, Z_j)}{\mathbb{D}(Z_i) \cdot \mathbb{D}(Z_j)} = \text{cov}\left(\rho_i X + \sqrt{1 - \rho_i^2} \epsilon_i, \rho_j X + \sqrt{1 - \rho_j^2} \epsilon_j\right) = \rho_i \rho_j \tag{24}$$

as $Z_i, X \sim N(0, 1)$ and ϵ_i are independent of each other and X also. During the simulation in the section 4 we assume one same correlation between the reference entities, ρ and then we can set the parameters as $\rho_i := \sqrt{\rho}$ so the model is:

$$Z_i = \sqrt{\rho} X + \sqrt{1 - \rho} \epsilon_i. \tag{25}$$

Before the factor models and copulas we wanted to determine the joint loss distribution of Z_i:

$$\mathbb{P}(Z_1 \leq B_1(t), \dots, Z_n \leq B_n(t)), \tag{26}$$

where $B_i(t)$ is the boundary below which the i^{th} reference entity will default. Because of that Z_i is from the one-factor gaussian copula model hereby $\mathbb{P}(Z_i \leq B_i(t)) = \Phi(B_i(t)) = p_i(t)$ and

$$B_i(t) = \Phi^{-1}(p_i(t)), \tag{27}$$

$p_i(t)$ denote the default probability of i. reference entity.

By replacing these variables into equation (21), we get the following:

$$\mathbb{P}\left(Z_1 \leq B_1(t), \dots, Z_n \leq B_n(t)\right) = \Phi_{n,\Sigma}\left(\Phi^{-1}\left(p_1(t)\right), \dots, \Phi^{-1}\left(p_n(t)\right)\right), \tag{28}$$

where Σ covariance matrix is:

$$\Sigma = \begin{bmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \dots & \rho & 1 \end{bmatrix}. \tag{29}$$

Large homogenous portfolio model

In the general model we use a simplified approach, namely the large homogenous portfolio model. It means that the N_i notional and R_i recovery rate the same for each name, and $N_i = \frac{N}{n}$ and let R denote the identical recovery rate. Hereby the Loss Given Default the same all of the reference entities so in the pricing process it is enough to calculate the number of defaults. However we don't assume that the default probabilities are equal for every name. Beside these assumptions the expected tranche loss can be written in the following form:

$$\mathbb{E}\left[L_{[a,d]}(t)\right] = \mathbb{E}\left[\max\left(\min\{D^{\#}_{[a,d]}(t) \cdot (1-R)\frac{N}{n}, dN\} - aN, 0\right)\right], \tag{30}$$

where $D^{\#}_{[a,d]}(t)$ denotes the number of defaults in a $[a, d]$ tranche which occurred before t.

3 Theoretical results

In the previous section we built the base model and now we would like to do some developments on it. Firstly, we have to introduce the correlation skew and the base correlation. After that, we will show what happens if stochastic recovery function is used instead of deterministic recovery rate and finally, we will see other copula models as well. Our fundamental problem is that the Gaussian copula has light tails so it gives less weight to the rare events, which have main role in the equity and senior tranches. Our goal is to test our model and make it conform to be more in line with observed market phenomena.

3.1 Base correlation

In this subsection we briefly describe the implied correlation and therefore the correlation smile. And to solve this we introduce the base correlation concept.

In general, the one factor Gaussian copula model directly isn't able to match the market because of its light tails property. To overcome this restriction one solution is to modify the correlation. Increasing correlation leads to an increasing probability in the tails, thus leading a very few or a very large number of defaults. With this modification we are capable to fit our model's result to the market quotes obtaining the so called "implied correlation" or "compound correlation" based on [6] and [7].

If we want to get the implied correlations, we extract the hazard rates from the market spreads and from that with the survival probability and the Gaussian copula functions we examine the tranche expected losses. To compare this simulation prices with the market prices the adequate correlations to each tranches are necessary. With an approximation method for the correlations we can get back the implied correlation for the given slices of contract. The next figures we get from the iTraxx Crossover index tranche simulations.

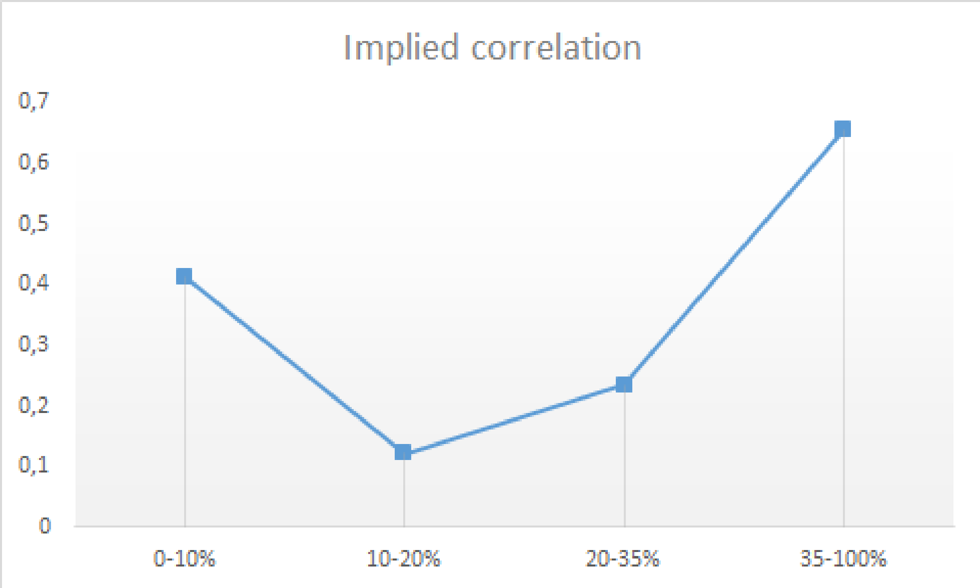


Figure 4: Correlation skew

We can see the problem of the implied correlation in figure 4, this is the so called *correlation smile*. The expectation is that the correlations are the same for each tranche, but the chart doesn't show that. A solution, the *base correlation* was introduced by McGinthy. The main view of this method is that we describe all tranches with equity tranches and we order the correlation for the detachment point, not for the tranches. We can examine the mezzanine and the senior tranches with the difference of 2 equity tranche, where these equity tranches are priced with the correlation of their detachment points (as we see, these are different). The process for calculating the expected loss ($\mathbb{E}L$) for each first loss tranche is, for example:

$$\mathbb{E}L[0, 0.2] = \mathbb{E}L[0, 0.1] + \mathbb{E}L[0.1, 0.2], \tag{31}$$

where $\mathbb{E}L[0.1, 0.2]$ comes from the market spread on the $[0.1, 0.2]$ tranche and $\mathbb{E}L[0, d]$ is the expected losses on a $[0, d]$ equity tranche. Therefore the correlation adherent to $[0.1, 0.2]$ tranche comes from equation (31). Once we have calculated the expected losses and the correlation for the sequence of first loss tranches, we can solve for each tranche. The results are shown in figure 5.

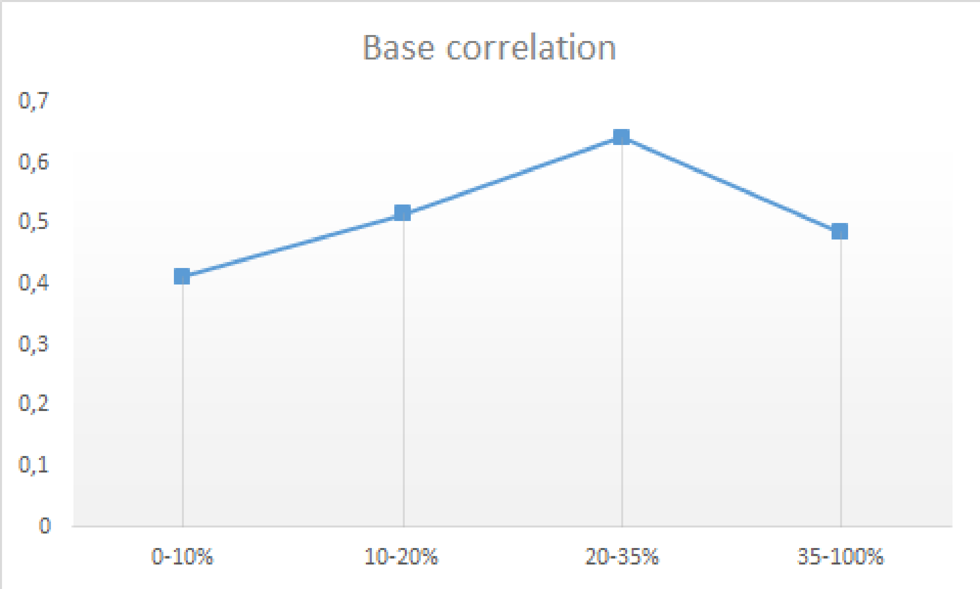


Figure 5: Base correlation

3.2 Stochastic Recovery Model

In this subsection we consider a special case of the general model based on the [5] and [8] articles. We change the recovery rate from the deterministic r_i value for stochastic recovery and recalibrate our base model. To specify this, let $0 \leq a_i \leq 1$ and b_i be vectors and ϵ_i sequence of independent, 0 mean, 1 variance random variables for all $i = 1, \ldots, n$ names which are independent from X common market factor. Let R_i^S be the stochastic recovery function for all $i = 1, \ldots, n$ reference entities which is defined as:

$$Z_i = a_i X + \sqrt{1 - a_i^2}\epsilon_i \tag{32}$$

$$R_i^S = C_i(\mu_i + b_i X + \xi_i) \tag{33}$$

where

- $C_i : \mathbb{R} \to [0, 1]$ – arbitrary mappping functions,
- μ_i – constants,
- ξ_i – sequence of independent, 0 mean, 1 variance random variables which is independent of X and ϵ_i.

In the model a_i is contained the correlation dependency and controlled it, so we can say that let a_i be equal with ρ_i. Later we would like to compare the results of deterministic and stochastic recovery calibrated models thereby we choose $b_i = a_i = \rho_i$. With these choices the modified model can be determined as:

$$Z_i = \rho_i X + \sqrt{1-\rho_i^2}\epsilon_i \tag{34}$$

$$R_i^S = C_i(\mu_i + \rho_i X + \xi_i) \tag{35}$$

In addition, $\rho_i \equiv 0$ corresponds to the constant recovery case.

As our base model use Gaussian copula function, therefore we can specify this stochastic recovery extension with Gaussian copula functions. As defined in the subsection 2.2.2 let $\epsilon_i \sim N(0,1)$, $\xi_i \sim N(0,1)$ be independent variables and also independent of $X \sim N(0,1)$ common factor. Moreover it is assumed that C_i are the standard cumulative Gaussian distribution, Φ for all $i = 1, \ldots, n$. Finally, we define the stochastic recovery in equation (36) that we wanted to work with:

$$R_i^S = \Phi(\mu_i + a_{\rho_i} X + \xi_i). \tag{36}$$

To compare the two recovery rates during the simulation we have to know the parameters. For this in the following we're writing about the properties of this improvement.

Let $\sigma_i = \sqrt{a_{\rho_i}^2 + 1}$ and $Y_i = \mu_i + a_{\rho_i} X + \xi_i$, then the following statements are satisfied and are proved by Andersen and Sidenius's [8] paper:

$$\mathbb{P}(R_i^S \leq x) = \mathbb{P}\left(Y_i \leq \Phi^{-1}(x)\right) = \Phi\left(\frac{\Phi^{-1}(x) - \mu_i}{\sigma_i}\right) \tag{37}$$

$$\mathbb{E}[R_i^S] = \mathbb{E}\left[\Phi(Y_i)\right] = \Phi\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}\right). \tag{38}$$

During the implementation we would like to determine the parameters that the expected values of the recovery rates will be equal:

$$R_i = \mathbb{E}[R_i] = \mathbb{E}[R_i^S] = \Phi\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}\right),$$

Therefore

$$\mu_i = \sqrt{2 + a_{\rho_i}^2}\Phi^{-1}(R_i), \quad i = 1, \ldots, n$$

If we want another (constant) multiplier to the individual factor, $b_i \cdot \xi_i$ then $\mu_i = \sqrt{1 + a_{\rho_i}^2 + b_i^2}\Phi^{-1}(R_i)$, because $\sigma_i = \sqrt{a_{\rho_i}^2 + b_i^2}$, and b_i depends on ρ correlation.

The main properties of the stochastic recovery function are driven by the coefficient a_{ρ_i} from which we derive that if $\rho \rightarrow +\infty$ then $a_{\rho_i} \rightarrow 0$. The following figure 6 shows the relationship between the conditional default probability and the stochastic recovery functions:

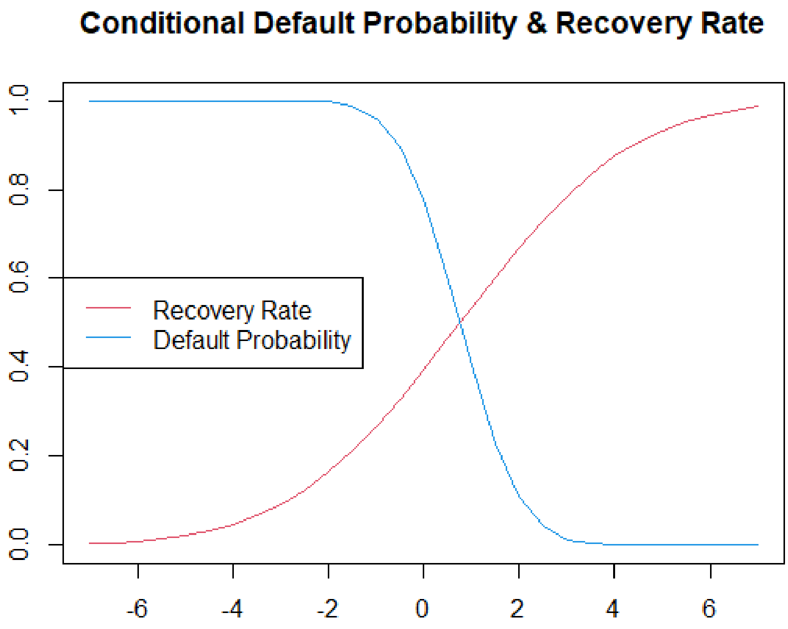


Figure 6: Default Probability and Recovery Rate

This ensures that when the common factor is negative (the market is "bad"), then the recovery rate is low, i.e. the default probability is high and conversely when the factor is positive (the market is "evolving"), then there is very low default probabilities of the companies.

We will show other numerical results for expected loss in the different tranches with the association of stochastic and deterministic recovery rate functions in section 4.2.

3.3 Copula extensions

In this subsection we would like to add some other properties to our model to the more exact matching with the market prices. It is based on the [6] article, in which Claudio Ferrarese compare the market implied loss distribution and the loss distribution from the Gaussian copula.

We can generate our model by using different distributions for the common factor and the individual factors. The aim is to use copulas which have fat-tailed distributions. In figure 7 there are the density functions of the NIG, Gaussian, alpha-stable and Student t distributions and figure 8 clearly shows that every lower tails are higher than the Gaussian. The illustrated density functions are parameterized as to be symmetrical. It means that both of the NIG, alpha-stable and Student t distributions put higher probability to the rare events like senior or super senior tranche losses.

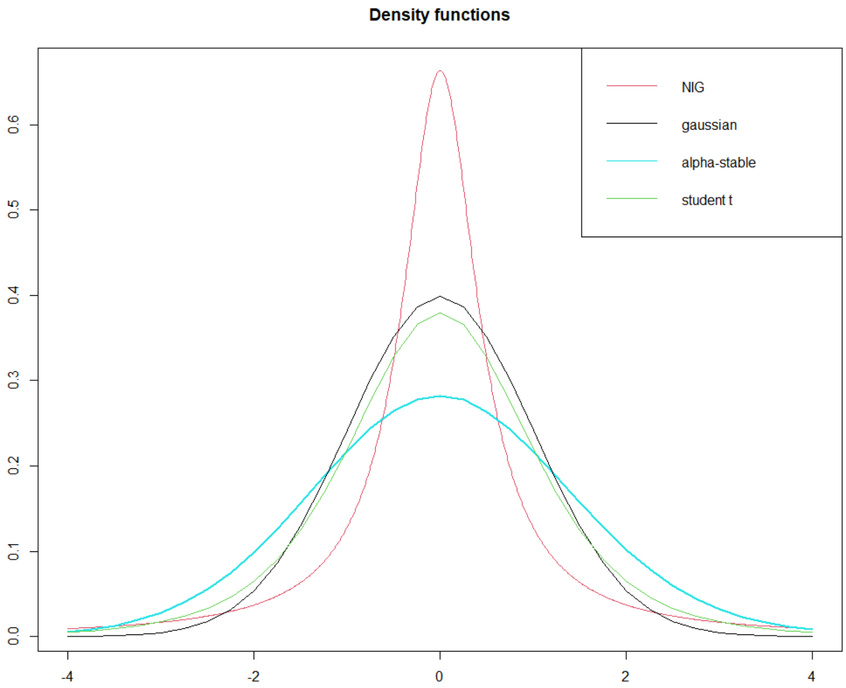


Figure 7: Density functions

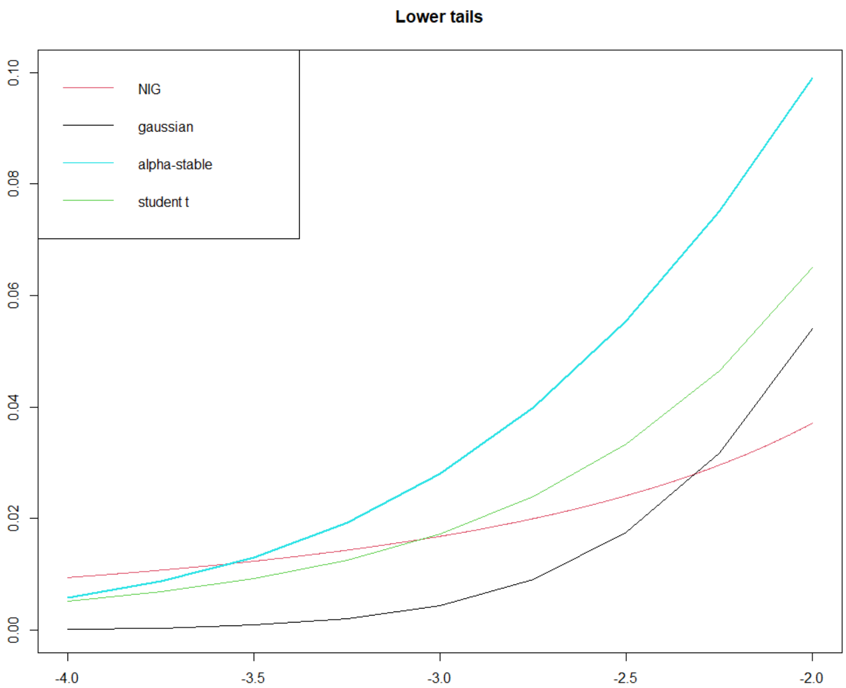


Figure 8: Lower tails of density functions

Following sections we will discuss the theoretical framework of Student t and α-stable distributions.

3.3.1 Student's t-distribution

The Student's t-distribution is very similar to the normal distribution, but it is fat-tailed. This is why we would like to use it in our model. Its probability density function is given by:

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2},$$

where ν is the degrees of freedom parameter and Γ is the gamma function.

The probability of density function is symmetric and we represented this distributions with different degrees of freedom, which is shown in figure 9.

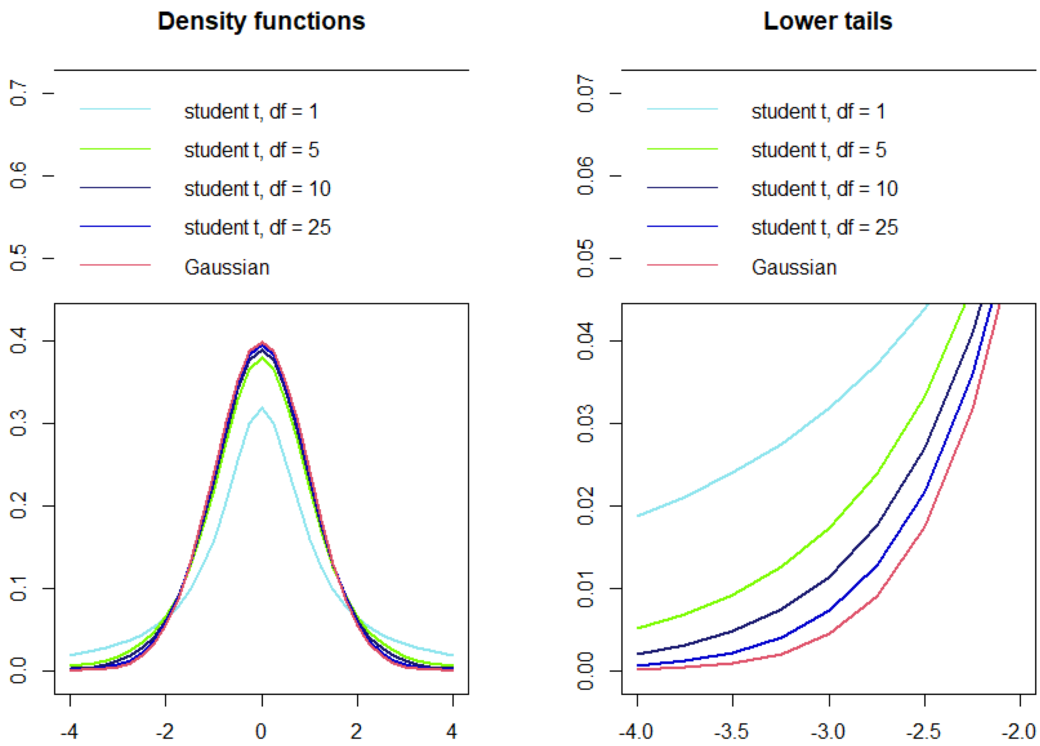


Figure 9: Student t distribution and their lower tails

As we increase the degrees of freedom, the student's t distribution is approaching the normal distribution. We like the heavy-tailed ones, so during the implementation we are testing with df = 1 and df = 5 degrees of freedom. Therefore in the factor model X and ϵ_i are generated from the student t distribution with ν degrees of freedom, so Z_i is also student t distributed with ν parameter.

3.3.2 α-stable

The α-stable distribution can be used for fat-tailed problems and usually during the CDO pricing. By definition, X has stable distribution if to any constants $a > 0, b$ exist $c > 0, d$ constants that $aX + bY$ has the same distribution as $cZ + d$, where X, Y are independent and X, Y, Z are from same distributions. Its characteristic function:

$$\mathbb{E}\left[e^{iuZ}\right] = \begin{cases} e^{-|u|^{\alpha}\cdot[1-i\beta\tan\left(\frac{\pi\alpha}{2}\right)(u)]}, & \text{if } \alpha \neq 1 \\ e^{-|u|\cdot[1+i\beta\tan\left(\frac{2}{\pi}\right)(u)\ln|u|]}, & \text{if } \alpha = 1 \end{cases}$$

where $0 < \alpha \leq 2$ index and $-1 < \beta < leq 1$ skewness parameters.

X is α-stable with $S_{\alpha}(\alpha, \beta, \gamma, \delta, 1)$ parameterization, if:

$$X := \begin{cases} \gamma Z + \gamma, & \text{if } \alpha \neq 1 \\ \gamma Z + \left(\delta + \beta\frac{2}{\pi}\gamma\ln\gamma\right), & \text{if } \alpha = 1 \end{cases}$$

γ is the scale parameter ($\gamma > 0$) and δ represent the location. In the standard normal distribution, the parameters are $S_{\alpha}(\alpha, \beta, 1, 0, 1) := S_{\alpha}(\alpha, \beta, 1)$ and we can match this distributions with other widely used distributions, line Gaussian ($\alpha = 2$), Cauchy ($\alpha = 1$) and Lévy ($\alpha = 0.5$). In our model we used it with $\beta = 0$ parameter – because we would like to use symmetric distributions – and some different α to observe different results. Figure 10 shows the density functions and zoom in their lower tails for the different α parameterization. See that each of them are heavier tailed than the Gaussian distribution.

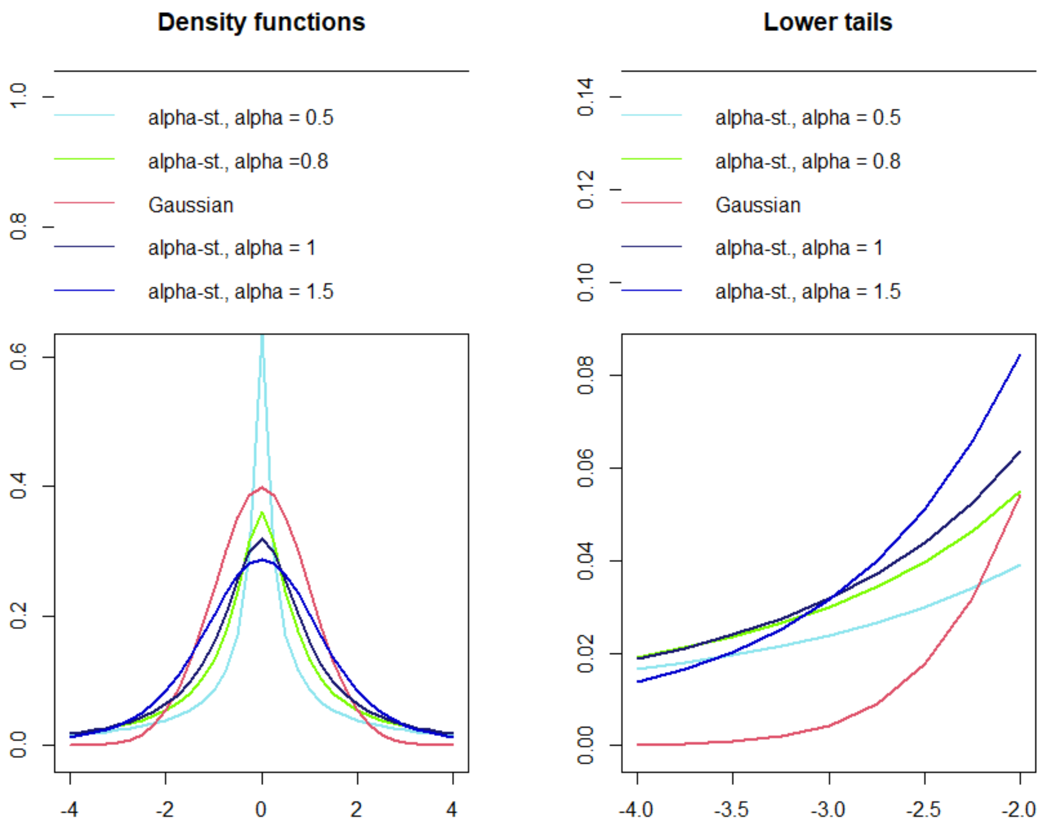


Figure 10: α-stable distributions and their lower tails

In our base model we can change the normal distributions of the common factor and the individual factors to α-stable distribution in the factor model, so X and ϵ_i follow two independent α-stable distributions:

$$X \sim S_\alpha(\alpha, \beta, 1)$$
$$\epsilon_i \sim S_\alpha(\alpha, \beta, 1).$$

From that we can rewrite the equation (22) and get that Z_i will be also from α-stable distribution:

$$Z_i \sim S_\alpha(\alpha, \beta, 1).$$

4 Simulation results

In the previous sections we saw the pricing methods of CDS and Synthetic CDO contracts, and some improvements to the basic model. In this section our aim is to show numerical results of the above mentioned models and explain the outputs. We implemented the simulation using the R 3.5.2 version and used the *"bootsrapCDS"* and *"priceCDS"* functions in "credule" package of R for the pricing.

4.1 Implementation results

In the financial market the two biggest CDX families are the CDX North American and Emerging Markets; and the iTraxx indices so we worked with these market data. These indices are standardized so their attachment and detachment points, and their premiums, recovery rate are fixed. Given that the expected loss has the main role in CDO pricing, in the following paragraphs we discuss it in more details.

4.1.1 Simple examples

To start the testing we simulate the expected loss of a CDS index tranche and check the correctness of the code. We assumed the following simple properties for the test portfolio: $n = 100$ names, $R = 0$ constant recovery rate, low (10%) and high (90%) default probability for every name and ρ correlation goes from 0 to 1 by 0.005. The attachment and detachment points are coming from the CDX.NA.IG index. The next figures show the results of the simulations which are in line with our expectations.

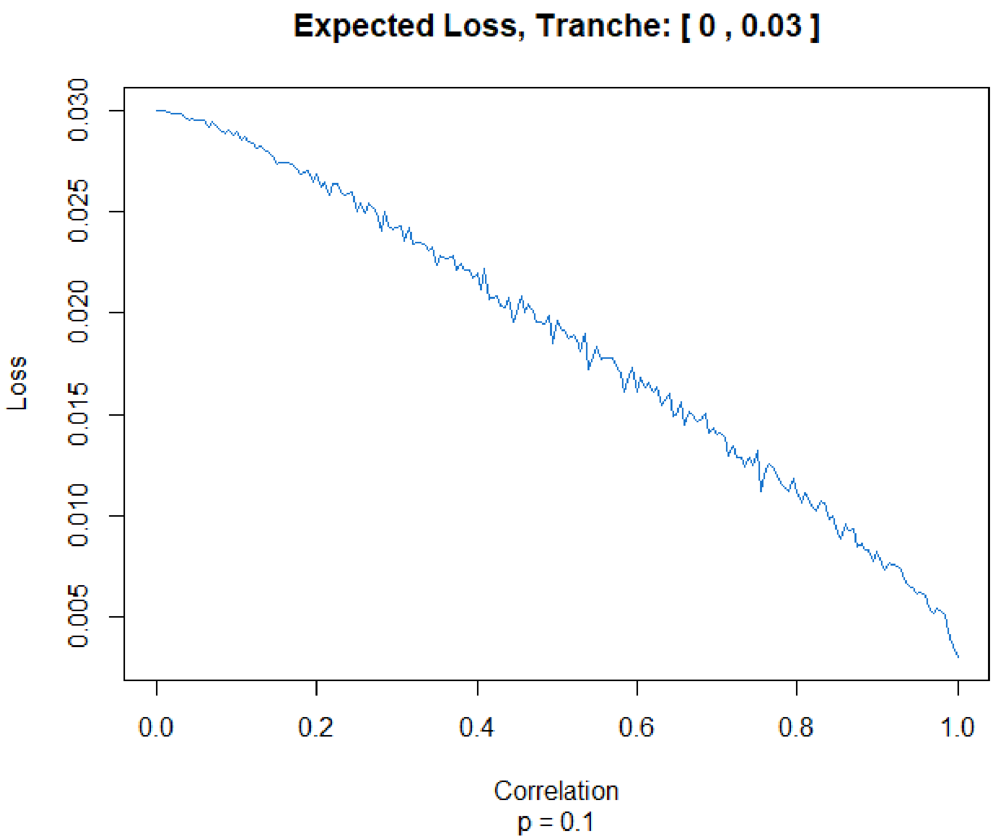


Figure 11: Equity Tranche

In figure 11 we selected the $[0, 0.03]$ equity tranche and 10% probability of default. As we expected when the correlation is zero the expected tranche loss is around 3% and as the correlation increases, the expected loss decreases to almost 0. This is because when there isn't correlation between the reference entities, so they don't move together, and the default probability is 10%, it means that around 10 names will default out of 100, so the expected loss is also around 10% for the total index, and from that we see that the $[0, 0.03]$ equity tranche is fully wiped out, so their expected loss is the detachment point, 3%. In addition, when there is a strong correlation between the reference entities, so they move together and the default probability stays at 10%, it means that 10% of the names have chance to default in the total portfolio and in the equity tranche also, so the expected loss of the $[0, 0.03]$ tranche is going to $3\% \cdot 10\% = 0.003$.

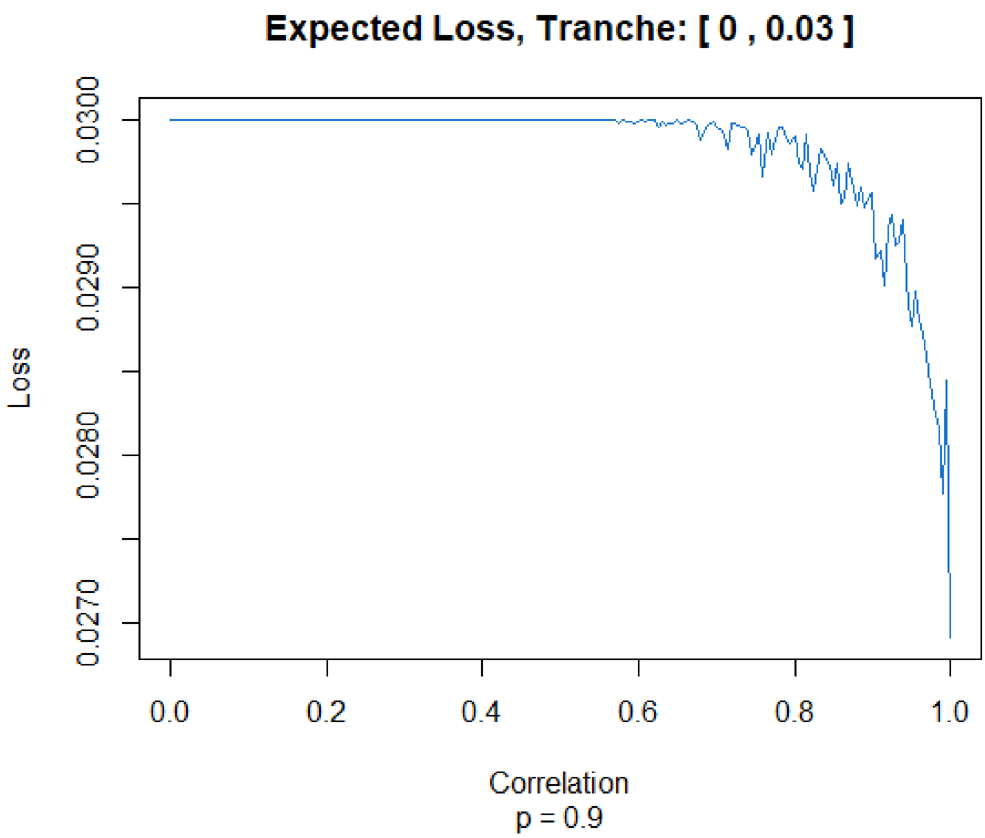


Figure 12: Equity Tranche

In figure 12 we stayed with the previous example, but change the default probability to 90%. The same argument can be explained in the zero correlation case as before, in which the 90% default probability causes 90 name default out of 100, so the expected loss of the $[0\%, 0.03]$ equity tranche is the detachment point. For now, the strong correlation and high probability of default mean that 90% of the names have chance to default in the total portfolio, so the expected loss in $[0, 0.03]$ equity tranche is closely about the detachment point, 0.03, like in the lower correlation, so we can say that the expected loss in the 0-3% tranche is independent from the correlation.

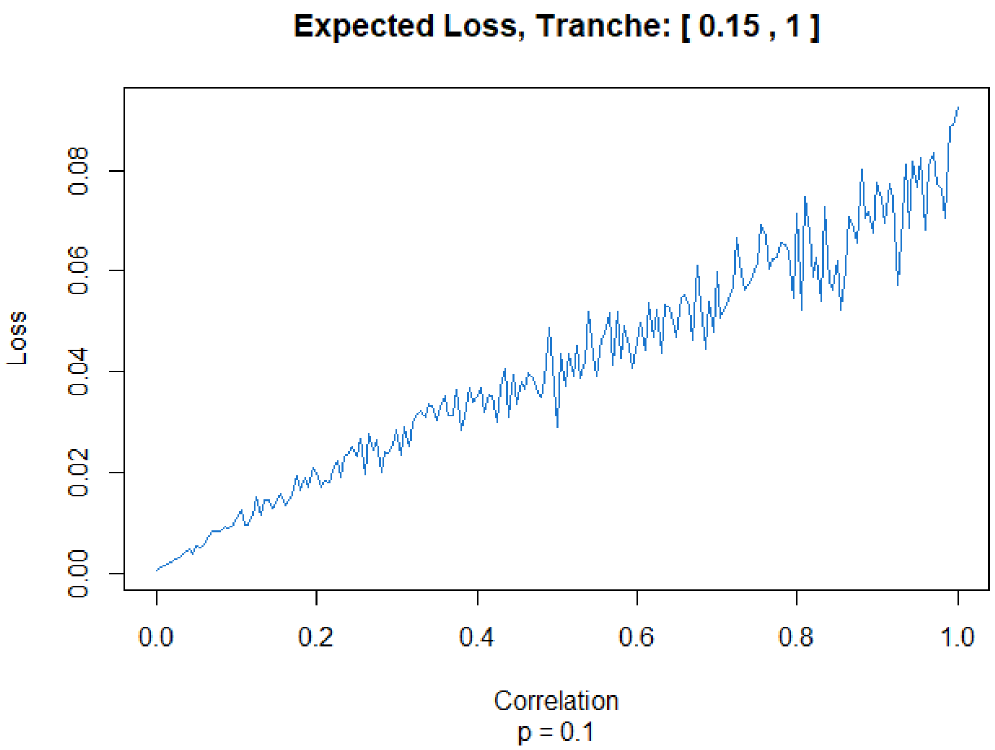


Figure 13: Senior Tranche

In figure 13, we chose a senior tranche in [0.15, 1] interval and 10% probability of default. In that case when the correlation is zero, the expected loss is 0, because only 10 names will be defaulted in the total portfolio and that is lower than the attachment point of the given senior tranche. As the correlation increases, the expected loss increases too, because if the correlation is 1, the expected loss has to be around the multiplication of the 10% and the difference of detachment and attachment point of the senior tranche, which is 0.085. As we approach to the stronger correlation, more names will default.

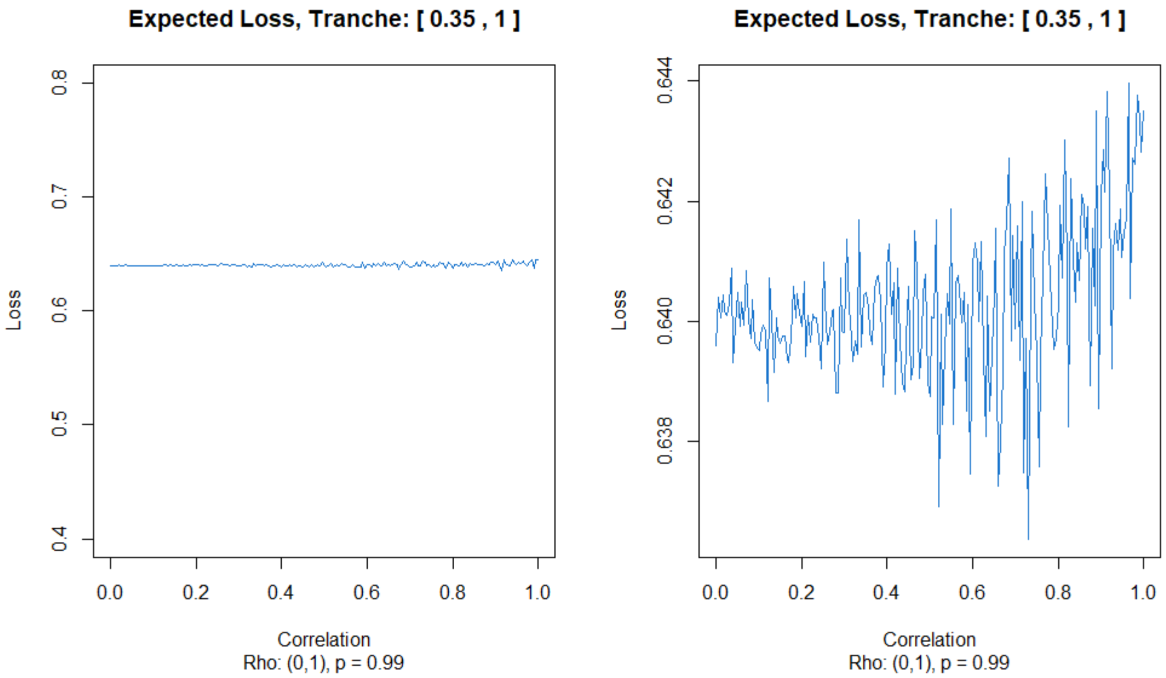


Figure 14: Super Senior Tranche

Figure 14 shows that the expected loss for a $[0.35, 1]$ senior tranche with 90% default probability looks like the above mentioned, that on the left side it is clearly shows that the loss is around the difference of the detachment and attachment point as a function of the correlation, but as we saw on the right side that the expected loss isn't as smooth as we saw in figure on the left, there is a volatility around the exact value.

For now, we saw the behavior of an equity and a senior tranche and we can conclude that these work in the opposite direction. Among them in the structure there are the mezzanine tranches, which inherit both of the equity and senior tranche behaviors.

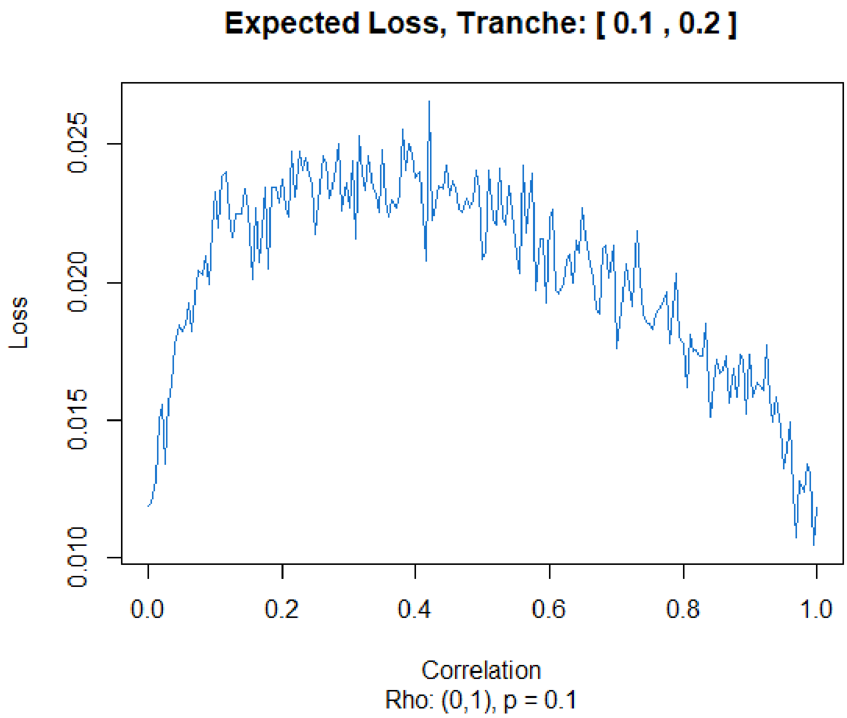


Figure 15: Mezzanine Tranche

Figure 15 shows a $[0.1, 0.2]$ mezzanine tranche. It can be seen that there is a part of the mezzanine tranche which is following the equity tranche methodology and there is another part, which is following the senior tranche methodology. So we can't establish a clear behavior for the mezzanine tranches.

4.2 Stochastic Recovery

As discussed in section 3.2 we implemented the theoretical framework of stochastic recovery model. As mentioned the common factor, X is a random variable from the standard normal distribution, which characterizes the financial market. Therefore first of all, we want to compare the results of two contrary examples of X. For that we simulated the stochastic recovery function and the number of reference entity defaults as a function of the correlation.

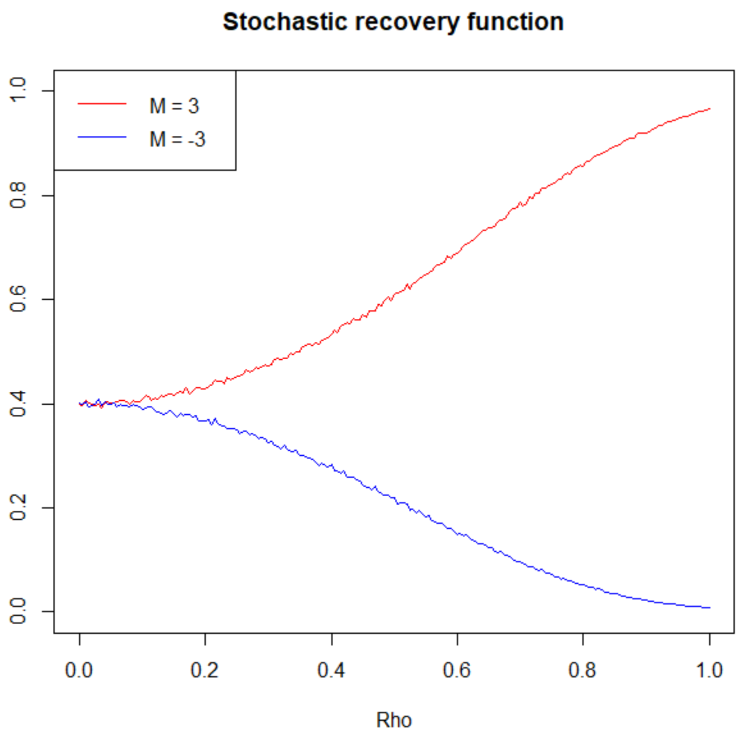


Figure 16: Stochastic Recovery in correlation

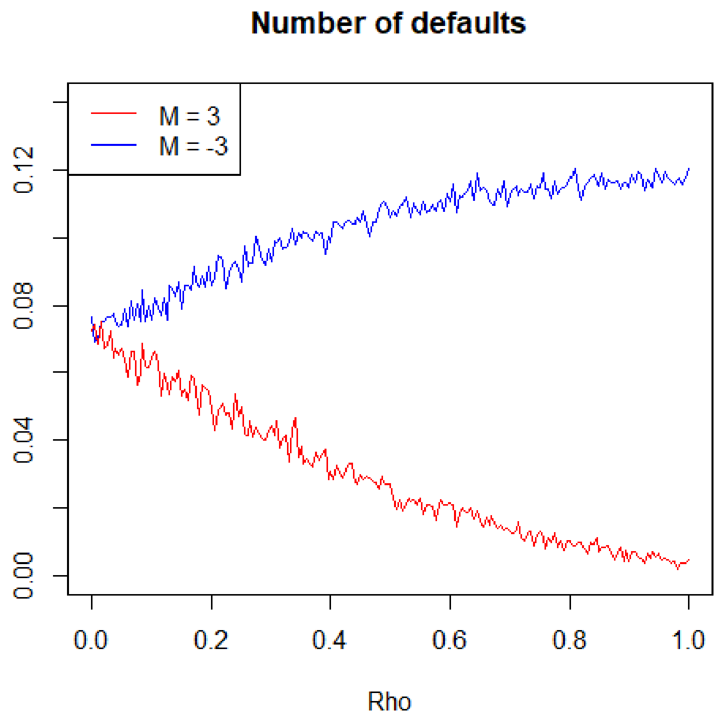


Figure 17: Number of defaults in correlation

We saw in figure 16 and 17 that if the M common factor is positive, so the market is evolving, than the recovery rate function is increasing; and the number of defaults are decreasing as a function of the correlation. Inversely, if the M common factor is negative, so the market is bad, than the recovery rate function is decreasing; and the number of defaults are increasing as a function of the correlation.

However the common factor is a random variable, hence in the simulation we averaged the expected losses for the different tranches and get the following plots.

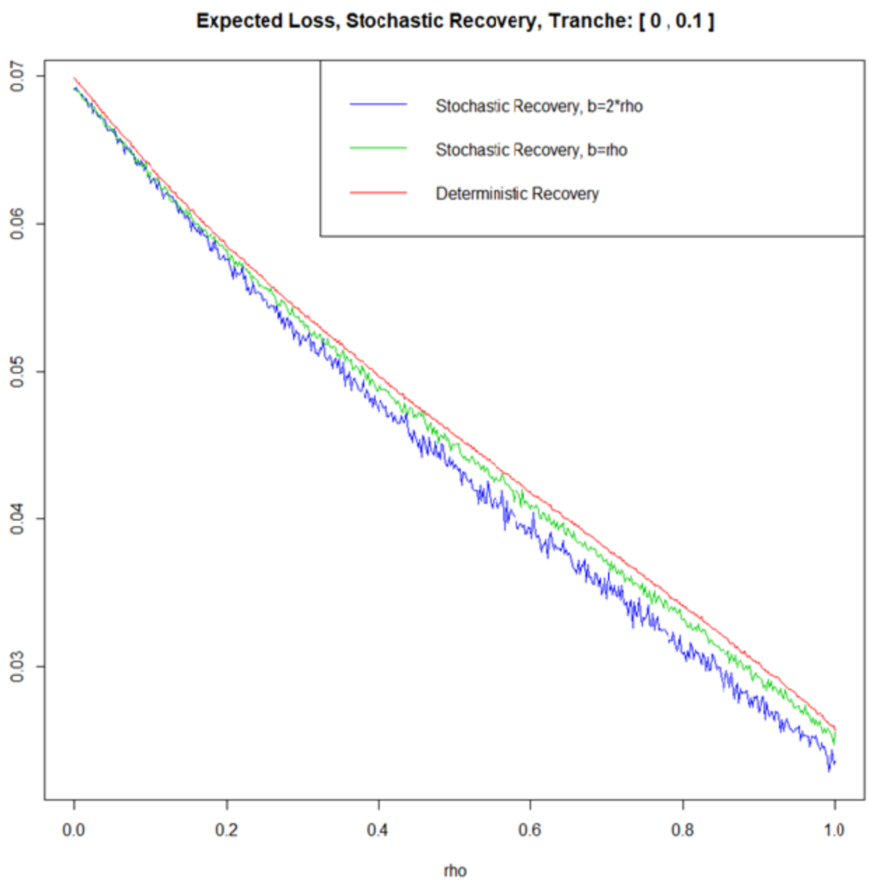


Figure 18: Expected losses of equity tranche with stochastic and deterministic recovery functions

Figure 18 shows the deterministic and stochastic recovery functions for the equity tranche. We set up the parameter of the stochastic recovery to be equal with the correlation and to be two times that. As we saw the expected loss is lower with the stochastic recovery than the deterministic and as we increase the parameter, the expected loss belongs to stochastic recovery will be lower and lower.

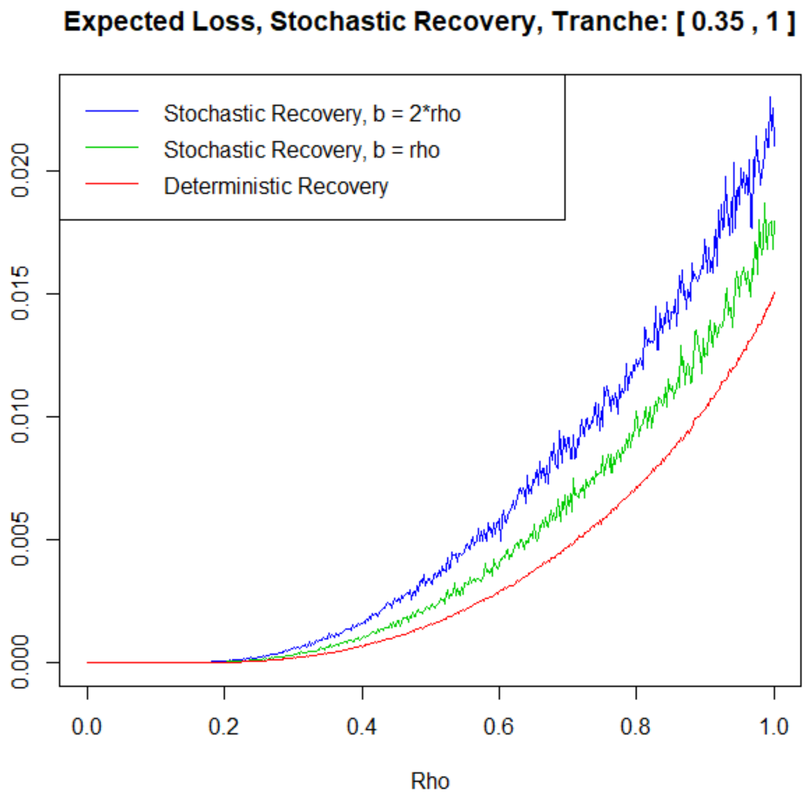


Figure 19: Expected losses of senior tranche with stochastic and deterministic recovery functions

Figure 19 shows the deterministic and stochastic recovery functions for the senior tranche and the parameterization is same as above. As we saw we can reach higher losses with the stochastic recovery function for the senior tranche.

Widen the range of expected loss either for the senior tranche by increasing it or for the equity tranche by decreasing.

Moreover as we saw in the section 4.1.1, there isn't clear behavior for the mezzanine tranche and it is presented in the next figure:

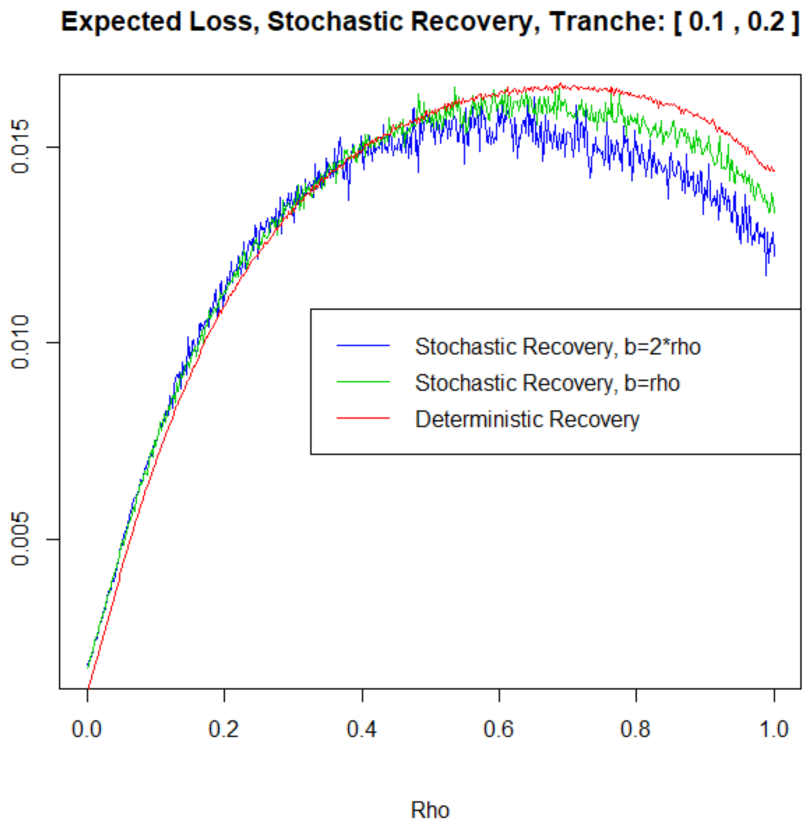


Figure 20: Expected losses of a mezzanine tranche with stochastic and deterministic recovery functions

As we see in figure 20 there is a part of the mezzanine tranche, which inherit the equity tranche behavior and there is another part, which is following the senior tranche behavior.

4.3 Copula extensions

As we wrote in the section 3.3 we generate our model with different distributions. During the numerical testing we used more parameterization to the copulas whose density functions are shown in figure 21:

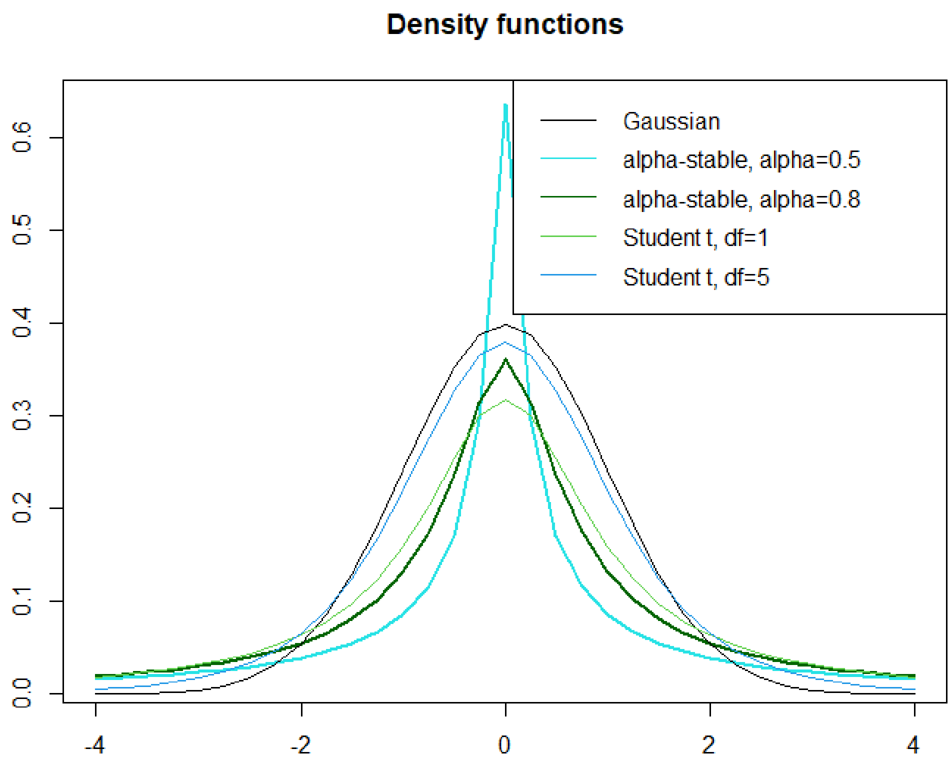


Figure 21: Density functions

Continuing the representation of the expected losses we calculate it for each tranches. As we expected with these heavy-tailed distributions we could generate higher expected losses then with the Gaussian. These result are shown in figure 22 for an equity tranche; figure 23 for a mezzanine tranche; and figure 24 for the senior tranche.

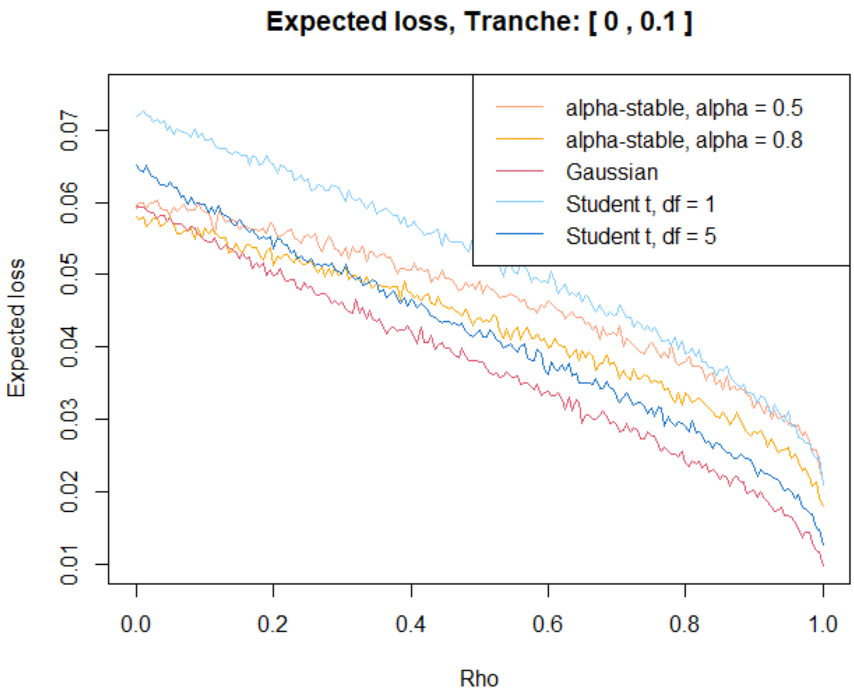


Figure 22: Different copulas for the equity tranche

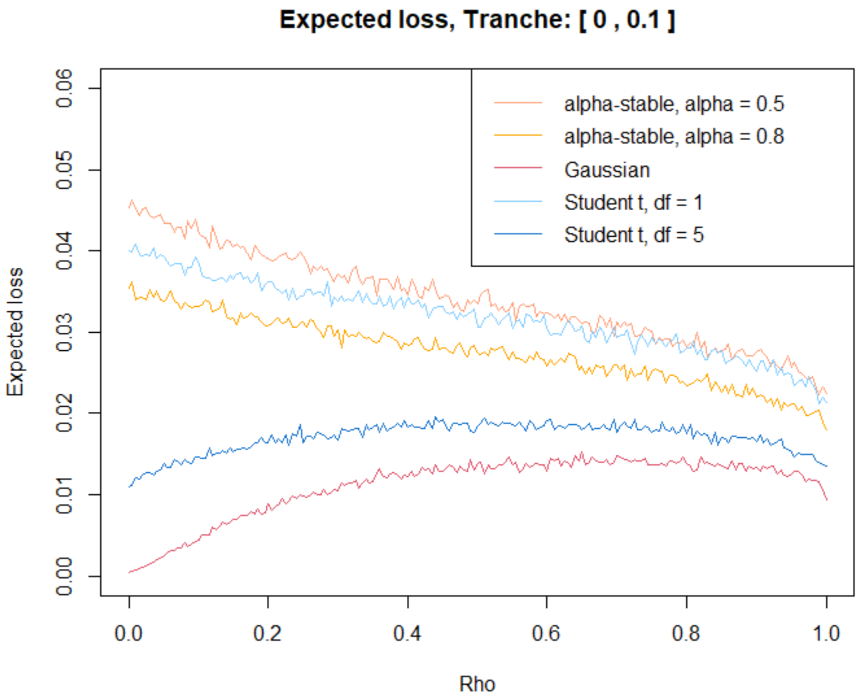


Figure 23: Different copulas for a mezzanine tranche

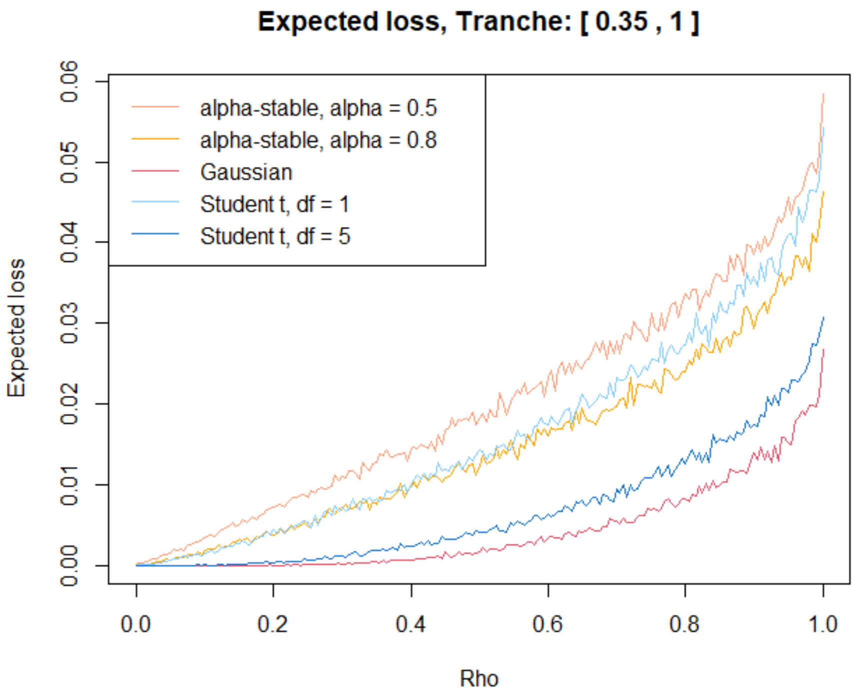


Figure 24: Different copulas for the senior tranche

Each figures clearly show that the generated losses are higher using the new copulas than the Gaussian copula for each tranches.

5 Conclusion

In this thesis, we have introduced a base model for pricing, implementation and testing the CDS index and the synthetic CDO tranches. Our goal was to improve the model's background that the model results will be better in line with the market prices.

In order to reach that firstly, we have presented the base correlation method which is the solution for the correlation skew's problem. Secondly, the stochastic recovery rate function have been introduced which is strengthen the effect of the expected losses as a function of the correlation. Finally, we used new copulas such as Student's t and α-stable distributions.

In section 4 we tested the original and the extended model, mainly via the expected losses in the different tranches. We observed that the equity tranche and the senior tranche work in the opposite direction while the mezzanine tranche inherit both of these behaviours. We saw that we can widen the range of the tranche expected losses as a function of correlation by using stochastic recovery. Moreover we saw that we can reach higher losses with different copula models at same correlation, which is better in line with the market prices.

For further research we would recommend the bespoke CDO contract pricing and testing at more generalized models, such as the stochastic correlation method.

As a conclusion, we saw during the implementation and simulations that is worth to use more general models in order to be closer to the real world. However the draw back is that we also introduce more parameters which can not be easily calibrated.

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A Appendix

Összefoglalás

A szakdolgozat célja, hogy bemutassa a CDS Index Tranchek modellezését és vizsgálja azt. A CDO-k és a CDS-ek a 2007-2008-as pénzügyi válságkor voltak a legnépszerűbb strukturált termékek, de azóta is jelen vannak a piacon. A dolgozatban a saját hozzájárulás az egyes modellek R programban történő implementálása, a várható veszteségek vizsgálata a különböző esetekben és a kapott eredmények elemzése.

Először az 1990-es években jelentek meg a származtatott termékek piacán a CDO-k, majd szépen nőtt a piaci jelentőségük és megjelentek a szintetikus CDO-k, melyek "kosarában" nem hitelek és kötvények voltak, mint eddig a CDO-knál, hanem más hitelderivatív termékek, mint például a CDS-ek. A 2000-es évek elején egyre elterjedtebbé váltak, míg eljutottunk oda, hogy a gazdasági világválság egyik főszereplői lettek. Ennek hatására szigorúbban kezdték szabályozni ezeket a termékeket, így ekkor került bevezetésre több sztenderdizálás is velük kapcsolatban. Rögzítették az egyes indexek szeletekre való felosztását, pl: CDX.NA.HY: (CDS index, észak amerikai magas hozamú termék) $[0, 0.15]$ – equity tranche; $[0.15, 0.25]$ – junior mezzanine tranche; $[0.25, 0.35]$ – senior mezzanine tranche; $[0.35, 1]$ senior tranche, valamint a termékek tulajdonságait is, mint például fix kupont, melyből adódott az upfront prémium használata.

Miután a szakdolgozat elején megismerkedtünk az egyes termékekkel, köztük az egynevű, többnevű CDS-ekkel, indexekkel és az index szeletekkel, megmutattuk hogyan történik ezek árazása. Először felírva az egyetlen referencia alanyból álló CDS termék árazását, majd a már több referencia alanyból álló szintetikus CDO-k árazását is. Az árazáshoz hozzátartozik a hazard ráta ismerete és az egyfaktoros Gauss kopula bevezetése. Ezzel fel is építettük az alap modellünket, melyben a várható veszteség kiszámolására nagy hangsúlyt fektettünk, hiszen az implemetálás során majd annak a viselkedését szeretnénk vizsgálni.

A 3. fejezetben foglalkoztunk azzal, hogy mikkel tudnánk fejleszteni a modellt ahhoz, hogy az jobban közelítse a piaci értékeket. Ennek első pontja a korrelációs mosoly kiküszöbölésére szolgáló úgynevezett alap korreláció bevezetése, melynek lényege, hogy mindegy egyes szeletet két equity tranche segítségével határozunk meg. Második pont a sztochasztikus megtérülési ráta bevezetése volt. Lényege, hogy várható értékben meg kell egyeznie a determinisztikus megtérülési rátával, azonban véletlen faktorral dolgozik, melyet a Gauss kopula által, normális eloszlásból generálunk. Harmadik pontban felírtuk a modellünket más kopulákból generálva, Student t és α-stabilis eloszlásokból a nagyobb veszteségek elérésének céljából. Ezek az eloszlások ugyanis vastagabb farkúak, mint a normális eloszlás, ezáltal nagyobb valószínűséget tulajdonítanak a szélsőséges eseteknek, mint az equity vagy a senior tranche.

Végül a 4. részben implemetáltuk és teszteltük, hogyan viselkednek azok a metódusok, melyeket az előzőekben tárgyaltunk. Pár példán ellenőriztük a korreláció függvényében a várható veszteség értékét, hogy azt kapjuk az alapmodelltől, amit várunk tőle, majd kiterjesztettük a sztochasztikus megtérülési rátával, ott is 2 paraméterrel. Ahogy vártuk, a korreláció függvényében a várható veszteség hatása felerősödik sztochasztikus megtérülési rátával. Vagyis equity tranche esetén a veszteség kisebb lesz, senior tranche esetén nagyobb. Tudjuk, hogy a mezzanine tranche ezen 2 ötvözete, ezért ott ezt nem tudjuk egyértelműen megállapítani. Van olyan része, ahol kevesebb lesz a veszteség sztochasztikus rátával és van olyan, ahol nagyobb.

Ugyanígy bemutatva azt is, mikor nem Gauss kopulából generálunk, hanem Student *t* és α-stabilisból, mindegyikből két értéket adva a paraméternek. A kapott 5 eredményt vizsgálva egyértelműen látszik mindegyik szelet esetén, hogy a Gauss kopulából generált várható veszteség a legkisebb, és ahogy haladunk a minél vastagabb farkú eloszlások felé, úgy nőnek az egyes veszteségek is a korreláció függvényében.

Összefoglalva azt mondhatjuk, hogy az egyes kiterjesztések szükségesek a modell jobb piachoz való illeszkedése szempontjából, azonban minden egyes ilyen fejlesztéssel egyre több paraméter jön be és egyre nehezebb lesz a kalibráció. További fejlesztési terület lehet még a sztochasztikus korreláció esetének vizsgálata, illetve akár a bespoke CDO-k bemutatása is.

Szószedet

attachment point: csatlakozási pont

base correlation: alap korreláció

Basket Default Swap: több referencia entitásból álló CDS termék

CDO (Collateralized Debt Obligation): fedezett adósságjellegű kötelezettség

CDS (Credit Default Swap): hitel-nemteljesítési csereügylet – egy olyan hitelderivatív termék, mely

correlation skew: korrelációs mosoly

detachment point: lekapcsolódási pont

expected loss: várható veszteség

recovery rate: megtérülési ráta

reference entity: referencia entitás

Single-Name CDS: egyetlen referencia entitásból álló CDS termék

upfront premium: a rögzített prémium feletti érték az egyes tranche-ekhez, hogy az árak igazságosak maradjanak

tranche: szelet

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