Eötvös Loránd University
Faculty of Science
Institute of Mathematics

Classification of Tensors and Tensor Rank

Szabolcs Andó-Kinorányi

Thesis Advisor:
Mátyás Domokos, Alfréd Rényi Institute of Mathematics

Internal Advisor:
Péter Frenkel, Department of Algebra and Number Theory

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A **diplomamunka** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 20 22.12.31.

a hallgató aláírása

Abstract

A direct product of general linear groups acts on a tensor product of a finite number of finite dimensional vector spaces. This thesis is about the orbits of this group action. In the case of 2 by m by n tensors, we reduce the problem of classifying these orbits to a different problem about classifying the orbits of multisets of pairs of binary forms and integers. For algebraically closed fields, and the field of real numbers we classify the orbits of 2 by m by n tensors for $m \leq n$ and $m \leq 4$. When $m = 2$ then the classification is computed over arbitrary fields. These results also help in giving a somewhat different proof to a known result by Parfenov about when the number of orbits is finite.

Acknowledgement

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I would like to thank my wife for supporting me throughout my Master's degree, and for helping me emotionally during the writing of this thesis. I am also indebted to my family for all the help in my education.

Contents

1 Introduction

"In the age of big data, the role of matrices is increasingly played by tensors, that is, multidimensional arrays of numbers." This quote is from the book [18] written by Mateusz Michałek and Bernd Sturmfels, on page 137. Tensors are very interesting objects naturally showing up in many different areas of mathematics. The classification of tensors is the main subject of this thesis. What we shall consider is the problem of finding canonical forms for tensors in $K^{k_1} \otimes ... \otimes K^{k_r}$ with respect to the action of $\mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_r, K)$. It is clearly sufficient to deal with the case $2 \leq k_1 \leq ... \leq k_r$. Of course, this is very difficult in general, so we can look at simpler questions, such as finding all r-tuples $(k_1, ..., k_r)$ for which there are finitely many orbits, or classifying the orbits for small values of r and for specific k_i-s.

In 1890, Kronecker ([14]) classified all the orbits of pairs of $m \times n$ matrices (or alternatively tensors in $K^2 \otimes K^m \otimes K^n$) with respect to the action of $\mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$. This can be extended to take into account the effect of $\mathrm{GL}(2, K)$ on the first component of the tensor product. This observation appears in the literature (see [21]), but here we exploit this opportunity to a greater extent. The problem of classifying the orbits in $K^2 \otimes K^m \otimes K^n$ is reduced to a different problem concerning $\mathrm{GL}(2, K)$-orbits of multisets of pairs of binary forms and integers in Corollary 4.3.15. To the author's knowledge, this is a new result.

The following result is due to Parfenov ([21]), although it follows from more general results by Kac ([13]). In this thesis it is Theorem 3.1.1. We will present a proof to this in section 5, that differs from the original to some extent.

Theorem. Assume $2 \leq k_1 \leq ... \leq k_r$ and $K = \mathbb{C}$. The number of $\mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_r, K)$-orbits in $K^{k_1} \otimes ... \otimes K^{k_r}$ is finite if and only if the r-tuple $(k_1, ..., k_r)$ is one of the following: $(n), (m, n), (2, 2, n), (2, 3, n)$.

What about the case when K is an arbitrary infinite field? As explained in Remark 3.1.2, K^n and $K^m \otimes K^n$ have finitely many orbits, and there are fields, where these are the only cases. One example is $\mathbb{Q}$, as stated and proved in Corollary 4.4.10. This is done by classifying the orbits in $K^2 \otimes K^2 \otimes K^n$ over arbitrary K in Propositions 4.4.7 and 4.2.11. The author could not find this result elsewhere in the literature.

Section 2 contains Kronecker's classification theorem and related results. Here the theory of matrix pencils is discussed, and their connection to quiver representations is pointed out. Kronecker's theorem can be interpreted in both of these categories. This serves as theoretical background for the rest of the thesis.

Section 3 is about the orbits in $K^{k_1} \otimes ... \otimes K^{k_r}$. It contains a result that states that if we embed a tensor product of vector spaces in a bigger tensor space canonically, then the orbits do not change. This is Corollary 3.2.7, and it is stated in [21] as Corollary 2.1 (a), but we present it with a different, more elementary proof. The end of this section indroduces the notion of the rank of a tensor. This is an interesting invariant of the orbits. A major motivation for the study of tensor rank is that the rank of the multiplication tensor of matrices approximates very well the number of multiplications one has to perform in the base field in order to compute the product of two matrices (see Chapter I in [15] and Chapter 15 in [4]).

In section 4, we make use of the results of the previous two sections as we point out the connection between matrix pencils and tensors in $K^2 \otimes K^m \otimes K^n$. We prove important statements that will later help prove Parfenov's theorem, the most important of which is reducing the classification of tensors in $K^2 \otimes K^m \otimes K^n$ to the classification of a subset of tensors in $K^2 \otimes K^i \otimes K^i$ for $i \leq m$. This section also contains the reduction of the problem of classifying the orbits in $K^2 \otimes K^m \otimes K^n$ to classifying the $\mathrm{GL}(2, K)$-orbits of multisets of pairs of binary forms and integers. In the last subsection we present the classification of orbits in $K^2 \otimes K^2 \otimes K^n$ for arbitrary K.

Section 5 contains the proof of Parfenov's theorem about the finiteness of the number of orbits in a tensor product of vector spaces. In this section K is algebraically closed. Because of the results of the previous two sections we only need to consider the spaces $K^2 \otimes K^2 \otimes K^2, K^2 \otimes K^3 \otimes K^3, K^2 \otimes K^4 \otimes K^4, K^3 \otimes K^3 \otimes K^3, K^2 \otimes K^2 \otimes K^2 \otimes K^2$. During the proof we classify (in Propositions 4.2.11 and 5.1.10) all the $\mathrm{GL}(k_1, K) \times \mathrm{GL}(k_2, K) \times \mathrm{GL}(k_3, k)$-orbits in $K^2 \otimes K^2 \otimes K^n, K^2 \otimes K^3 \otimes K^n$ and $K^2 \otimes K^4 \otimes K^n$, the last of which is a new result, as far as the author knows.

In section 6, we present another application of the results of sections 3 and 4 as we prove the results of the previous section also for $\mathbb{R}$ (which is not algebraically closed), namely the classification of orbits in $\mathbb{R}^2 \otimes \mathbb{R}^m \otimes \mathbb{R}^n$ for $m \leq 4$ (Propositions 4.2.11, 5.1.10, 6.1.1, 6.2.1, 6.3.5), and the theorem about the finiteness of orbits which holds unchanged for $\mathbb{R}$ (Theorem 6.4.1). This last result was proved earlier by different methods in [6].

2 Kronecker's classification theorem

2.1 Generalized Jordan normal form

To be able to state Kronecker's classification theorem in its most general form, we will need to introduce the concept of the generalized Jordan normal form of a matrix. This subsection contains a brief summary of section 22.5 of the book [20].

Definition 2.1.1 (Companion matrix)**.** If $p(x) = c_0 + c_1 x + c_2 x^2 + ... + c_{n-1} x^{n-1} + x^n \in K[x]$ is a monic polynomial, then its *companion matrix* is the matrix

$$C(p) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}.$$

Definition 2.1.2 (Generalized Jordan block)**.** Let U be a square matrix whose only nonzero entry is a 1 in the top right corner, and let $C(p)$ be the companion matrix of an irreducible monic polynomial. A *generalized Jordan block* is a block matrix of the form

$$\begin{pmatrix} C(p) & 0 & 0 & \cdots & 0 \\ U & C(p) & 0 & \cdots & 0 \\ 0 & U & C(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & U & C(p) \end{pmatrix}.$$

Theorem 2.1.3 ([20] Chapter 22. Section 5. Theorem 5.5)**.** Every square matrix $M \in K^{n \times n}$ is similar to a block diagonal matrix whose blocks are generalized Jordan blocks. These blocks are unique up to reordering.

2.2 Matrix pencils

This subsection contains the summary of Chapter XII. of The Theory of Matrices Vol. II. [10] by Felix Gantmacher.

Question 2.2.1 ([10])**.** Let $A, B, A', B' \in K^{n \times m}$ be matrices over a field K. The main question of this subsection is finding necessary and sufficient conditions for the existence of matrices $P \in \mathrm{GL}(n, K)$ and $Q \in \mathrm{GL}(m, K)$ such that

$$PAQ = A', \quad PBQ = B'.$$

We can reformulate this problem by defining the action of the group $\mathrm{GL}(n, K) \times \mathrm{GL}(m, K)$ on the set $K^{n \times m} \times K^{n \times m}$ in the following way. If $(P, Q) \in \mathrm{GL}(n, K) \times \mathrm{GL}(m, K)$ and $(A, B) \in K^{n \times m} \times K^{n \times m}$, then

$$(P, Q) \cdot (A, B) = (PAQ, PBQ).$$

Answering question 2.2.1 is equivalent to classifying the orbits of this group action.

Definition 2.2.2 (Matrix pencil)**.** A *matrix pencil* is an expression of the form $sA + tB$, where $A, B \in K^{n\times m}$ are matrices and s, t are commuting variables over the field K. In other words, a matrix pencil is a matrix whose entries are homogeneous linear elements of the polynomial algebra $K[s,t]$.

Definition 2.2.3 (Regular matrix pencil)**.** A matrix pencil $sA + tB$ is called *regular* if A and B are square matrices and $\det(sA + tB)$ as a polynomial in s and t is not the zero polynonial.

The group action above is the same as the action of $\mathrm{GL}(n, K) \times \mathrm{GL}(m, K)$ on the space of matrix pencils over K of size $n \times m$. Here a pair of matrices (P, Q) acts on a matrix pencil $sA + tB$ as

$$(P, Q)(sA + tB) = P(sA + tB)Q = sPAQ + tPBQ.$$

Definition 2.2.4 (The category of matrix pencils, [3] Chapter I, Example 2.5)**.** Fix an arbitrary field K. A *morphism* between matrix pencils $sA + tB$ and $sA' + tB'$ is a pair of matrices (P, Q) over K, for which $Q(sA+tB) = (sA'+tB')P$. The composition of morphisms (P, Q) and (P', Q') is $(P'P, Q'Q)$. The identity morphisms are (I, I), where I is the identity matrix. Matrix pencils together with these morphisms form a category.

Remark 2.2.5. Let us fix natural numbers n and m. Then the matrix pencils $sA + tB$ where $A, B \in K^{n\times m}$ form a subcategory in the category of matrix pencils. One can observe that two pencils in this subcategory are on the same $\mathrm{GL}(n, K) \times \mathrm{GL}(m, K)$-orbit if and only if there is an invertible morphism between them. This means that answering question 2.2.1 is equivalent to classifying the isomorphism classes of this subcategory.

To be able to state the classification theorem we need some additional notations.

Notation 2.2.6. For a fixed integer $\epsilon > 0$ let

$$L_\epsilon = s \cdot A_\epsilon + t \cdot B_\epsilon,$$

where

$$A_\epsilon = \left.\overbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}}^{\epsilon+1}\right\}\epsilon$$

$$B_\epsilon = \left.\overbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}}^{\epsilon+1}\right\}\epsilon$$

Also, let

$$N_\epsilon = sH_\epsilon + tI_\epsilon,$$

where H_ϵ is the Jordan block of size $\epsilon \times \epsilon$ corresponding to the eigenvalue 0, and I_ϵ is the identity matrix of size $\epsilon \times \epsilon$.

The classification theorem below is due to Kronecker and its proof can be found in chapter XII. of the book [10].

Theorem 2.2.7 (Kronecker, [10])**.** Every matrix pencil is on the same orbit as a block diagonal matrix pencil with blocks of the form

$$0, \text{ or } L_\epsilon, \text{ or } L_\nu^T, \text{ or } N_u, \text{ or } sI + tJ,$$

where 0 is a block of zeroes of arbitrary size, J is and arbitrary generalized Jordan block and I is the identity matrix of the corresponding size. The above form is unique up to reordering of the blocks. Furthermore a matrix pencil is regular if and only if its normal form only contains blocks of types N_u and $sI + tJ$.

Remark 2.2.8. A matrix pencil X is directly decomposable in the category of matrix pencils if and only if there exist matrices $(P, Q) \in \mathrm{GL}(n, K) \times \mathrm{GL}(m, K)$ such that PXQ is block diagonal with at least two blocks. This is because $\mathrm{diag}(Y_1, Y_2)$ is a direct sum of matrix pencils Y_1 and Y_2 in the categorical sense. This means that the theorem above says that the directly indecomposable matrix pencils up to the action of $\mathrm{GL}(n, K) \times \mathrm{GL}(m, K)$ are exactly the matrices of type $0, L_\epsilon, L_\nu^T, N_u, sI + tJ$, and every matrix pencil is the direct sum of finitely many indecomposable ones.

2.3 Representations of the Kronecker quiver

For the following subsection I used section III.2. of the book [3] by Assem, Simson and Skowronski.

Definition 2.3.1 (Quiver)**.** A *quiver* is a directed graph possibly with loops and multiple edges. The notation $G = (V, E)$ for a quiver G means that V is the set of vertices and E is the set of edges. A quiver is *finite* if it has finitely many vertices and edges.

Definition 2.3.2 (Representation)**.** Let $G = (V, E)$ be a quiver and K be an arbitrary field. For every $v \in V$ let M_v be a vector space over K and for every $e = (v_1, v_2) \in E$ let $\varphi_e : M_{v_1} \to M_{v_2}$ be a (K-) linear map. Then $M = (M_v, \varphi_e)_{(v \in V, e \in E)}$ is called a *representation* of G. A representation is *finite dimensional* if M_v is finite dimensional for all $v \in V$.

Definition 2.3.3 (Category of representations)**.** Let $M = (M_v, \varphi_e)_{(v\in V, e\in E)}$ and $N = (N_v, \psi_e)_{(v\in V, e\in E)}$ be representations of the quiver $G = (V, E)$ over the field K. A *morphism* between them is a collection of linear maps $(\alpha_v)_{v\in V}$ such that $\alpha_v : M_v \to N_v$ and $\alpha_{v_2} \circ \varphi_e = \psi_e \circ \alpha_{v_1}, \forall e = (v_1, v_2) \in E$. This yields the category of representations of G over K.

Definition 2.3.4 (Kronecker-quiver)**.** The *Kronecker-quiver* is a graph with two egdes u and v, and two vertices α, β going from u to v.

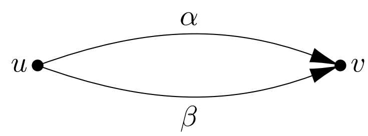

Figure 2.3.1: The Kronecker-quiver

We will use the following theorem from the book titled Categories for the Working Mathematician [17] written by Saunders Mac Lane, where the proof of it can be found.

Theorem 2.3.5 ([17] Chapter IV Section 4 Theorem 1)**.** A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if and only if it is full and faithful, and for each object $d \in \mathscr{D}$ there exists an object $c \in \mathscr{C}$ such that d is isomorphic to Fc.

Proposition 2.3.6 ([3], Chapter I, Example 2.5)**.** The category $\mathscr{C}$ of matrix pencils over a fixed field K is equivalent to the category $\mathscr{D}$ of **finite dimensional** representations of the Kronecker quiver over K via the functor

$$F : \mathscr{C} \to \mathscr{D}$$

that acts in the following way. On the objects

$$\begin{aligned} &F : \mathrm{Ob}\mathscr{C} \to \mathrm{Ob}\mathscr{D} \\ &sA + tB \mapsto (M_v, \varphi_e)_{(v\in V, e\in E)}, \end{aligned}$$

where if $A, B \in K^{n\times m}$, then $M_u = K^m, M_v = K^n, \varphi_\alpha = A, \varphi_\beta = B$ (here we identify matrices with the operators that multiply vectors by these matrices from the left). If $(P, Q) : sA + tB \to sA' + tB'$ is a morphism, then

$$F((P, Q)) = (\alpha_u = P, \alpha_v = Q),$$

again identifying matrices with the corresponding linear maps.

Proof. F is a functor, because on one hand

$$F((I, I')) = (\mathrm{id}_{K^m}, \mathrm{id}_{K^n}),$$

and on the other,

$$\begin{aligned} F((P, Q) \circ (P', Q')) = F((PP', QQ')) = (PP', QQ') = \\ (P, Q) \circ (P', Q') = F((P, Q)) \circ F((P', Q')). \end{aligned}$$

It is clear that F is an additive functor that is full and faithful, i.e. for $sA+tB, sA'+tB' \in \mathscr{C}$, the map

$$\mathrm{Hom}_{\mathscr{C}}(sA+tB, sA'+tB') \to \mathrm{Hom}_{\mathscr{D}}(F(sA+tB), F(sA'+tB'))$$

is an isomorphism of Abelian groups, because it maps a pair of matrices to a pair of the corresponding linear maps. For a representation M such that $\dim M_u = m, \dim M_v = n$, let us take bases $b_1, ..., b_m$ and $b'_1, ..., b'_n$ in M_u and M_v respectively, and let A, B be the matrices of $\varphi_\alpha, \varphi_\beta$ respectively. Then M and $F(sA+tB)$ are isomorphic via (α_u, α_v):

$$\begin{aligned} \alpha_u : M_u &\to K^m \\ b_i &\mapsto e_i, \quad i = 1, ..., m \\ \alpha_v : M_v &\to K^n \\ b'_i &\mapsto e'_i, \quad i = 1, ..., n, \end{aligned}$$

where $e_1, ..., e_m$ and $e'_1, ..., e'_n$ are the standard bases. From Theorem 2.3.5 we conclude that F is an equivalence of categories. □

Remark 2.3.7. There is a third category that is equivalent to the ones above, and that is the category of modules over the path algebra KG of the Kronecker quiver G. The interested reader can find the theory of path algebras and their connection to representations in [3] Chapters II and III.

From Theorem 2.2.7, Remark 2.2.8, and Proposition 2.3.6 we can conclude the following theorem.

Theorem 2.3.8 (Kronecker)**.** *The following representations of the Kronecker quiver are indecomposable, and every indecomposable representation is isomorphic to one of these:*

M_u	M_v	φ_A	φ_B
K	0	0	0
0	K	0	0
K^{n+1}	K^n	A_n	B_n
K^n	K^{n+1}	A_n^T	B_n^T
K^n	K^n	H_n	I_n
K^n	K^n	I_n	J

Here n is an arbitrary positive integer and J is a generalized Jordan block of size $n \times n$. A_n, B_n, H_n, I_n are as in Notation 2.2.6. Every finite dimensional representation is a direct sum of finitely many indecomposable ones, and this direct sum is unique up to reordering the summands.

3 Orbits in $K^{k_1} \otimes ... \otimes K^{k_r}$

Throughout this section K will denote an arbitrary field, and by *orbits in* $K^{k_1} \otimes ... \otimes K^{k_r}$ we shall mean $\mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_r, K)$-orbits.

In this section we will investigate the orbits of the natural action of $\mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_r, K)$ on the space $K^{k_1} \otimes ... \otimes K^{k_r}$. Our aim is to find the r-tuples $(k_1, ..., k_r)$ for which there are finitely many orbits, and also to classify the orbits in these cases. We may assume that $k_1 \leq ... \leq k_r$ because permutations of the k_i-s will not change the structure of the orbits. We can also assume that all $k_i > 1$, because $V \otimes_K K = V$ for all K-vector spaces V.

3.1 Parfenov's theorem

The following result is due to Parfenov and the proof can be found in the article [21]. As it is noted in [21], it follows from more general results in [13]. We shall give a somewhat different proof of it below.

Theorem 3.1.1 ([21] Theorems 3 and 5.)**.** If $K = \mathbb{C}$, then there are finitely many orbits for the following r-tuples: $(n), (m, n), (2, 2, n), (2, 3, n)$. For every other r-tuple there are infinitely many orbits.

Remark 3.1.2. Let us examine the cases where $r < 3$.

- When $r = 1$ then we are looking for the orbits in K^n under the action of $\mathrm{GL}(n, K)$. Clearly there are two orbits: $\{0\}$ and $K^n \backslash \{0\}$.
- In the case of $r = 2$ we ought to classify matrices in $K^{m \times n}$ under the action $\mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$. We know from elementary linear algebra that if the rank of M is i, then M is on the same orbit as $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ where I is the $i \times i$ identity matrix. We also know that matrices of different ranks are on different orbits. This means that there are $\min\{m, n\} + 1$ orbits: one for every possible value of the rank.

3.2 The case of infinitely many orbits

Notation 3.2.1. Let r be a positive natural number and fix $i \in \{1, ..., r\}$. Let us define the following correspondence on elementary tensors:

$$\begin{aligned} \varphi^i : K^{k_1} \otimes ... \otimes K^{k_r} &\to \mathrm{Hom}_K \left(K^{k_1} \otimes ... \otimes K^{k_{i-1}} \otimes K^{k_{i+1}} \otimes ... \otimes K^{k_r}, K^{k_i} \right) \\ a_1 \otimes .... \otimes a_r &\mapsto \varphi^i_{a_1 \otimes .... \otimes a_r} \end{aligned}$$

where

$$\varphi^i_{a_1 \otimes ... \otimes a_r}(b_1 \otimes ... \otimes b_{i-1} \otimes b_{i+1} \otimes ... \otimes b_r) = (b_1^T a_1) \cdot ... \cdot (b_{i-1}^T a_{i-1}) \cdot (b_{i+1}^T a_{i+1}) \cdot ... \cdot (b_r^T a_r) \cdot a_i.$$

Then $\varphi^i_{a_1 \otimes ... \otimes a_r}$ extends as a linear map to $K^{k_1} \otimes ... \otimes K^{k_{i-1}} \otimes K^{k_{i+1}} \otimes ... \otimes K^{k_r}$ and φ^i extends as a linear map to $K^{k_1} \otimes ... \otimes K^{k_r}$. Furthermore φ is a linear isomorphism. This is clear from the canonical isomorphisms $V \cong V^*$ and $V^* \otimes W \cong \mathrm{Hom}(V, W)$.

Notation 3.2.2. If $\psi : V \to W$ is a linear map, then $\mathrm{Ran}(\psi)$ shall denote the range of ψ.

Definition 3.2.3. If $T \in K^{k_1} \otimes ... \otimes K^{k_r}$ then $\mathrm{rk}_i(T)$ is the rank of φ_T^i as a linear map, i.e. $\mathrm{rk}_i(T) = \dim(\mathrm{Ran}(\varphi_T^i))$.

Lemma 3.2.4. If $S, T \in K^{k_1} \otimes ... \otimes K^{k_i} \otimes ... \otimes K^{k_r}$ are on the same orbit, then $\mathrm{rk}_i(S) = \mathrm{rk}_i(T)$.

Proof. If $T = (Q_1, ..., Q_r)S$, then $\varphi_T^i = Q_i \circ \varphi_S^i \circ (Q_1^T, ..., Q_{i-1}^T, Q_{i+1}^T, ..., Q_r^T)$, and composing with invertible linear maps does not change the rank. □

Lemma 3.2.5. Let $S \in K^{k_1} \otimes ... \otimes K^{k_r}$, and assume $i \in \{1, ...r\}$ is fixed. Then there exists a decomposition $S = \sum_{j=1}^m a_1^j \otimes ... \otimes a_r^j$ such that $a_i^j \in \mathrm{Ran}(\varphi_S^i)$ for all $j \in \{1, ..., m\}$.

Proof. Let $w_1, ..., w_n$ be a basis in $\mathrm{Ran}(\varphi_S^i)$ and let us extend it to a basis $w_1, ...w_n, v_{n+1}, ..., v_{k_i}$ of K^{k_i}. We can express S in the basis

$$\{e_{j_1} \otimes ... \otimes w_{j_i} \otimes ... \otimes e_{j_r} : j_1, ..., j_r\} \cup \{e_{j_1} \otimes ... \otimes v_{j_i} \otimes ... \otimes e_{j_r} : j_1, ..., j_r\}$$

as

$$S = \sum_{j_1=1}^{k_1} ... \sum_{j_i=1}^{n} ... \sum_{j_r}^{k_r} \lambda_{j_1,...,j_r} e_{j_1} \otimes ... \otimes w_{j_i} \otimes ... \otimes e_{j_r} +$$
$$+ \sum_{j_1=1}^{k_1} ... \sum_{j_i=n+1}^{k_i} ... \sum_{j_r}^{k_r} \mu_{j_1,...,j_r} e_{j_1} \otimes ... \otimes v_{j_i} \otimes ... \otimes e_{j_r}.$$

Of course

$$\mathrm{Ran}(\varphi_S^i) \ni \varphi_S^i(e_{j_1} \otimes ... \otimes e_{j_{i-1}} \otimes e_{j_{i+1}} \otimes ... \otimes e_{j_r}) = \sum_{j_i=1}^{n} \lambda_{j_1,...j_r} w_{j_i} + \sum_{j_i=n}^{k_i} \mu_{j_1,...j_r} v_{j_i},$$

so all $\mu_{j_1,...j_r} = 0$, which means that

$$S = \sum_{j_1=1}^{k_1} ... \sum_{j_i=1}^{n} ... \sum_{j_r}^{k_r} \lambda_{j_1,...,j_r} e_{j_1} \otimes ... \otimes w_{j_i} \otimes ... \otimes e_{j_r}.$$

This decomposition proves the lemma. □

Theorem 3.2.6 (Same as [21] Corollary 2.2, with different proof)**.** Let $S, T \in K^{k_1} \otimes ... \otimes K^{k_i} \otimes ... \otimes K^{k_r} \subseteq K^{k_1} \otimes ... \otimes K^{k_i+1} \otimes ... \otimes K^{k_r}$. S and T are on the same orbit in $K^{k_1} \otimes ... \otimes K^{k_i+1} \otimes ... \otimes K^{k_r}$ if and only if they are on the same orbit in $K^{k_1} \otimes ... \otimes K^{k_i} \otimes ... \otimes K^{k_r}$.

Proof. If S and T are on the same orbit in $K^{k_1} \otimes ... \otimes K^{k_i} \otimes ... \otimes K^{k_r}$, then that means that there exist $P_j \in \mathrm{GL}(k_j, K)$ for $j = 1, ..., r$ such that $(P_1, ..., P_r)S = T$. Let

$$\hat{P} = \begin{pmatrix} P_i & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(k_i + 1, K),$$

then

$$(P_1, ..., P_{i-1}, \hat{P}, P_{i+1}, ..., P_r)S = T,$$

which means that S and T are on the same orbit in $K^{k_1} \otimes ... \otimes K^{k_i+1} \otimes ... \otimes K^{k_r}$.

For the other direction let

$$(Q_1, ..., Q_r) \in \mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_i + 1, K) \times ... \times \mathrm{GL}(k_r, K)$$

such that $(Q_1, ..., Q_r)S = T$. We know from Lemma 3.2.5 that there exists $S = \sum_{j=1}^{m} a_1^j \otimes ... \otimes a_r^j$ such that $a_i^j \in \mathrm{Ran}(\varphi_S^i)$ for $j = 1, ..., r$. Of course $\mathrm{Ran}(\varphi_S^i) \leq K^{k_i}$, because $S \in K^{k_1} \otimes ... \otimes K^{k_i} \otimes ... \otimes K^{k_r}$. Let $w_1, ..., w_n$ be a basis of $\mathrm{Ran}(\varphi_S^i)$ and let us extend it to a basis $w_1, ..., w_n, v_{n+1}, ..., v_{k_i}$ of K^{k_i}. Then $w_1, ...w_n, ...v_{n+1}, ..., v_{k_i}, e_{k_i+1}$ is a basis of $K^{k_{i+1}}$.

Also by Lemma 3.2.4 $\dim \mathrm{Ran}(\varphi_T^i) = n$, and clearly $\mathrm{Ran}(\varphi_T^i) \leq \langle Q_i w_1, ..., Q_i w_n \rangle$, so $Q_i w_1, ..., Q_i w_n$ is a basis of the subspace $\mathrm{Ran}(\varphi_T^i) \leq K^{k_i}$. Then we can extend it to a basis $Q_i w_1, ...Q_i w_n, u_{n+1}, ...u_{k_i}$ of K^{k_i}. Let us define the linear map $\tilde{Q}$ in the following way:

$$\begin{aligned} \tilde{Q} : K^{k_i+1} &\to K^{k_i+1} \\ w_j &\mapsto Q_i w_j \quad j = 1, ..., n \\ v_j &\mapsto u_j \quad j = n+1, ...k_i \\ e_{k_i+1} &\mapsto e_{k_i+1}. \end{aligned}$$

Clearly $\tilde{Q} \in \mathrm{GL}(k_i + 1, K)$ and there exists some $\hat{Q} \in \mathrm{GL}(k_i, K)$ such that $\tilde{Q} = \begin{pmatrix} \hat{Q} & 0 \\ 0 & 1 \end{pmatrix}$. Also if $a \in \mathrm{Ran}(\varphi_S^i)$ then $Q_i(a) = \tilde{Q}(a)$, therefore

$$T = (Q_1, ..., Q_r)S = \sum_{j=1}^{m} Q_1 a_1^j \otimes ... \otimes \underbrace{Q_i a_i^j}_{=\tilde{Q} a_i^j} \otimes ... \otimes Q_r a_r^j = (Q_1, ..., \tilde{Q}, ..., Q_r)S.$$

Then $(Q_1, ..., \hat{Q}, ..., Q_r)S = T$ in $K^{k_1} \otimes ... \otimes K^{k_i} \otimes ... \otimes K^{k_r}$. This concludes the proof of the theorem. □

Corollary 3.2.7 ([21] Corollary 2.1 (a))**.** If $k_j \leq k_j'$ for $j = 1, ..., r$ and there are infinitely many orbits in $K^{k_1} \otimes ... \otimes K^{k_r}$, then there are infinitely many orbits in $K^{k_1'} \otimes ... \otimes K^{k_r'}$.

Proof. If for a fixed i $k_i = k_i' + 1$ and $k_j = k_j'$ for $j \neq i$, then the statement follows from Theorem 3.2.6. For arbitrary k_j, k_j' the result follows by induction on $\sum k_j' - \sum k_j$. □

Corollary 3.2.8. Suppose $k_j \leq k_j'$ for $j = 1, ..., r$ and $k_j' \geq 1$ for $j = r+1, ..., s$. Assume moreover that there are infinitely many orbits in $K^{k_1} \otimes ... \otimes K^{k_r}$. Then there are infinitely many orbits in $K^{k_1'} \otimes ... \otimes K^{k_r'} \otimes K^{k_{r+1}'} \otimes ... \otimes K^{k_s'}$.

Proof. We know that

$$K^{k_1} \otimes ... \otimes K^{k_r} = K^{k_1} \otimes ... \otimes K^{k_r} \otimes \underbrace{K \otimes ... \otimes K}_{s-r \text{ times}}$$

and the $\mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_r, K)$-orbits are the same as $\mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_r, K) \times \mathrm{GL}(1, K) \times ... \times \mathrm{GL}(1, K)$-orbits since the only difference is multiplying by a scalar in the last $s - r$ components. Then the statement follows from Corollary 3.2.7. □

This means that in order to prove Theorem 3.1.1 we need to show two things:

- for tuples $(2, 2, n), (2, 3, n)$ there is a finite number of orbits;
- for tuples $(2, 4, 4), (3, 3, 3), (2, 2, 2, 2)$ there are infinitely many orbits,

and the rest of the theorem follows from Corollary 3.2.8 and Remark 3.1.2. We shall prove these in the next two sections. We will classify the orbits in the cases $(2, 2, n), (2, 3, n), (2, 4, n)$ using Kronecker's theory of matrix pencils.

3.3 Tensor rank

In this subsection we will introduce the notion of the rank of a tensor, which, as we will see, is invariant under the action of $\mathrm{GL}(k_1, K) \times ... \times \mathrm{GL}(k_r, K)$.

Definition 3.3.1 (Tensor rank)**.** Let $T \in V_1 \otimes ... \otimes V_r$ for vector spaces V_i over the field K. Then the tensor rank $\mathbf{R}(T)$ of T is the smallest number q such that T is a sum of q elementary tensors, i.e.

$$\mathbf{R}(T) = \min \left\{ q \in \mathbb{N} \middle| \exists a_i^j \in K^{k_i}, i = 1, ..., r, j = 1, ..., q : T = \sum_{j=1}^{q} a_1^j \otimes ... \otimes a_r^j \right\}.$$

Remark 3.3.2. We define the empty sum as the 0 tensor, so $\mathbf{R}(0) = 0$.

Remark 3.3.3. When $T \in K^{k_1} \otimes K^{k_2}$, then T is a $k_1 \times k_2$ matrix. Its tensor rank is the number q such that T is the sum of q rank one matrices, i.e. matrices that are the product of a column vector and a row vector. But this is the definition of the matrix rank, so in this case the matrix rank and the tensor rank coincide.

Proposition 3.3.4. The tensor rank is invariant under the action of $\mathrm{GL}(V_1) \times ... \times \mathrm{GL}(V_r)$, i.e. for all $(P_1, ..., P_r) \in \mathrm{GL}(V_1) \times ... \times \mathrm{GL}(V_r)$ and all $T \in V_1 \otimes ... \otimes V_r$ it stands that

$$\mathbf{R}(T) = \mathbf{R}((P_1, ..., P_r)T).$$

Proof. Because of the fact that

$$(P_1, ..., P_r) \left(\sum_{j=1}^{q} a_1^j \otimes ... \otimes a_r^q \right) = \sum_{j=1}^{q} P_1 a_1^j \otimes ... \otimes P_r a_r^q,$$

it stands that $\mathbf{R}(T) \geq \mathbf{R}((P_1, ..., P_r)T)$, and also

$$\mathbf{R}((P_1, ..., P_r)T) \geq \mathbf{R}((P_1^{-1}, ..., P_r^{-1})(P_1, ..., P_r)T) = \mathbf{R}(T).$$

□

Another important question is whether the tensor rank changes if we extend the tensor spaces V_i. The proposition below answers this question.

Proposition 3.3.5 ([23] Proposition 3.1)**.** If $V_i \leq V_i'$ for $i = 1, ..., r$, and $T \in V_1 \otimes ... \otimes V_r$, then let R_1 denote the rank of T, and let R_2 denote the rank of T considered as a tensor in $V_1' \otimes ... \otimes V_r'$. Then $R_1 = R_2$.

Proof. It is clear that $R_2 \leq R_1$. Let $\pi_i : V_i' \to V_i$ denote a projection. If $T = \sum_{j=1}^{q} a_1^j \otimes ... \otimes a_r^j$ for $a_i^j \in V_i'$, then it is easy to see that $T = \sum_{j=1}^{q} \pi_1(a_1^j) \otimes ... \otimes \pi_r(a_r^j)$, which proves $R_1 \leq R_2$. □

Computing the tensor rank is not straightforward in general, indeed it is NP-complete (see [11]). One can find explicit decompositions to prove an upper bound, however finding lower bounds is not easy. A method for it is the so-called Substitution method explained below. The proof for it can be found in the article [2].

Theorem 3.3.6 (Substitution method, [2] Appendix B, Corollary B.2)**.** Let $T \in K^{k_1} \otimes ... \otimes K^{k_r}$. Assume that $a_1, ..., a_{k_r}$ is a basis in K^{k_r} and

$$T = \sum_{j=1}^{k_r} T_j \otimes a_j$$

for $T_j \in K^{k_1} \otimes ... \otimes K^{k_{r-1}}$, where for some $m < k_r$, $T_1, ..., T_m$ are linearly independent as vectors. Then there exist constants $\lambda_{i,j} \in K$ for $i = 1, ..., m$, $j = m + 1, ..., k_r$ such that

$$\mathbf{R}(T) \geq \mathbf{R}\left(\sum_{j=m+1}^{k_r} \left(T_j + \sum_{i=1}^{m} \lambda_{i,j} T_i\right) \otimes a_j\right) + 1.$$

Example 3.3.7. Let $\mathbf{a}, \mathbf{b}$ be a basis of the two dimensional vector space K^2. We shall compute the tensor rank of $T = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a}$. It is clear that $\mathbf{R}(T) \leq 3$. We will use four different techniques to show that $\mathbf{R}(T) \geq 3$.

- This technique is based on an answer to the question posed in [1]. Let us assume that $\mathbf{R}(T) \leq 2$, i.e. for $u_{i,j}, v_{i,j}, w_{i,j} \in K$ it stands that

$$\begin{aligned} T =& (u_{1,1}\mathbf{a} + u_{1,2}\mathbf{b}) \otimes (v_{1,1}\mathbf{a} + v_{1,2}\mathbf{b}) \otimes (w_{1,1}\mathbf{a} + w_{1,2}\mathbf{b}) + \\ &+ (u_{2,1}\mathbf{a} + u_{2,2}\mathbf{b}) \otimes (v_{2,1}\mathbf{a} + v_{2,2}\mathbf{b}) \otimes (w_{2,1}\mathbf{a} + w_{2,2}\mathbf{b}). \end{aligned}$$

  This means that we can write the following equation system for the coefficients of this tensor.

$$\begin{aligned} u_{1,1}v_{1,1}w_{1,1} + u_{2,1}v_{2,1}w_{2,1} &= 0 \\ u_{1,1}v_{1,1}w_{1,2} + u_{2,1}v_{2,1}w_{2,2} &= 0 \\ u_{1,1}v_{1,2}w_{1,1} + u_{2,1}v_{2,2}w_{2,1} &= 0 \\ u_{1,1}v_{1,2}w_{1,2} + u_{2,1}v_{2,2}w_{2,2} &= 1 \\ u_{1,2}v_{1,1}w_{1,1} + u_{2,2}v_{2,1}w_{2,1} &= 0 \\ u_{1,2}v_{1,1}w_{1,2} + u_{2,2}v_{2,1}w_{2,2} &= 1 \\ u_{1,2}v_{1,2}w_{1,1} + u_{2,2}v_{2,2}w_{2,1} &= 1 \\ u_{1,2}v_{1,2}w_{1,2} + u_{2,2}v_{2,2}w_{2,2} &= 0. \end{aligned}$$

If we introduce the notations

$$U = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix}, W = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix},$$

then the system above is equivalent to the following matrix equations.

$$U^T \begin{pmatrix} v_{1,1} & 0 \\ 0 & v_{2,1} \end{pmatrix} W = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$U^T \begin{pmatrix} v_{1,2} & 0 \\ 0 & v_{2,2} \end{pmatrix} W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we multiply the first equation by the inverse of the second one from the left then we get

$$W^{-1} \begin{pmatrix} \frac{v_{1,1}}{v_{1,2}} & 0 \\ 0 & \frac{v_{2,1}}{v_{2,2}} \end{pmatrix} W = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which means that the matrix

$$D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is diagonalizable. This is a contradiction because D is a Jordan block.

- The second method is from the article [12] where the authors use it to compute the tensor rank of the multiplication tensor of the $\mathbb{R}$-algebra $\mathbb{H}$. Let $\langle \cdot, \cdot \rangle$ denote the dot product. Let us assume again that $\mathbf{R}(T) \leq 2$, i.e. $T = \mathbf{u}_1 \otimes \mathbf{v}_1 \otimes \mathbf{w}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2 \otimes \mathbf{w}_2$. Clearly $\mathrm{Ran}(\varphi_T^1) = K^2$ so $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent. This means that there exists $\mathbf{z} \in K^2$ such that $\langle \mathbf{z}, \mathbf{b} \rangle \neq 0$ and either $\langle \mathbf{z}, \mathbf{u}_1 \rangle = 0$ or $\langle \mathbf{z}, \mathbf{u}_2 \rangle = 0$. We might assume that $\langle \mathbf{z}, \mathbf{u}_1 \rangle = 0$. Let us define the linear map

$$\psi : K^2 \otimes K^2 \to K$$
$$S \mapsto \langle \mathbf{z}, \varphi_T^1(S) \rangle.$$

This means $\psi \in (K^2 \otimes K^2)^* = K^2 \otimes K^2$. The matrix corresponding to ψ in $K^2 \otimes K^2$ is

$$\begin{pmatrix} \psi(\mathbf{a} \otimes \mathbf{a}) & \psi(\mathbf{a} \otimes \mathbf{b}) \\ \psi(\mathbf{b} \otimes \mathbf{a}) & \psi(\mathbf{b} \otimes \mathbf{b}) \end{pmatrix} = \begin{pmatrix} \langle \mathbf{z}, 0 \rangle & \langle \mathbf{z}, \mathbf{b} \rangle \\ \langle \mathbf{z}, \mathbf{b} \rangle & \langle \mathbf{z}, \mathbf{a} \rangle \end{pmatrix}.$$

This is a rank 2 matrix because $\langle \mathbf{z}, 0 \rangle = 0$, so the determinant is $-\langle \mathbf{z}, \mathbf{b} \rangle^2 \neq 0$. But

$$\psi = \underbrace{\langle \mathbf{z}, \mathbf{u}_1 \rangle}_{=0} \cdot (\mathbf{v}_1 \otimes \mathbf{w}_1) + \langle \mathbf{z}, \mathbf{u}_2 \rangle \cdot (\mathbf{v}_2 \otimes \mathbf{w}_2) = \langle \mathbf{z}, \mathbf{u}_2 \rangle \cdot (\mathbf{v}_2 \otimes \mathbf{w}_2)$$

which contradicts the fact that the rank of ψ is 2.

- The third method is from the proof of Lemma 4.7 in [23]. Let us assume that $T = \mathbf{u}_1 \otimes \mathbf{v}_1 \otimes \mathbf{w}_1 + \mathbf{u}_2 \otimes \mathbf{v}_2 \otimes \mathbf{w}_2$ for contradiction. Then $\mathbf{u}_1$ and $\mathbf{u}_2$ span $\mathrm{Ran}(\varphi_T^1)$, which is clearly two dimensional. This means that they are linearly independent. Let $\psi : K^2 \to K$ be a nonzero linear map whose kernel is spanned by $\mathbf{u}_1$. Then

$$\psi(\mathbf{u}_2) \cdot (\mathbf{v}_2 \otimes \mathbf{w}_2) = \psi(\mathbf{a}) \cdot (\mathbf{b} \otimes \mathbf{b}) + \psi(\mathbf{b}) \cdot (\mathbf{a} \otimes \mathbf{b}) + \psi(\mathbf{b}) \cdot (\mathbf{b} \otimes \mathbf{a}),$$

so $\psi(\mathbf{b}) = 0$, because if it were not, then on the left hand side there would be a rank 1, and on the right hand side, a rank 2 matrix. This means that $\mathbf{b}$ and $\mathbf{u}_1$ are linearly dependent. The same way $\mathbf{b}$ and $\mathbf{u}_2$ are linearly dependent, and therefore so are $\mathbf{u}_1$ and $\mathbf{u}_2$, but we have already seen that this is not the case, which is a contradiction.

- The fourth technique is the 3.3.6 substitution method. If $\mathbf{R}(T) \leq 2$, then, according to the substitution method, there exists $\lambda \in K$ such that

$$\mathbf{R}(\mathbf{b} \otimes (\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a} + \lambda \mathbf{b} \otimes \mathbf{b})) \leq 1.$$

But this tensor is in $K \otimes K^2 \otimes K^2 = K^2 \otimes K^2$, so it is a matrix. Its entries (computed in the basis $\{\mathbf{a}, \mathbf{b}\}$) are $\begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}$, so it is clearly a rank 2 matrix, which is a contradiction.

The value of the tensor rank is known for tensors in $K^2 \otimes K^m \otimes K^n$ when K is algebraically closed. For this we need to introduce a correspondence between these tensors and matrix pencils of size $m \times n$. This result is Theorem 4.1.5 on page 20. The question is, does that help over arbitrary fields? Interestigly, yes, it does, although it only yields an inequality via the following proposition. This inequality can be found in Corollary 4.1.6.

Proposition 3.3.8. If $K \leq F$ and $T \in K^{k_1} \otimes ... \otimes K^{k_r}$, then

$$\mathbf{R}_K(T) \geq \mathbf{R}_F(T),$$

where $\mathbf{R}_K(T), \mathbf{R}_F(T)$ denote the tensor rank of T over the fields K, F respectively.

Proof. Follows from the fact that any decomposition of T over K is also a decomposition over F. □

Another question arises, and that is whether the tensor rank is affected by changing the base field. This is answered by the following example.

Example 3.3.9. Let $\mathbf{a}, \mathbf{b}$ denote a basis of $\mathbb{Q}^2$, and let us examine the rank tensor

$$T = \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} + 2\mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a}$$

over the fields $\mathbb{Q}$ and $\mathbb{R}$. Of course

$$T = \frac{1}{2}(\mathbf{a} + \mathbf{b}) \otimes (\mathbf{a} + \mathbf{b}) \otimes (\mathbf{a} + \mathbf{b}) + \frac{1}{2}(\mathbf{a} - \mathbf{b}) \otimes (\mathbf{a} - \mathbf{b}) \otimes (\mathbf{a} - \mathbf{b}) + \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a},$$

so $\mathbf{R}_K(T) \leq 3$ for $K = \mathbb{Q}, \mathbb{R}$. Also $\mathbf{R}_K(T) \geq 2$ for $K = \mathbb{Q}, \mathbb{R}$, since image of T in the projection $K^2 \otimes K^2 \otimes K^2 \to K \otimes K^2 \otimes K^2$ is a rank 2 matrix (in the first component we use the projection $\mathbf{a} \mapsto 1, \mathbf{b} \mapsto 0$).

Also it holds that

$$\begin{aligned} T = &\frac{1}{2}\left(\frac{1}{\sqrt{2}}\mathbf{a} + \mathbf{b}\right) \otimes \left(\mathbf{a} + \sqrt{2}\mathbf{b}\right) \otimes \left(\sqrt{2}\mathbf{a} + \mathbf{b}\right) \\ &+ \frac{1}{2}\left(\frac{1}{\sqrt{2}}\mathbf{a} - \mathbf{b}\right) \otimes \left(\mathbf{a} - \sqrt{2}\mathbf{b}\right) \otimes \left(\sqrt{2}\mathbf{a} - \mathbf{b}\right), \end{aligned}$$

so $\mathbf{R}_{\mathbb{R}}(T) = 2$, but, as we shall prove using the substitution method, $\mathbf{R}_{\mathbb{Q}}(T) = 3$. Indeed, if it were 2, then there would exist $\lambda \in \mathbb{Q}$ such that

$$\mathbf{R}_{\mathbb{Q}}(\mathbf{b} \otimes (\mathbf{a} \otimes \mathbf{b} + 2\mathbf{b} \otimes \mathbf{a} + \lambda \mathbf{a} \otimes \mathbf{a} + \lambda \mathbf{b} \otimes \mathbf{b})) \leq 1,$$

but this is a matrix with determinant $\lambda^2 - 2$, which is nonzero, so it is a contradiction.

4 Orbits in $K^2 \otimes K^m \otimes K^n$

4.1 Tensors and matrix pencils

Notation 4.1.1. $\mathrm{GL}_{2,m,n}$ shall denote $\mathrm{GL}(2,K) \times \mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$ in the rest of the thesis. We will also use the notation π_1 for the projection to the first component:

$$\begin{aligned} \pi_1 : \mathrm{GL}_{2,m,n} &\to \mathrm{GL}(2,K) \\ (M,P,Q) &\mapsto M. \end{aligned}$$

Notation 4.1.2. We will also use the following notations throughout the rest of the thesis. If $S, T \in K^2 \otimes K^m \otimes K^n$ then we will denote them being on the same $\mathrm{GL}(2,K) \times \mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$-orbit by $S \approx T$, and them being on the same $\{I\} \times \mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$-orbit by $S \approxeq T$. If two matrix pencils $sA + tB$ and $sA' + tB'$ are on the same $\mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$-orbit, then we will also denote this by $sA + tB \approxeq sA' + tB'$.

If $T \in K^2 \otimes K^m \otimes K^n$ then we can write T in the basis $\{e_i \otimes e_j \otimes e_k : i,j,k\}$

$$T = \sum_{i=1}^{2} \sum_{j=1}^{m} \sum_{k=1}^{n} \lambda_{i,j,k} e_i \otimes e_j \otimes e_k,$$

which means that $T = e_1 \otimes M_1 + e_2 \otimes M_2$ with $M_i = \sum_{j=1}^{m} \sum_{k=1}^{n} \lambda_{i,j,k} e_j \otimes e_k \in K^{m \times n}$. M_1 and M_2 are unique because if $e_1 \otimes M_1 + e_2 \otimes M_2 = e_1 \otimes M_1' + e_2 \otimes M_2'$ then $e_1 \otimes (M_1 - M_1') + e_2 \otimes (M_2 - M_2') = 0$ and by writing it in the basis $\{e_i \otimes e_j \otimes e_k : i,j,k\}$ we can see that $M_i - M_i' = 0$ for $i = 1,2$. This means that

$$K^2 \otimes K^m \otimes K^n = \left\{ e_1 \otimes M_1 + e_2 \otimes M_2 : M_1, M_2 \in K^{m \times n} \right\}.$$

Proposition 4.1.3 ([16] Section 3.11)**.** The correspondence

$$\begin{aligned} \Phi : K^2 \otimes K^m \otimes K^n &\to \left\{ sA + tB : A, B \in K^{m \times n} \right\} \\ e_1 \otimes M_1 + e_2 \otimes M_2 &\mapsto sM_1 + tM_2 \end{aligned}$$

is a linear isomorphism. Furthermore if we consider the action of $\{I\} \times \mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$ on $K^2 \otimes K^m \otimes K^n$ and the action of $\mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$ on matrix pencils then the following is true for $P \in \mathrm{GL}(m,K)$ and $Q \in \mathrm{GL}(n,K)$:

$$\Phi((I,P,Q)T) = \left(P, Q^T\right) \Phi(T).$$

Proof. Clearly Φ is a linear map, and we have already seen that it is bijective. Let $T = \sum_{i=1}^{2} \sum_{j=1}^{m} \sum_{k=1}^{n} \lambda_{i,j,k} e_i \otimes e_j \otimes e_k$. Then

$$\Phi((I,P,Q)T) = \Phi\left(\sum_{i=1}^{2}\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{i,j,k}e_i \otimes Pe_j \otimes Qe_k\right) =$$

$$= s\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{1,j,k}(Pe_j)\cdot(Qe_k)^T\right) + t\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{2,j,k}(Pe_j)\cdot(Qe_k)^T\right) =$$

$$= P\left(s\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{1,j,k}e_j\cdot e_k^T\right) + t\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{1,j,k}e_j\cdot e_k^T\right)\right)Q^T =$$

$$= P\Phi(T)Q^T = \left(P,Q^T\right)\Phi(T).$$

□

Corollary 4.1.4 ([16] Section 3.11)**.** If $S,T \in K^2 \otimes K^m \otimes K^n$, then S and T are on the same $\{I\} \times \mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$-orbit if and only if $\Phi(S)$ and $\Phi(T)$ are on the same $\mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$-orbit. For this reason from now on we shall consider matrix pencils (of size $m \times n$) as elements of $K^2 \otimes K^m \otimes K^n$ via the isomorphism Φ.

Now we can state the following theorem about the tensor rank of matrix pencils in $K^2 \otimes K^m \otimes K^n$ for K algebraically closed.

Theorem 4.1.5 (Grigoriev, Ja'Ja, Teichert, [16] Theorem 3.11.1.1)**.** Assume K is an algebraically closed field, and let $X \in K^2 \otimes K^m \otimes K^n$ such that the Kronecker normal form of X is

$$\mathrm{diag}\left(0, L_{\epsilon_1}, ..., L_{\epsilon_q}, L_{\nu_1}^T, ..., L_{\nu_p}^T, N_{u_1}, ..., N_{u_r}, sI_f + tF_f\right),$$

where F_f is in Jordan normal form, and I_f is the identity matrix, both of size $f \times f$. Let d be the number of N_u-s that are at least of size 2×2, and for each eigenvalue λ of F, let $d(\lambda)$ denote the number of Jordan blocks of size at least 2 asssociated to λ, and let

$$M = \max\left\{\max_{\lambda}(d(\lambda)), d\right\}.$$

Then

$$\mathbf{R}(X) = \sum_{i=1}^{q}\epsilon_i + \sum_{j=1}^{p}\mu_j + q + p + r + f + M.$$

Together with Proposition 3.3.8 this yields the following.

Corollary 4.1.6. Let $\overline{K}$ denote the algebraic closure of K. If K is an arbitrary field, and the Kronecker normal form of $X \in K^2 \otimes K^m \otimes K^n$ over $\overline{K}$ is

$$\mathrm{diag}\left(0, L_{\epsilon_1}, ..., L_{\epsilon_q}, L_{\nu_1}^T, ..., L_{\nu_p}^T, N_{u_1}, ..., N_{u_r}, sI_f + tF_f\right),$$

then, using the same notation as in Theorem 4.1.5,

$$\mathbf{R}(X) \geq \sum_{i=1}^{q}\epsilon_i + \sum_{j=1}^{p}\mu_j + q + p + r + f + M$$

over the field K.

Proposition 4.1.7. If $T \approx S$ for $T, S \in K^2 \otimes K^n \otimes K^n$ then T is regular if and only if S is regular as a matrix pencil.

Proof. We only need to prove that if T is not regular, then S is not regular either, then the other direction is trivial (because of symmetry). Let $(M, P, Q) \in \mathrm{GL}_{2,n,n}$ such that $(M, P, Q)T = S$, then $f = \det(T)$ is polynomial in two variables that is equal to the zero polynomial. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\det(S) = \det(P)\det(Q) f(as + ct, bs + dt) = 0,$$

and we are done. □

4.2 Kronecker's normal form for tensors

Proposition 4.2.1. If the projection of stabilizer of $T \in K^2 \otimes K^m \otimes K^n$ with respect to the action of $\mathrm{GL}_{2,m,n}$ to the first component of the product is surjective, i.e.

$$\pi_1(\mathrm{Stab}_{\mathrm{GL}_{2,m,n}}(T)) = \mathrm{GL}(2, K),$$

then for any tensor S, $T \approx S$ if and only if $T \approxeq S$.

Proof. Trivially if $T \approxeq S$, then $T \approx S$. Conversely assume that $T \approx S$. Because the projection of the stabilizer to the first component is surjective, this means that for any $M \in \mathrm{GL}(2, K)$ there exists $P \in \mathrm{GL}(m, K^m), Q \in \mathrm{GL}(n, K^n)$ such that $(M, P, Q)T = T$, i.e. $(M, I, I)T \approxeq T$. Because $T \approx S$ there exists M', P', Q' such that $(M', P', Q')T = S$. But then $T \approxeq (M', I, I)T \approxeq (I, P', Q')(M', I, I)T = (M', P', Q')T = S$. □

Lemma 4.2.2. If $sA+tB \approxeq sA'+tB'$ and $sC+tD \approxeq sC'+tD'$ then $\begin{pmatrix} sA+tB & 0 \\ 0 & sC+tD \end{pmatrix} \approxeq \begin{pmatrix} sA'+tB' & 0 \\ 0 & sC'+tD' \end{pmatrix}$.

Proof. If $P(sA + tB)Q = sA' + tB'$ and $P'(sC + tD)Q' = sC' + tD'$ then

$$\begin{pmatrix} P & 0 \\ 0 & P' \end{pmatrix} \begin{pmatrix} sA+tB & 0 \\ 0 & sC+tD \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q' \end{pmatrix} = \begin{pmatrix} sA'+tB' & 0 \\ 0 & sC'+tD' \end{pmatrix}.$$

□

Proposition 4.2.3. If $T \in K^2 \otimes K^n \otimes K^m$ is an indecomposable matrix pencil, and $M \in \mathrm{GL}(2, K)$, then $(M, I, I)T$ is also indecomposable as a matrix pencil.

Proof. Let us assume that $P \cdot ((M, I, I)T) \cdot Q = \mathrm{diag}(X, Y)$. Then

$$(I, P, Q)T = (M^{-1}M, P, Q)T = (M^{-1}, I, I) \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} (M^{-1}, I, I)X & 0 \\ 0 & (M^{-1}, I, I)Y \end{pmatrix}.$$

T is indecomposable, so this means that either $(M^{-1}, I, I)X$, or $(M^{-1}, I, I)Y$ is zero, but then one of X, Y is zero. We have thus proved that $(M, I, I)T$ has only one decomposition, which is the trivial one. □

Proposition 4.2.4. For all $\epsilon > 0$ $\pi_1(\mathrm{Stab}_{\mathrm{GL}_{2,m,n}}(L_\epsilon)) = \mathrm{GL}(2, K)$.

Proof. Let $M \in \mathrm{GL}(2, K)$. Then from Proposition 4.2.3 it follows that $(M, I, I)L_\epsilon$ is indecomposable. But from Theorem 2.2.7 we know that the only indecomposable matrix pencil of size $\epsilon \times (\epsilon + 1)$ is L_ϵ, so $(M, I, I)L_\epsilon \approxeq L_\epsilon$. □

Remark 4.2.5. This proposition can also be proved constructively using the following lemma.

Lemma 4.2.6. If $[\lambda : \mu], [\alpha : \beta], [\alpha' : \beta'] \in K\mathbb{P}^1$ such that $[\lambda : \mu] \neq [\alpha : \beta]$ and $[\lambda : \mu] \neq [\alpha' : \beta']$ then

(1) $(\alpha s + \beta t)A_\epsilon + (\lambda s + \mu t)B_\epsilon \approxeq (\alpha' s + \beta' t)A_\epsilon + (\lambda s + \mu t)B_\epsilon$;

(2) $(\lambda s + \mu t)A_\epsilon + (\alpha s + \beta t)B_\epsilon \approxeq (\lambda s + \mu t)A_\epsilon + (\alpha' s + \beta' t)B_\epsilon$;

(3) $(\alpha s + \beta t)H_\epsilon + (\lambda s + \mu t)I_\epsilon \approxeq (\alpha' s + \beta' t)H_\epsilon + (\lambda s + \mu t)I_\epsilon$.

Proof. We will only prove the first assertion as the other ones can be derived similarly.

The equation system

$$\begin{aligned} \alpha' y - \lambda x &= \alpha \\ \beta' y - \mu x &= \beta \end{aligned}$$

has a unique solution (x, y) because $[\lambda : \mu] \neq [\alpha' : \beta']$ and $y \neq 0$ because $[\lambda : \mu] \neq [\alpha : \beta]$. If we add x times the last column of $(\alpha s + \beta t)A_\epsilon + (\lambda s + \mu t)B_\epsilon$ to the one before it, then we have

$$\begin{pmatrix} (\alpha s+\beta t) & (\lambda s+\mu t) & & & & \\ & (\alpha s+\beta t) & (\lambda s+\mu t) & & & \\ & & \ddots & \ddots & & \\ & & & (\alpha s+\beta t) & (\lambda s+\mu t) & \\ & & & & (\alpha' ys+\beta' yt) & (\lambda s+\mu t) \end{pmatrix}.$$

If we now add x times the penultimate column to the one before it, then we obtain

$$\begin{pmatrix} (\alpha s+\beta t) & (\lambda s+\mu t) & & & & \\ & (\alpha s+\beta t) & (\lambda s+\mu t) & & & \\ & & \ddots & \ddots & & \\ & & & (\alpha' ys+\beta' yt) & (\lambda s+\mu t) & \\ & & & (\alpha' xys+\beta' xyt) & (\alpha' ys+\beta' yt) & (\lambda s+\mu t) \end{pmatrix}.$$

Now if we subtract x times the penultimate row from the last one then we will get

$$\begin{pmatrix} (\alpha s+\beta t) & (\lambda s+\mu t) & & & & \\ & (\alpha s+\beta t) & (\lambda s+\mu t) & & & \\ & & \ddots & \ddots & & \\ & & & (\alpha' ys+\beta' yt) & (\lambda s+\mu t) & \\ & & & & (\alpha s+\beta t) & (\lambda s+\mu t) \end{pmatrix}.$$

We can now see that we might propagate that $(\alpha' ys + \beta' yt)$ upwards to obtain

$$\begin{pmatrix} (\alpha' ys + \beta' yt) & (\lambda s + \mu t) & & & & \\ & (\alpha s + \beta t) & (\lambda s + \mu t) & & & \\ & & \ddots & \ddots & & \\ & & & (\alpha s + \beta t) & (\lambda s + \mu t) & \\ & & & & (\alpha s + \beta t) & (\lambda s + \mu t) \end{pmatrix}.$$

By adding x times the last column to the penultimate one we can create another $(\alpha' ys + \beta' yt)$, and we can propagate it up with the method above. Iterating this we get

$$\begin{pmatrix} (\alpha' ys + \beta' yt) & (\lambda s + \mu t) & & & & \\ & (\alpha' ys + \beta' yt) & (\lambda s + \mu t) & & & \\ & & \ddots & \ddots & & \\ & & & (\alpha' ys + \beta' yt) & (\lambda s + \mu t) & \\ & & & & (\alpha' ys + \beta' yt) & (\lambda s + \mu t) \end{pmatrix}.$$

Now we multiply the ith row by $\frac{1}{y^i}$ and the jth column by y^{j-1} for $i = 1, ..., \epsilon$ and $j = 1, ..., \epsilon + 1$ to obtain the first statement of the lemma. □

Proposition 4.2.7. For every tensor of the form $T = \operatorname{diag}\left(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T\right)$ it stands that $\pi_1(\operatorname{Stab}_{\mathrm{GL}_{2,m,n}}(T)) = \mathrm{GL}(2, K)$.

Proof. Let $M \in \mathrm{GL}(2, K)$. Then $(M, I, I)L_{\epsilon_i} \approxeq L_{\epsilon_i}$ for $i \in \{1, ..., p\}$ from Proposition 4.2.4 and similarly $(M, I, I)L_{\nu_i}^T \approxeq L_{\nu_i}^T$ for $i \in \{1, ..., q\}$. By Lemma 4.2.2 $(M, I, I)T \approxeq T$. □

Proposition 4.2.8. If K is an infinite field, then every tensor $T \in K^2 \otimes K^n \otimes K^n$ that is regular as a matrix pencil is on the same $\mathrm{GL}_{2,n,n}$-orbit as some $sI + tJ$ with J in generalized Jordan normal form.

Proof. Clearly it is enough to prove this for tensors T already in Kronecker normal form. Let $T = \operatorname{diag}(N_{u_1}, ..., N_{u_r}, sI' + tJ')$, where I', J' are $f \times f$ matrices. Let $0 \neq \mu \in K$ such that $-\frac{1}{\mu}$ is not an eigenvalue of J' (this exists since K is infinite), and let $M = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$. Let $sA' + tB' = (M, I, I)\operatorname{diag}(N_{u_1}, ..., N_{u_r})$, $sA'' + tB'' = (M, I, I)(sI' + tJ')$, and $sA + tB = \operatorname{diag}(sA' + tB', sA'' + tB'') = (M, I, I)T$.

A' and A'' are both square matrices. A' has maximal rank because it is a lower triangular matrix whose diagonal elements are all μ. We claim that A'' is also of maximal rank. Indeed, if $k_{J'}$ is the characteristic polynomial of J', then $k_{J'}\left(-\frac{1}{\mu}\right) \neq 0$ since $-\frac{1}{\mu}$ is not an eigenvalue of J', and

$$\det(A'') = \det(I' + \mu J') = \det\left(\mu \cdot \left(\frac{1}{\mu} I' + J'\right)\right) = \mu^k k_{J'}\left(-\frac{1}{\mu}\right) \neq 0.$$

This means that $A = \operatorname{diag}(A', A'')$ is a square matrix of maximal rank. Then by Theorem 2.2.7 $sA + tB \approxeq sC + tD$ where $sC + tD$ is in Kronecker normal form and C is a square matrix of maximal rank. The only possibility is $sC + tD = sI + tJ$ for some J in generalized Jordan normal form. Then $(M, I, I)T \approxeq sI + tJ$. □

Proposition 4.2.9. If K is an infinite field, then every tensor $T \in K^2 \otimes K^m \otimes K^n$ is on the same $\mathrm{GL}_{2,m,n}$-orbit as some $\mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T, sI + tJ)$ (a tensor in Kronecker normal form which has no blocks of the type N_u).

Proof. Clearly it is enough to prove this for tensors T already in Kronecker normal form. Let $T = \mathrm{diag}(0, L_{\epsilon_1'}, ..., L_{\epsilon_p'}, L_{\nu_1'}^T, ..., L_{\nu_q'}^T, N_{u_1}, ..., N_{u_r}, sI' + tJ')$. Then from Proposition 4.2.8 there is some $M \in \mathrm{GL}(2, K)$ such that $(M, I, I)\mathrm{diag}(N_{u_1}, ..., N_{u_r}, sI' + tJ') \approxeq sI + tJ$. From Proposition 4.2.7, Proposition 4.2.1 and Lemma 4.2.2

$$(M, I, I)T \approxeq \mathrm{diag}(0, L_{\epsilon_1'}, ..., L_{\epsilon_p'}, L_{\nu_1'}^T, ..., L_{\nu_q'}^T, sI + tJ).$$

□

Proposition 4.2.10. Let

$$T = \mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T, S), T' = \mathrm{diag}(0, L_{\epsilon_1'}, ..., L_{\epsilon_p'}, L_{\nu_1'}^T, ..., L_{\nu_q'}^T, S')$$

with S, S' regular. Then $T \approx T'$ if and only if the zero blocks in the beginning are of the same size, $\epsilon_i = \epsilon_i'$ for $i = 1, ..., p$, $\nu_j = \nu_j'$ for $j = 1, ..., q$ (after possibly reordering the blocks), and $S \approx S'$.

Proof. For the "only if" part assume that $T \approx T'$. Then for some $M \in \mathrm{GL}(2, K)$

$$T' \approxeq (M, I, I)T \approxeq \begin{pmatrix} (M, I, I)\mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T) & 0 \\ 0 & (M, I, I)S \end{pmatrix}$$

so from Proposition 4.2.7 and Lemma 4.2.2

$$\mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T, (M, I, I)S) \approxeq T' = \mathrm{diag}(0, L_{\epsilon_1'}, ..., L_{\epsilon_p'}, L_{\nu_1'}^T, ..., L_{\nu_q'}^T, S').$$

Then it is clear from Theorem 2.2.7 that $(M, I, I)S \approxeq S'$ (so $S \approx S'$), the zero blocks in the beginning are the same, and the rest of the blocks are the same up to reordering.

Now we shall prove the "if" statement. Assume that $(M, I, I)S \approxeq S'$ and the zero blocks in the beginning are the same size (we might assume that we do not need to reorder them). Then from Proposition 4.2.7 and Lemma 4.2.2 it follows that $(M, I, I)T \approxeq T'$. □

Proposition 4.2.11. Let us assume that the set of tensors $(S_\alpha^i)_{\alpha \in A_i}$ classifies the orbits of the regular matrix pencils in $K^2 \otimes K^i \otimes K^i$ (this makes sense because of Proposition 4.1.7) for all $i \in \mathbb{N}$, i.e. every regular matrix pencil of size $i \times i$ is on the same $\mathrm{GL}_{2,i,i}$-orbit as exactly one of the S_α^i. Then every tensor in $K^2 \otimes K^m \otimes K^n$ is on the same $\mathrm{GL}_{2,m,n}$-orbit as a block diagonal pencil of the following form: $\mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T, S_\alpha^i)$ with S_α^i optional. This form is unique up to the reordering of the blocks.

Proof. Let $T \in K^2 \otimes K^m \otimes K^n$ be an arbitrary tensor. Then by Theorem 2.2.7 either we have that $T \approxeq \mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T)$ (in which case we have reduced T to the necessary form), or $T \approxeq \mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T, S)$ for regular pencil S. Clearly $S \approx S_\alpha^i$ for some $\alpha \in A_i$, $i \in \mathbb{N}$, so from Proposition 4.2.10 $T \approx \mathrm{diag}(0, L_{\epsilon_1}, ..., L_{\epsilon_p}, L_{\nu_1}^T, ..., L_{\nu_q}^T, S_\alpha^i)$. This form is unique from Proposition 4.2.10. □

This means that we only need to classify orbits of regular matrix pencils in $K^2 \otimes K^i \otimes K^i$ and from this everything else will follow.

Corollary 4.2.12. There are finitely many $\mathrm{GL}_{2,m,n}$-orbits in $K^2 \otimes K^m \otimes K^n$ $(m \leq n)$ if and only if there are finitely many $\mathrm{GL}_{2,i,i}$-orbits in the space of regular pencils in $K^2 \otimes K^i \otimes K^i$ for $i = 1, ..., m$.

4.3 Regular matrix pencils

Notation 4.3.1. Let $\widehat{\mathcal{J}}$ denote the set of homogeneous binary forms $f(s,t)$ over the field K such that either

- $f(s,1)$ is irreducible and t does not divide f;
- or $f(s,t) = \lambda \cdot t$ for $\lambda \in K \backslash \{0\}$.

Let $\sim$ denote the equivalence relation on $\widehat{\mathcal{J}}$ that equates forms that are scalar multiples of one another, and let

$$\mathcal{J} = \widehat{\mathcal{J}}/\sim.$$

Remark 4.3.2. Observe that if $f(s,t)$ is a homogeneous binary form then $f \in \widehat{\mathcal{J}}$ if and only if f is irreducible.

Notation 4.3.3. If

$$q(t) = a_0 + a_1 s + ... + a_n s^n \in K[t],$$

then let

$$\overline{q}(s,t) = a_0 t^n + a_1 t^{n-1} s + ... + a_n s^n$$

be the binary form of degree n associated to q.

We will use the following corollary of Kroneckers classification theorem.

Corollary 4.3.4. If $X \in K^2 \otimes K^n \otimes K^n$ is an indecomposable regular matrix pencil, then there exists

- either a u positive integer such that $X \approx N_u$;
- or a generalized Jordan block J such that $X \approx sI - tJ$.

Furthermore this normal form is unique.

Proof. For the uniqueness observe that $N_u \not\approx sI - tJ$ since if $sA + tB \approx sC + tD$ then the ranks of A and C have to be the same. If $sI - tJ \approx sI - tJ'$, then $P(sI - tJ)Q = sI - tJ'$, so $Q = P^{-1}$, which means $PJP^{-1} = J'$, so $J = J'$ from Theorem 2.1.3.

As for the existence, if the first case does not hold, then we know from Theorem 2.2.7 that $X \approx sI + tJ'$ where J' is a generalized Jordan block. Then $-J'$ is similar to a matrix J in generalized Jordan form. J has to be a generalized Jordan block, because $-J$ is similar to J' and J' is a generalized Jordan block, so it cannot have a block diagonal form. Then

$$X \approx sI + tJ' = sI - t(-J') \approx sI - tJ.$$

□

Notation 4.3.5. Let $\mathcal{R}$ denote the set of $\mathrm{GL}(\cdot, K) \times \mathrm{GL}(\cdot, K)$-orbits in the set of all regular matrix pencils. Let $\mathcal{R}_0 \subseteq \mathcal{R}$ denote the set of indecomposable orbits. (This makes sense because if $X \approx Y$, and X is indecomposable and regular, then so is Y.)

Proposition 4.3.6. Let $q(t)$ be a monic polynomial, then clearly $\det(sI - tC(q)) = \overline{q}(s,t)$. Let $p(s) = f(s,1)$ for $f \in \mathcal{J}$, where $f(s,t) \neq \lambda \cdot t$. We can assume that $p(s)$ is monic. If

$$X = sI - \begin{pmatrix} C(p) & 0 & 0 & \cdots & 0 \\ U & C(p) & 0 & \cdots & 0 \\ 0 & U & C(p) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & U & C(p) \end{pmatrix} t,$$

with U as in Definition 2.1.2, then $\det(X) = \overline{p}(s,t)^n = f(s,t)^n$ (there are n $C(p)$-s in the main diagonal).

Proof. Clear. □

Definition 4.3.7. Let

$$\begin{aligned} \psi_0 : \mathcal{J} \times \mathbb{N}^+ &\to \mathcal{R}_0 \\ (f,n) &\mapsto X \text{ if } f(s,t) \neq \lambda \cdot t \\ (f,n) &\mapsto N_n \text{ if } f(s,t) = \lambda \cdot t \text{ for } \lambda \in K \backslash \{0\} \end{aligned}$$

with X as in Proposition 4.3.6.

Proposition 4.3.8. The mapping ψ_0 is well defined and bijective.

Proof. It is clearly well defined, and the bijective property follows from Corollary 4.3.4. □

Remark 4.3.9. It is clear from the multiplicative property of the determinant that if $X \approx Y$, then $\det(X) = \lambda \det(Y)$ for some $\lambda \in K\backslash\{0\}$. Moreover one can observe that the determinants of different indecomposable normal forms (in the sense of Corollary 4.3.4) are not scalar multiples of one another, so for **indecomposable** regular matrix pencils, it is true that

$$X \approx Y \Leftrightarrow \frac{\det(X)}{\det(Y)} \in K.$$

This means that if X is an arbitrary regular indecomposable matrix pencil, and Y is its normal form in Corollary 4.3.4, then we can compute $\psi_0^{-1}(X)$ by factoring $\det(X)$.

Definition 4.3.10. We will now define an action of $\mathrm{GL}(2, K)$ on $\mathcal{J}$. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, K)$, and $f \in \mathcal{J}$, then $M \cdot f(s,t) = f(as + ct, bs + dt)$. This makes sense because on one hand if f is reducible, then $M \cdot f$ is reducible, but also $f = M^{-1} \cdot (M \cdot f)$, so if $M \cdot f$ is reducible, then so is f, which means that

$$f \in \mathcal{J} \Leftrightarrow f \text{ is irreducible as a binary form} \Leftrightarrow M \cdot f \text{ is irreducible} \Leftrightarrow M \cdot f \in \mathcal{J}.$$

Proposition 4.3.11.

$$\psi_0(M \cdot f, n) \approxeq (M, I, I)\psi_0(f, n).$$

Proof. We have seen that $\det(\psi_0(f,n)) = f(s,t)^n$. If $\psi_0(f,n) = sA + tB$, then

$$\det((M,I,I)\psi_0(f,n)) = \det((M,I,I)(sA+tB)) = \det((as+ct)A + (bs+dt)B) = \\ = M \cdot \det(\psi_0(f,n)) = M \cdot f(s,t)^n,$$

and

$$\det(\psi_0(M \cdot f, n)) = (M \cdot f)^n = f(as+ct, bs+dt)^n = M \cdot f(s,t)^n.$$

Thus the statement is clear from Remark 4.3.9. □

Definition 4.3.12. Let $\mathcal{M}$ denote all multisets with elements from $\mathcal{J} \times \mathbb{N}^+$. Let

$$\psi : \mathcal{M} \to \mathcal{R}$$
$$[(f_1,n_1),...,(f_k,n_k)] \mapsto \mathrm{diag}(\psi_0(f_1,n_1),...,\psi_0(f_k,n_k)).$$

The action of $\mathrm{GL}(2,K)$ introduced in Definition 4.3.10 induces an action of $\mathrm{GL}(2,K)$ on the set $\mathcal{M}$ in the obvious way.

Proposition 4.3.13. If $m \in M$, and $M \in \mathrm{GL}(2,K)$, then

$$\psi(M \cdot m) \approxeq (M,I,I)\psi(m).$$

Proof. Clear from Proposition 4.3.11. □

Proposition 4.3.14. If $m_1, m_2 \in \mathcal{M}$, then m_1 and m_2 are on the same $\mathrm{GL}(2,K)$-orbit if and only if $\psi(m_1) \approx \psi(m_2)$.

Proof. Let $m_1 = [(f_1,k_1),...,(f_p,k_p)]$, and $m_2 = [(g_1,l_1),...,(g_n,l_n)]$. Then $\psi(m_1) \approx \psi(m_2)$ is equivalent to saying that for some $M \in \mathrm{GL}(2,K)$, it stands that $\psi(m_1) \approxeq (M,I,I)\psi(m_2)$, but from Proposition 4.3.13 $(M,I,I)\psi(m_2) \approxeq \psi(M \cdot m_2)$, so that is the same as $\psi(M \cdot m_2) \approxeq \psi(m_1)$. Because of the fact that the indecomposable summands of $\psi(m_1)$ are the matrix pencils $\psi_0(f_1,k_1),...,\psi_0(f_p,k_p)$, and the indecomposable summands of $\psi(M \cdot m_2)$ are $\psi_0(M \cdot g_1, l_1),...,\psi_0(M \cdot g_n, l_n)$, this means from Theorem 2.2.7, that it is equivalent to the following two conditions:

(1) $p = n$;

(2) and there is a bijection $\varphi : \{1,...,p\} \to \{1,...,p\}$ such that $\psi_0(f_i,k_i) \approxeq \psi_0(M \cdot g_{\varphi(i)}, l_{\varphi(i)})$ for $i = 1,...,p$.

Then from Corollary 4.3.4 it follows that condition (2) is equivalent to $k_i = l_{\varphi(i)}$ and $f_i = \lambda_i \cdot M \cdot g_{\varphi(i)}$ for some $\lambda_i \in K$, so $f_i = M \cdot g_{\varphi(i)}$ in $\mathcal{J}$, but this is the same as m_1 being on the same $\mathrm{GL}(2,K)$-orbit as $[(g_{\varphi(1)}, l_{\varphi(1)}),...,(g_{\varphi(p)}, l_{\varphi(p)})] = [(g_1,l_1),...(g_n,l_n)] = m_2$, so we are done. □

Corollary 4.3.15. The regular $\mathrm{GL}_{2,k,k}$-orbits in $K^2 \otimes K^k \otimes K^k$ are in a natural bijection (explained above) with their $\mathrm{GL}(2,K)$-orbits in $\mathcal{M}_k$, where

$$\mathcal{M}_k = \left\{ [(f_1, l_1), ..., (f_p, l_p)] \in \mathcal{M} : \sum_{i=1}^{p} l_i \deg(f_i) = k \right\}.$$

In particular, from Proposition 4.2.11, the problem of classifying $\mathrm{GL}_{2,m,n}$-orbits in $K^2 \otimes K^m \otimes K^n$ is equivalent to the problem of classifying the $\mathrm{GL}(2,K)$-orbits in $\mathcal{M}_k$ for $k \leq \min\{m,n\}$.

Since for algebraically closed fields, $\mathcal{J} = \{s\lambda + t\mu : [\lambda : \mu] \in K\mathbb{P}^1\} = K\mathbb{P}^1$ (the projective line over K), we have the following for K algebraically closed.

Corollary 4.3.16. Let K be algebraically closed. Let us consider the multisets with elements from $K\mathbb{P}^1 \times \mathbb{N}^+$. By the *size* of a multiset we mean the sum of the natural numbers in the second component of its elements. The set of regular $\mathrm{GL}_{2,k,k}$-orbits in $K^2 \otimes K^k \otimes K^k$ is in bijection with the set of $\mathrm{PGL}(2,K)$-orbits of multisets of size k with elements from $K\mathbb{P}^1 \times \mathbb{N}^+$. In particular, the problem of classifying $\mathrm{GL}_{2,m,n}$-orbits in $K^2 \otimes K^m \otimes K^n$ is equivalent to the problem of classifying the $\mathrm{PGL}(2,K)$-orbits of multisets with elements from $K\mathbb{P}^1 \times \mathbb{N}^+$ of size $k \leq \min\{m,n\}$.

Notation 4.3.17. Let K be algebraically closed. If $m \in \mathcal{M}$, then let us write $m = [(p_1, k_1), ..., (p_n, k_n)]$ for $p_i \in K\mathbb{P}^1$. The p_i-s need not be pairwise distinct, so let $\mathcal{H}_m$ denote the set of p_i-s. If $p \in \mathcal{H}_m$, then let $\mathcal{H}_m^p$ denote the set of pairs $(a,b) \in \mathbb{N}^+ \times \mathbb{N}^+$ such that (p,a) appears b times in m.

Corollary 4.3.18. Let K be algebraically closed. If $\psi(m_1) \approx \psi(m_2)$, then there is a bijective correspondence

$$\varphi : \mathcal{H}_{m_1} \to \mathcal{H}_{m_2}$$

such that for all $p \in \mathcal{H}_{m_1}$ it stands that $\mathcal{H}_{m_1}^p = \mathcal{H}_{m_2}^{\varphi(p)}$.

Proposition 4.3.14 also yields essentially the following Proposition, but we will present a more constructive proof to help find the matrices that transform a given matrix pencil into its normal form.

Proposition 4.3.19. If K is algebraically closed, then let $X \in K^2 \otimes K^k \otimes K^k$ be a regular matrix pencil. Then from Proposition 4.2.8 we know that there exist matrices H_i in Jordan canonical form of size $\epsilon_i \times \epsilon_i$, such that every eigenvalue is zero (i.e. all the diagonal elements are zero) for $i = 1, ..., k$, and there exist $\lambda_1, ..., \lambda_k \in K$ pairwise distinct such that the following holds. If $\lambda \in K$ then let

$$J_i(\lambda) = (s + \lambda t)\, I_{\epsilon_i} + tH_i,$$

and

$$N_i = tI_{\epsilon_i} + sH_{\epsilon_i},$$

then

$$\operatorname{diag}\left(J_1(\lambda_i), ..., J_k(\lambda_k)\right) \approx X.$$

The statement of this proposition is that

$$\operatorname{diag}\left(J_1(\lambda_i), ..., J_k(\lambda_k)\right) \approx \operatorname{diag}\left(N_1, J_2(0), J_3(1), J_4(\mu_4), ..., J_k(\mu_k)\right)$$

with $0, 1, \mu_4, ..., \mu_k \in K$ pairwise distinct.

Proof. If $k < 3$, then let us define either λ_2, λ_3, or just λ_3 such that $\lambda_1, \lambda_2, \lambda_3$ are pairwise distinct. Let

$$M = \begin{pmatrix} -\lambda_1(\lambda_3 - \lambda_2) & (\lambda_3 - \lambda_2) \\ -\lambda_2(\lambda_3 - \lambda_1) & (\lambda_3 - \lambda_1) \end{pmatrix},$$

then we can see that $\det M = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$, so $M \in \mathrm{GL}(2, K)$, and

$$M(s + \lambda_1 t) = \nu_1 t$$
$$M(s + \lambda_2 t) = \nu_2 s$$
$$M(s + \lambda_3 t) = \nu_3(s + t)$$

where $\nu_1 = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)$, $\nu_2 = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)$, $\nu_3 = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$, so $\nu_1, \nu_2, \nu_3 \in K \backslash \{0\}$. Clearly

$$\left(M, \frac{1}{\nu_1} I, I\right) J_1(\lambda_1) = tI_{\epsilon_1} + (\eta s + \eta' t) H_1$$

with $\eta = \frac{\lambda_3 - \lambda_2}{\nu_1} \neq 0$, and from (3) of Lemma 4.2.6

$$tI_{\epsilon_1} + (\eta s + \eta' t) H_1 \approxeq N_1.$$

Similarly

$$(M, I, I) J_2(\lambda_2) \approxeq J_2(0), (M, I, I) J_3(\lambda_3) \approxeq J_3(1),$$

and

$$(M, I, I) J_i(\lambda_i) \approxeq J_i(\mu_i) \text{ for } i = 4, ..., k$$

for some $\mu_4, ..., \mu_k$. Clearly $0, 1, \mu_4, ..., \mu_k$ are pairwise distinct, so from Lemma 4.2.2 we are done. ☐

Remark 4.3.20. The proposition says that if J is a matrix in Jordan normal form with k different eigenvalues, then we can choose the values of the first three arbitrarily to obtain a new matrix J' such that $sI + tJ \approx sI + tJ'$. By choosing these values the other eigenvalues may change too, but the number of Jordan blocks, and their sizes will not. The block N_i corresponds to the "eigenvalue" ∞.

Remark 4.3.21. We can see that the proof does not use the fact that K is algebraically closed, so over any field (using the same notation as in the proposition) it holds that

$$\mathrm{diag}\,(J_1(\lambda_i), ..., J_k(\lambda_k)) \approx \mathrm{diag}\,(N_1, J_2(0), J_3(1), J_4(\mu_4), ..., J_k(\mu_k))$$

with $0, 1, \mu_4, ..., \mu_k \in K$ pairwise distinct.

4.4 Classification of orbits in $K^2 \otimes K^2 \otimes K^n$

Notation 4.4.1. In this subsection $f(s, t)$ will denote a homogeneous quadratic irreducible binary form. Let $D(f)$ be the discriminant of the polynomial $f(s, 1)$, which is quadratic since f is irreducible.

Lemma 4.4.2. Denote by $\overline{K}$ the algebraic closure of K. Let $\lambda, \mu \in \overline{K}$. If $\mu \neq 0$, then $f(\lambda, \mu) = 0$ if and only if $f\left(\frac{\lambda}{\mu}, 1\right) = 0$.

Proof. Clearly $\mu^2 \cdot f\left(\frac{\lambda}{\mu}, 1\right) = f(\lambda, \mu)$. □

Proposition 4.4.3. If for $\lambda \in \overline{K}$ it holds that $f(\lambda, 1) = 0$, then $K(\lambda) = K\left(\sqrt{D(f)}\right)$.

Proof. Follows from the facts that because of the quadratic formula, $\lambda \in K\left(\sqrt{D(f)}\right)$ and $\sqrt{D(f)} \in K(\lambda)$. □

Proposition 4.4.4. If f and g are on the same $\mathrm{GL}(2, K)$-orbit, then $K\left(\sqrt{D(f)}\right) = K\left(\sqrt{D(g)}\right)$.

Proof. Because of symmetry it is enough to prove $K\left(\sqrt{D(g)}\right) \leq K\left(\sqrt{D(f)}\right)$. Let $\lambda \in \overline{K}$ be a root of $f(s, 1)$, and let $M \in \mathrm{GL}(2, K)$ be such that $M \cdot g = f$. Then

$$0 = f(\lambda, 1) = (M \cdot g)(\lambda, 1) = g(a\lambda + c, b\lambda + d).$$

If $b\lambda + d = 0$, then $a\lambda + c \neq 0$ because $M \in \mathrm{GL}(2, K)$, so since $g(s, 0)$ is a scalar multiple of s^2 it follows that $g(s, 0) = 0$, but then this means that t divides $g(s, t)$, which contradicts the irreducibility of g. Consequently $b\lambda + d \neq 0$, so from Lemma 4.4.2 $g(\mu, 1) = 0$ with $\mu = \frac{a\lambda+c}{b\lambda+d}$. This means that $\mu \in K(\lambda)$, so using Proposition 4.4.3

$$K\left(\sqrt{D(g)}\right) = K(\mu) \leq K(\lambda) = K\left(\sqrt{D(f)}\right).$$

□

Lemma 4.4.5. If $d_1, d_2 \in K$, then $K\left(\sqrt{d_1}\right) = K\left(\sqrt{d_2}\right)$ if and only if for some $g \in K$, $d_1 = g^2 d_2$.

Proof. The "if" part is obvious.

As for the "only if" part, if $\sqrt{d_1}$ or $\sqrt{d_2}$ is in K then the proof is clear so we can assume that this is not the case. We claim that

$$K\left(\sqrt{d_2}\right) = \{a + b\sqrt{d_2} : a, b \in K\}.$$

The containment $\supseteq$ is trivial, and the other direction follows from the fact that the right side is a field (this is easily checked). Then

$$\sqrt{d_1} = a + b\sqrt{d_2}$$

for some $a, b \in K$. Then

$$K \ni d_1 = \left(a + b\sqrt{d_2}\right)^2 = a^2 + b^2 d_2 + 2ab\sqrt{d_2},$$

so $2ab = 0$. If $b = 0$ then $\sqrt{d_1} \in K$, which would be a contradiction, so $a = 0$. This means that $\sqrt{d_1} = b\sqrt{d_2}$, so by choosing $g = b$ we have completed the proof. □

Proposition 4.4.6. If $d_1, d_2 \in K\backslash\{0\}$, then

$$\begin{pmatrix} s & d_1 t \\ t & s \end{pmatrix} \approx \begin{pmatrix} s & d_2 t \\ t & s \end{pmatrix}$$

if and only if for some $g \in K$, it holds that $d_1 = g^2 d_2$.

Proof. The "if" part follows from the fact that

$$\left(\begin{pmatrix} \frac{1}{g} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \right) \begin{pmatrix} s & d_2 t \\ t & s \end{pmatrix} = \begin{pmatrix} s & d_1 t \\ t & s \end{pmatrix}.$$

For the "only if" statement assume that $\begin{pmatrix} s & d_1 t \\ t & s \end{pmatrix} \approx \begin{pmatrix} s & d_2 t \\ t & s \end{pmatrix}$. Then, applying the triple $\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I, I \right)$ to both sides we have that $\begin{pmatrix} s & -d_1 t \\ -t & s \end{pmatrix} \approx \begin{pmatrix} s & -d_2 t \\ -t & s \end{pmatrix}$. From Proposition 4.3.14 it follows that $s^2 - d_1 t^2$ and $s^2 - d_2 t^2$ are on the same $\mathrm{GL}(2, K)$-orbit. Then, after substituting $t = 1$, their discriminants are respectively $4d_1, 4d_2$. Then from Proposition 4.4.4 it follows that

$$K\left(\sqrt{d_1}\right) = K\left(2\sqrt{d_1}\right) = K\left(2\sqrt{d_2}\right) = K\left(\sqrt{d_2}\right).$$

Then Lemma 4.4.5 completes the proof. □

Proposition 4.4.7. Let us consider the factor group $K^\times / (K^\times)^2$. Let us exclude from this the equivalence class of 1 and let H denote the remaining set (we can think of H as the set containing exactly one representative of each equivalence class except the identity's). Then the following tensors classify the regular orbits in $K^2 \otimes K^2 \otimes K^2$:

$$\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} t & 0 \\ s & t \end{pmatrix}, \begin{pmatrix} s & dt \\ t & s \end{pmatrix} (d \in H),$$

where the last element is a family of tensors parametrized by H.

Proof. Proposition 4.3.14 and Proposition 4.4.6 yield the fact that the listed tensors are on different $\mathrm{GL}_{2,2,2}$-orbits, using the fact that clearly $\begin{pmatrix} s & dt \\ t & s \end{pmatrix} \approx \begin{pmatrix} s & -dt \\ -t & s \end{pmatrix}$.

What remains of the proof is showing that each regular matrix pencil $T \in K^2 \otimes K^2 \otimes K^2$ is on the same $\mathrm{GL}_{2,2,2}$-orbit as one of the listed tensors. Proposition 4.2.8 yields that we can assume $T = sI + tJ$ with J in generalized Jordan normal form. If J is in regular Jordan normal form then from Proposition 4.3.19 and Remark 4.3.21 we know that T is on the same orbit as one of the first three items in the list, so we can assume that this is not the case.

Therefore $T = \begin{pmatrix} s & -c_0 t \\ t & s - c_1 t \end{pmatrix}$ with $s^2 - c_1 st + c_0 t^2$ irreducible, which means that $c_1^2 - 4c_0$ is not a square. Clearly

$$T \approx \begin{pmatrix} s & c_0 t \\ -t & s + c_1 t \end{pmatrix},$$

and $s^2 + c_1st + c_0t^2$ is also irreducible, and since

$$\left(s - \frac{c_1}{2}t\right)^2 + c_1\left(s - \frac{c_1}{2}t\right)t + c_0t^2 = s^2 + \left(c_0 - \frac{c_1^2}{4}\right)t^2,$$

it holds that $s^2 + c_1st + c_0t^2$ is on the same $\mathrm{GL}(2, K)$-orbit as $s^2 + \left(c_0 - \frac{c_1^2}{4}\right)t^2$, so from Proposition 4.3.14 it follows that

$$\begin{pmatrix} s & c_0t \\ -t & s + c_1t \end{pmatrix} \approx \begin{pmatrix} s & \left(c_0 - \frac{c_1^2}{4}\right)t \\ -t & s \end{pmatrix} = \begin{pmatrix} s & -dt \\ -t & s \end{pmatrix} \approx \begin{pmatrix} s & dt \\ t & s \end{pmatrix},$$

where $d = \frac{c_1^2}{4} - c_0$ is clearly not a square, since $d = \frac{c_1^2 - 4c_0}{4}$. □

Remark 4.4.8. In $K^2 \otimes K^1 \otimes K^1$ there is only one regular orbit: if $\lambda s + \mu t \in K^2 \otimes K^1 \otimes K^1$ and either λ or μ is nonzero, then there is a matrix $M \in \mathrm{GL}(2, K)$ such that $M\begin{pmatrix}\lambda \\ \mu\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}$, so $(M, 1, 1)(\lambda s + \mu t) = s$.

Corollary 4.4.9. The classification of orbits in $K^2 \otimes K^2 \otimes K^n$ now follows from Proposition 4.4.7, Remark 4.4.8 and Proposition 4.2.11.

Corollary 4.4.10. $\mathbb{Q}^2 \otimes \mathbb{Q}^2 \otimes \mathbb{Q}^2$ has infinitely many $\mathrm{GL}_{2,2,2}$-orbits, and therefore $\mathbb{Q}^{k_1} \otimes \ldots \otimes \mathbb{Q}^{k_r}$ has infinitely many orbits, if $r \geq 3$ and $k_i \geq 2$ for $i = 1, \ldots, r$.

Proof. The first observation follows from Proposition 4.4.7 and the fact that $\left|\mathbb{Q}^\times / (\mathbb{Q}^\times)^2\right| = \aleph_0$. The second observation follows from Corollary 3.2.8. □

5 Proof of Parfenov's theorem

5.1 Classification of regular orbits in $K^2 \otimes K^2 \otimes K^2$, $K^2 \otimes K^3 \otimes K^3$, $K^2 \otimes K^4 \otimes K^4$ for K algebraically closed

In this subsection we shall classify all the $\mathrm{GL}_{2,i,i}$-orbits of the regular matrix pencils in the spaces $K^2 \otimes K^2 \otimes K^2, K^2 \otimes K^3 \otimes K^3, K^2 \otimes K^4 \otimes K^4$ when K is algebraically closed. This makes sense because of Proposition 4.1.7. Proposition 4.2.8 is very useful here as it states that every regular matrix pencil in $K^2 \otimes K^i \otimes K^i$ is on the same $\mathrm{GL}_{2,i,i}$-orbit as some $sI + tJ$ with J in Jordan canonical form, if K is algebraically closed.

Remark 5.1.1. Richard Ehrenborg classified all the $\mathrm{GL}_{2,2,2}$-orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ in his article [8].

Proposition 5.1.2. For arbitrary K (it need not be algebraically closed) let

$$T_i = \begin{pmatrix} \mu_i s + \mu_i' t & 0 & 0 & 0 \\ 0 & \lambda_i s + \lambda_i' t & 0 & 0 \\ 0 & 0 & \vartheta_i s + \vartheta_i' s & 0 \\ 0 & 0 & 0 & \alpha_i s + \alpha_i' t \end{pmatrix} \text{ for } i = 1, 2$$

such that $[\mu_i : \mu_i'], [\lambda_i : \lambda_i'], [\vartheta_i : \vartheta_i'], [\alpha_i : \alpha_i'] \in K\mathbb{P}^1$ are four distinct points for both $i = 1, 2$. Then there exists $M \in \mathrm{GL}(2, K)$ such that $(M, I, I)T_1 = T_2$ if and only if

$$([\mu_1 : \mu_1'], [\lambda_1 : \lambda_1'], [\vartheta_1 : \vartheta_1'], [\alpha_1 : \alpha_1']) = ([\mu_2 : \mu_2'], [\lambda_2 : \lambda_2'], [\vartheta_2 : \vartheta_2'], [\alpha_2 : \alpha_2']),$$

i.e. the cross-ratios of the coefficients coincide.

Proof. Follows from the fact that two pairs of four distinct points of the projective line have the same cross-ratio if and only if there exists a projective transformation that maps one of them to the other (see for example [9] Theorem 8.6.3). □

Notation 5.1.3. If

$$T = \begin{pmatrix} \mu s + \mu' t & 0 & 0 & 0 \\ 0 & \lambda s + \lambda' t & 0 & 0 \\ 0 & 0 & \vartheta s + \vartheta' s & 0 \\ 0 & 0 & 0 & \alpha s + \alpha' t \end{pmatrix},$$

where $[\mu : \mu'], [\lambda : \lambda'], [\vartheta : \vartheta'], [\alpha : \alpha'] \in K\mathbb{P}^1$ are all distinct points, then let

$$\mathrm{CrossRatio}(T) := ([\mu : \mu'], [\lambda : \lambda'], [\vartheta : \vartheta'], [\alpha : \alpha']).$$

Remark 5.1.4. If we have four projective points P_1, P_2, P_3, P_4 with cross-ratio a then with the reordering of the P_i-s we can get exactly the values $\left\{a, \frac{1}{a}, 1-a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}\right\}$ for the new cross-ratio ([9] Corollary 8.6.8).

Corollary 5.1.5. Let K be an arbitrary field. If $a, b \in K\backslash\{0,1\}$, and

$$T = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s+t & 0 \\ 0 & 0 & 0 & s+a\cdot t \end{pmatrix}, S = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s+t & 0 \\ 0 & 0 & 0 & s+b\cdot t \end{pmatrix},$$

then $T \approx S$ if and only if $b \in \left\{a, \frac{1}{a}, 1-a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}\right\}$.

Proof. Clearly CrossRatio$(T) = a$ and CrossRatio$(S) = b$. If $T \approx S$, then for some M, P, Q it stands that $(M, P, Q)S = T$. Then for $T' = (M, I, I)S$, CrossRatio$(T') =$ CrossRatio$(S) = b$ from Proposition 5.1.2. Also $T' \approxeq T$, so from Theorem 2.2.7 the diagonal elements in T' are a reordering of those in T, so by Remark 5.1.4,

$$\text{CrossRatio}(T') = b \in \left\{a, \frac{1}{a}, 1-a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}\right\}.$$

If $b \in \left\{a, \frac{1}{a}, 1-a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}\right\}$, then there exists a reordering of $[0:1], [1:0], [1:1], [1:b]$ such that their cross-ratio is a by Remark 5.1.4. Let us put this reordering into the main diagonal of the diagonal matrix pencil T''. Then $T'' \approxeq T$, and CrossRatio$(T'') = a$, so from Proposition 5.1.2 there exists an $M \in \text{GL}(2, K)$ such that $(M, I, I)S = T''$. This proves the statement of the corollary. □

Lemma 5.1.6.

$$\begin{pmatrix} t & 0 & 0 & \cdots & 0 \\ s & t & 0 & \cdots & 0 \\ 0 & s & t & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s & t \end{pmatrix} \approxeq \begin{pmatrix} -t & 0 & 0 & \cdots & 0 \\ s & -t & 0 & \cdots & 0 \\ 0 & s & -t & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s & -t \end{pmatrix}$$

Proof. If we multiply the right hand side by -1 then it follows from (3) in Lemma 4.2.6. □

Lemma 5.1.7.

$$\text{diag}\left(N_{u_1}, ..., N_{u_r}, sI + tJ\right) \approx \text{diag}\left(N_{u_1}, ..., N_{u_r}, sI - tJ\right).$$

Proof. Apply the triple $\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I, I\right)$ to both sides, then it follows from Lemmas 5.1.6 and 4.2.2. □

Remark 5.1.8. In Proposition 4.3.14, the representatives of the orbits are given in the form $\text{diag}\left(N_{u_1}, ..., N_{u_r}, sI - tJ\right)$, which is not in Kronecker normal form, but because of Lemma 5.1.7, we can substitute it easily to a tensor in Kronecker normal form.

Notation 5.1.9. Let us introduce an equivalence relation on the set $K \cup \{\infty\}$: $a \sim b$ if $b \in \left\{a, \frac{1}{a}, 1-a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}\right\}$. Remark 5.1.4 shows that this is indeed an equivalence. Clearly $\{0, 1, \infty\}$ is an equivalence class, and let us introduce the notation

$$H = {}^{K\backslash\{0,1\}}\!/\!_{\sim}.$$

Proposition 5.1.10. If K is algebraically closed, then the following tables contain exactly one tensor from each regular orbit in $K^2 \otimes K^2 \otimes K^2$, $K^2 \otimes K^3 \otimes K^3$, and $K^2 \otimes K^4 \otimes K^4$. To help with the proof the corresponding multisets from Definition 4.3.12 are included in the tables (because of Lemma 5.1.7 and Remark 5.1.8 these are indeed the multisets that correspond to the orbits).

Tensor	**The corresponding multiset**
$\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$	$[(\infty, 1), (\infty, 1)]$
$\begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix}$	$[(\infty, 1), (0, 1)]$
$\begin{pmatrix} t & 0 \\ s & t \end{pmatrix}$	$[(\infty, 2)]$

Tensor	**The corresponding multiset**
$\begin{pmatrix} t & 0 & 0 \\ s & t & 0 \\ 0 & s & t \end{pmatrix}$	$[(\infty, 3)]$
$\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & s & t \end{pmatrix}$	$[(\infty, 1), (\infty, 2)]$
$\begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & t & s \end{pmatrix}$	$[(\infty, 1), (0, 2)]$
$\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$	$[(\infty, 1), (\infty, 1), (\infty, 1)]$
$\begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}$	$[(\infty, 1), (0, 1), (0, 1)]$
$\begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+t \end{pmatrix}$	$[(\infty, 1), (0, 1), (1, 1)]$

Tensor	**The corresponding multiset**
$\begin{pmatrix} t & 0 & 0 & 0 \\ s & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \end{pmatrix}$	$[(\infty, 4)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \end{pmatrix}$	$[(\infty, 1), (\infty, 3)]$

$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & t & s & 0 \\ 0 & 0 & t & s \end{pmatrix}$	$[(\infty,1),(0,3)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ s & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & s & t \end{pmatrix}$	$[(\infty,2),(\infty,2)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ s & t & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & t & s \end{pmatrix}$	$[(\infty,2),(0,2)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & s & t \end{pmatrix}$	$[(\infty,1),(\infty,1),(\infty,2)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & t & s \end{pmatrix}$	$[(\infty,1),(0,1),(0,2)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & t & s \end{pmatrix}$	$[(\infty,1),(\infty,1),(0,2)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s+t & 0 \\ 0 & 0 & t & s+t \end{pmatrix}$	$[(\infty,1),(0,1),(1,2)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{pmatrix}$	$[(\infty,1),(\infty,1),(\infty,1),(\infty,1)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix}$	$[(\infty,1),(0,1),(0,1),(0,1)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix}$	$[(\infty,1),(\infty,1),(0,1),(0,1)]$
$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s+t & 0 \\ 0 & 0 & 0 & s+t \end{pmatrix}$	$[(\infty,1),(0,1),(1,1),(1,1)]$

$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s+t & 0 \\ 0 & 0 & 0 & s+a\cdot t \end{pmatrix}$	$(a \in H)$	$[(\infty,1),(0,1),(1,1),(a,1)]$

Proof. We will first prove that all regular matrix pencils can be reduced to the ones listed in the tables above. By Proposition 4.2.8 we only need to prove this for matrix pencils $sI+tJ$ with J in Jordan normal form, and for these the statement follows from Proposition 4.3.19.

The fact that all of the tensors listed are on different orbits follows from Proposition 4.3.18 and Corollary 5.1.5, after we switch the tensors out in the table using Lemma 5.1.7 and Remark 5.1.8. □

Remark 5.1.11. The elements of H are equivalence classes and not elements of K, so the notation $\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s+t & 0 \\ 0 & 0 & 0 & s+a\cdot t \end{pmatrix}$ does not strictly make sense for $a \in H$. We can think of this in two ways:

- we might consider H as a subset in K containing exactly one representative of each equivalence class in H;
- or we can say that we take the tensor

$$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s+t & 0 \\ 0 & 0 & 0 & s+a\cdot t \end{pmatrix}$$

  for each element $a \in K\backslash\{0,1\}$, and two of these are on the same orbit if and only if they are in the same equivalence class in H.

Corollary 5.1.12. If K is algebraically closed, then for the 3-tuples $(2,2,n),(2,3,n)$ there are finitely many $\mathrm{GL}_{k_1,k_2,k_3}$-orbits in $K^{k_1}\otimes K^{k_2}\otimes K^{k_3}$.

Proof. Follows from Corollary 4.2.12, Proposition 5.1.10 and Remark 4.4.8. □

Corollary 5.1.13. Let K be an arbitrary infinite field. If $4 \le m \le n$, then $K^2\otimes K^m\otimes K^n$ has infinitely many $\mathrm{GL}_{2,m,n}$-orbits.

Proof. Because of Corollary 3.2.8 we only need to prove this for $m=n=4$. For this case it follows from Corollary 5.1.5. □

5.2 Orbits in $K^3\otimes K^3\otimes K^3$

Definition 5.2.1 ([19] Definition 4.4)**.** Assume G acts on the K-vector space V, and $\mu : V\times G \to \mathrm{GL}(U)$ for K-vector space U, then $R: V\to U$ is a *relative invariant of weight* μ, if for all $g\in G, v\in V$

$$R(g\cdot v) = \mu(v,g)\cdot R(v).$$

Definition 5.2.2 ([7] Section 2)**.** Let us define the polynomial functions $f_{i,j,k} : K^3 \otimes K^3 \otimes K^3 \to K$ for $i, j, k \in \{1, 2, 3\}$ with the following equations:

$$\det(t_1A_1 + t_2A_2 + t_3A_3) = \sum_{i+j+k=3} t_1^i t_2^j t_3^k f_{i,j,k} \left(\mathbf{e}_1 \otimes A_1 + \mathbf{e}_2 \otimes A_2 + \mathbf{e}_3 \otimes A_3\right)$$

where $A_1, A_2, A_3 \in K^3 \otimes K^3$ are matrices and $(\mathbf{e}_i)_{i=1,2,3}$ is the standard basis of K^3. Let us also define $h, q : K^3 \otimes K^3 \otimes K^3 \to K$: let $h\left(\mathbf{e}_1 \otimes A_1 + \mathbf{e}_2 \otimes A_2 + \mathbf{e}_3 \otimes A_3\right)$ be the coefficient of $t_1^2 t_2^2 t_3^2$ in

$$\det \begin{pmatrix} t_2A_2 & t_1A_1 \\ t_1A_1 & t_3A_3 \end{pmatrix},$$

and let $q\left(\mathbf{e}_1 \otimes A_1 + \mathbf{e}_2 \otimes A_2 + \mathbf{e}_3 \otimes A_3\right)$ be the coefficient of $t_1^2 t_2 t_3^2 t_4 t_5^2 t_6$ in

$$\det \begin{pmatrix} 0 & t_1A_1 & t_2A_2 \\ t_4A_1 & 0 & t_3A_3 \\ t_5A_2 & t_6A_3 & 0 \end{pmatrix}.$$

Finally let

$$H = h - \frac{1}{3} f_{2,1,0} f_{0,1,2} - \frac{1}{3} f_{2,0,1} f_{0,2,1} + \frac{2}{3} f_{1,2,0} f_{1,0,2} + \frac{1}{12} f_{1,1,1}^2,$$

$$\begin{aligned} Q = q - \frac{1}{2} h f_{1,1,1} + \frac{3}{2} f_{3,0,0} f_{0,3,0} f_{0,0,3} - \frac{1}{2} f_{3,0,0} f_{0,2,1} f_{0,1,2} - \frac{1}{2} f_{0,3,0} f_{2,0,1} f_{1,0,2} \\ - \frac{1}{2} f_{0,0,3} f_{2,1,0} f_{1,2,0} - \frac{1}{2} f_{1,1,1} f_{1,2,0} f_{1,0,2} + \frac{1}{2} f_{2,1,0} f_{1,0,2} f_{0,2,1} + \frac{1}{2} f_{1,2,0} f_{2,0,1} f_{0,1,2}. \end{aligned}$$

Remark 5.2.3. The previous definitions are from [7], and it is established there that H and Q are $\mathrm{SL}(3, K) \times \mathrm{SL}(3, K) \times \mathrm{SL}(3, K)$ invariants on $K^3 \otimes K^3 \otimes K^3$, and as we shall point out below, they are relative invariants with respect to the action of $\mathrm{GL}(3, K) \times \mathrm{GL}(3, K) \times \mathrm{GL}(3, K)$. It is easy to see that if we divide two relative invariants of the same weight, then we get a rational absolute invariant.

Proposition 5.2.4. Let us assume that $\mathrm{char} K = 0$. If $T \in K^3 \otimes K^3 \otimes K^3$, and $(A, B, C) \in \mathrm{GL}(3, K) \times \mathrm{GL}(3, K) \times \mathrm{GL}(3, K)$, then

$$H((A, B, C)T) = \det(A)^2 \det(B)^2 \det(C)^2 H(T),$$

and

$$Q((A, B, C)T) = \det(A)^3 \det(B)^3 \det(C)^3 Q(T),$$

i.e. H and Q are relative invariants with weights

$$\mu_1 : ((A, B, C), T) \mapsto \det(A)^2 \det(B)^2 \det(C)^2,$$

$$\mu_2 : ((A, B, C), T) \mapsto \det(A)^3 \det(B)^3 \det(C)^3,$$

respectively.

Proof. It is enough to prove this for K algebraically closed, because any field can be embedded in an algebraically closed field. Then if A is a three by three matrix, then $A = \sqrt[3]{\det(A)} \cdot \frac{A}{\sqrt[3]{\det(A)}}$, where

$$\det\left(\frac{A}{\sqrt[3]{\det(A)}}\right) = \left(\frac{1}{\sqrt[3]{\det(A)}}\right)^3 \cdot \det(A) = 1,$$

so $\frac{A}{\sqrt[3]{\det(A)}} \in \mathrm{SL}(3,K)$. We know from [7] that H and Q are $\mathrm{SL}(3,K)\times\mathrm{SL}(3,K)\times\mathrm{SL}(3,K)$ invariants, so

$$H((A,B,C)T) = H\left(\sqrt[3]{\det(A)}\sqrt[3]{\det(B)}\sqrt[3]{\det(C)}T\right),$$

$$Q((A,B,C)T) = Q\left(\sqrt[3]{\det(A)}\sqrt[3]{\det(B)}\sqrt[3]{\det(C)}T\right),$$

This means that we need to prove that for $\lambda \in K$,

$$H(\lambda T) = \lambda^6 H(T),$$

$$Q(\lambda T) = \lambda^9 H(T),$$

and these are clear from the following three facts that follow directly from the definitions above

- $h(\lambda T) = \lambda^6 h(T)$;
- $q(\lambda T) = \lambda^9 q(T)$;
- $f_{i,j,k}(\lambda T) = \lambda^3 f_{i,j,k}(T)$ for $i,j,k \in \{1,2,3\}$.

□

Corollary 5.2.5. Let $\mathrm{char}K = 0$, and let $T, S \in K^3\otimes K^3\otimes K^3$ such that $Q(T) \neq 0 \neq Q(S)$. If $\frac{H(T)^3}{Q(T)^2} \neq \frac{H(S)^3}{Q(S)^2}$, then T and S are on different $\mathrm{GL}(3,K)\times\mathrm{GL}(3,K)\times\mathrm{GL}(3,K)$-orbits.

Notation 5.2.6. In what follows we shall introduce a notation for elements of $K^3\otimes K^3\otimes K^3$ similar to the notation of matrix pencils. If $T = \mathbf{e}_1\otimes A_1 + \mathbf{e}_2\otimes A_2 + \mathbf{e}_3\otimes A_3 \in K^3\otimes K^3\otimes K^3$, then we shall denote

$$T = s\cdot A_1 + t\cdot A_2 + u\cdot A_3$$

for variables s, t, u.

Corollary 5.2.7. If $\mathrm{char}K = 0$, then there are infinitely many $\mathrm{GL}(3,K)\times\mathrm{GL}(3,K)\times\mathrm{GL}(3,K)$-orbits in $K^3\otimes K^3\otimes K^3$.

Proof. The function `invariant` at the end of the SageMath ([22]) code in Appendix A computes the value of $\frac{H^3}{Q^2}$. Using this code we can compute that

$$\frac{H^3}{Q^2}\begin{pmatrix} s & u & 0\\ t & s & \lambda u\\ 0 & t & s\end{pmatrix} = \frac{1}{432}\frac{\lambda^6 - 30\lambda^5 - 303\lambda^4 - 1060\lambda^3 + 303\lambda^2 - 30\lambda + 1}{\lambda^4 - 2\lambda^3 + \lambda^2}$$

(the notation is as in Appendix A). We will prove that for different choices of λ this will yield infinitely many different possible results, thereby proving the statement. Clearly since K is an infinite field ($\mathrm{char}K = 0$ implies the infiniteness of K) it is sufficient to prove that for any value $\mu \in K$ there are only finitely many λ-s such that

$$\frac{H^3}{Q^2}\begin{pmatrix} s & u & 0 \\ t & s & \lambda u \\ 0 & t & s \end{pmatrix} = \mu,$$

but this is clear because these λ-s are exactly the roots of the nonzero polynomial

$$\lambda^6 - 30\lambda^5 - 303\lambda^4 - 1060\lambda^3 + 303\lambda^2 - 30\lambda + 1 - 432\mu(\lambda^4 - 2\lambda^3 + \lambda^2).$$

□

5.3 Orbits in $K^2 \otimes K^2 \otimes K^2 \otimes K^2$

Notation 5.3.1. If A, B are two by two matrices, then let

$$A \cdot B = \det(A + B) - \det(A) - \det(B).$$

Notation 5.3.2 ([5])**.** Let $T \in K^2 \otimes K^2 \otimes K^2 \otimes K^2$, and let $(\mathbf{e}_i)_{i=1,2}$ be the standard basis of K^2. Then for some matrices $A, B, C, D \in K^2 \otimes K^2$

$$T = \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes A + \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes B + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes C + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes D.$$

We can think of T as an array in the following way

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

Then let

$$\begin{aligned} \det : K^2 \otimes K^2 \otimes K^2 \otimes K^2 &\to K \\ T &\mapsto \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \det_0 : K^2 \otimes K^2 \otimes K^2 \otimes K^2 &\to K \\ T &\mapsto A \cdot D - B \cdot C. \end{aligned}$$

The map $\det_0$ is Cayley's hyperdeterminant.

Remark 5.3.3 ([5])**.** The functions det and $\det_0$ are $\mathrm{SL}(2, K) \times \mathrm{SL}(2, K) \times \mathrm{SL}(2, K) \times \mathrm{SL}(2, K)$ invariants, and they are relative $\mathrm{GL}(2, K) \times \mathrm{GL}(2, K) \times \mathrm{GL}(2, K) \times \mathrm{GL}(2, K)$-invariants of the following weights

$$\begin{aligned} \mu_1 &: ((A, B, C, D), T) \mapsto \det(A)^2 \det(B)^2 \det(C)^2 \det(D)^2, \\ \mu_2 &: ((A, B, C, D), T) \mapsto \det(A) \det(B) \det(C) \det(D), \end{aligned}$$

respectively.

Corollary 5.3.4. The map $\frac{\det}{\det_0^2} : K^2 \otimes K^2 \otimes K^2 \otimes K^2 \to K$ is a rational absolute invariant.

Corollary 5.3.5. If K is an infinite field, then there are infinitely many $\mathrm{GL}(2,K) \times \mathrm{GL}(2,K) \times \mathrm{GL}(2,K) \times \mathrm{GL}(2,K)$-orbits in $K^2 \otimes K^2 \otimes K^2 \otimes K^2$.

Proof. Clearly

$$\frac{\det}{\det_0^2}\left(\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \frac{\lambda}{\lambda^2 + 2\lambda + 1}.$$

This can take on infinitely many values, because for each μ there are only finitely many λ-s such that

$$\frac{\lambda}{\lambda^2 + 2\lambda + 1} = \mu,$$

because if that is the case, then λ is the root of the nonzero polynomial

$$\mu\lambda^2 + (2\mu - 1)\lambda + \mu.$$

□

5.4 Finishing the proof

We have now proved Parfenov's Theorem 3.1.1 about the finiteness of the number of orbits in $\mathbb{C}^{k_1} \otimes \ldots \otimes \mathbb{C}^{k_r}$:

- if $r \leq 2$, then there are finitely many orbits by Remark 3.1.2;
- if $r = 3, k_1 = 2$ and $k_2 \leq 3$, then there are finitely many orbits by Proposition 5.1.10, Remark 4.4.8 and Corollary 4.2.12;
- if $r = 3, k_1 = 2$ and $k_2 \geq 4$, then there are an infinite number of orbits by Corollary 5.1.13;
- if $r = 3, k_1 \geq 3$, then there are infinitely many orbits by Corollary 3.2.7 and Corollary 5.2.7;
- if $r \geq 4$, then the number of orbits is infinite by Corollary 3.2.8 and Corollary 5.3.5.

We have proved a little bit more general result as K need not be the field of complex numbers, it only needs to be algebraically closed with characteristic 0.

6 Classification over $\mathbb{R}$

In this section we will classify the $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})$-orbits of the regular matrix pencils in the spaces $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$, $\mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ and $\mathbb{R}^2 \otimes \mathbb{R}^4 \otimes \mathbb{R}^4$. By Proposition 4.2.11 this means that we will have classified all the $\mathrm{GL}_{2,m,n}$-orbits in $\mathbb{R}^2 \otimes \mathbb{R}^m \otimes \mathbb{R}^n$, where $2 \leq m \leq n$ and $m \leq 4$. It will follow from the classification that there are finitely many orbits exactly when $m \leq 3$, which is the same as in the algebraically closed case.

Remark 6.0.1. In the article [23] the classification of $\mathrm{GL}_{2,m,n}$-orbits in $\mathbb{R}^2 \otimes \mathbb{R}^m \otimes \mathbb{R}^n$ (here $2 \leq m \leq n$) is computed for $m = n = 2$, and in [6] the same is done for $m \leq 3$.

Remark 6.0.2. From Proposition 4.2.8 we know that if $T \in \mathbb{R}^2 \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ is regular, then $T \approx sI + tJ$ with J in generalized Jordan normal form. If J is in regular Jordan normal form, then we know form Remark 4.3.21 that (if $n \leq 4$ then) T is on the same orbit as one of the tensors listed in Proposition 5.1.10. The tensors listed there are also clearly on different orbits as the proof for that never uses the fact that K is algebraically closed: Proposition 5.1.5 does not assume K to be algebraically closed and while Corollary 4.3.18 is stated for algebraically closed fields, the fact that the listed tensors are on different orbits really follows directly from Proposition 4.3.14.

This means that the task is to classify the orbits that are on the same $\mathrm{GL}_{2,n,n}$-orbit as some $sI + tJ$ where J is in generalized Jordan normal form, but not in regular Jordan normal form.

6.1 $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$

The following Proposition is a special case of Proposition 4.4.7.

Proposition 6.1.1 ([23], [6])**.** In $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ the tensor listed in Proposition 5.1.10 together with $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ classify all the regular orbits.

6.2 $\mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$

Proposition 6.2.1 ([6])**.** In $\mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ the tensors that classify the regular orbits are the ones listed in Proposition 5.1.10 and

$$\begin{pmatrix} s & -t & 0 \\ t & s & 0 \\ 0 & 0 & s \end{pmatrix}.$$

Proof. Again it follows from Proposition 4.3.14 that the new tensor is on a different orbit compared to the ones in Proposition 5.1.10 so together with Remark 6.0.2 this means that all the proposed tensors are on different orbits.

If $T \in \mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ is regular then $T \approx sI + tJ$ with J in generalized Jordan normal form (Proposition 4.2.8). Because of Remark 6.0.2 we can assume that J is not in regular Jordan normal form. Because J must have an eigenvalue in $\mathbb{R}$ (since every cubic polynomial with real coefficients has a root in $\mathbb{R}$) it follows that J is block diagonal with a 2×2 and a 1×1 block, and from Proposition 6.1.1 we can assume that the 2×2 block is $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$, so

$$sI + tJ \approx \begin{pmatrix} s & -t & 0 \\ t & s & 0 \\ 0 & 0 & \lambda s + \mu t \end{pmatrix}.$$

If $M = \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix}$, then

$$\det\left((M, I, I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix}\right) = (\lambda^2 + \mu^2)s^2 + (\lambda^2 + \mu^2)t^2,$$

so its Kronecker normal form can only be $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$, so

$$(M, I, I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \approx \begin{pmatrix} s & -t \\ t & s \end{pmatrix}.$$

But $(M, I, I)(\lambda s + \mu t) = (\lambda^2 + \mu^2)s$, so by Lemma 4.2.2

$$(M, I, I)\begin{pmatrix} s & -t & 0 \\ t & s & 0 \\ 0 & 0 & \lambda s + \mu t \end{pmatrix} \approx \begin{pmatrix} s & -t & 0 \\ t & s & 0 \\ 0 & 0 & s \end{pmatrix}.$$

□

6.3 $\mathbb{R}^2 \otimes \mathbb{R}^4 \otimes \mathbb{R}^4$

Lemma 6.3.1. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ and $\lambda > 0$ it holds that

$$(M, I, I)\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix} \approx \begin{pmatrix} s & -t \\ t & s \end{pmatrix}$$

if and only if

$$M \in \left\{\begin{pmatrix} -\sqrt{\lambda}d & b \\ \sqrt{\lambda}b & d \end{pmatrix}, \begin{pmatrix} \sqrt{\lambda}d & b \\ -\sqrt{\lambda}b & d \end{pmatrix}\right\}.$$

Proof. First we shall prove the "only if" part. From the multiplicative property of the determinant it is clear that

$$\det\left((M, I, I)\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix}\right) = \det\begin{pmatrix} as + ct & -\lambda bs - \lambda dt \\ bs + dt & as + ct \end{pmatrix}$$
$$= (a^2 + \lambda b^2)s^2 + (2ac + 2\lambda bd)st + (c^2 + \lambda d^2)t^2$$

and

$$\det\begin{pmatrix} s & -t \\ t & s \end{pmatrix} = s^2 + t^2$$

are scalar multiples of one another. This means that

$$a^2 + \lambda b^2 = c^2 + \lambda d^2$$

and

$$ac + \lambda bd = 0.$$

If we multiply the first equation with c^2 and use the fact that $a^2c^2 = \lambda^2 b^2 d^2$, which follows from the second equation, then we get $\lambda^2 b^2 d^2 + \lambda b^2 c^2 = c^4 + \lambda d^2 c^2$, which is equivalent to

$$(c^2 - \lambda b^2)(c^2 + \lambda d^2) = 0.$$

Because $M \in \mathrm{GL}(2, \mathbb{R})$ it is clear that $c^2 + \lambda d^2 > 0$ so $c = \pm\sqrt{\lambda}b$. Then $a = \mp\sqrt{\lambda}d$.

For the "if" part assume

$$M \in \left\{ \begin{pmatrix} -\sqrt{\lambda}d & b \\ \sqrt{\lambda}b & d \end{pmatrix}, \begin{pmatrix} \sqrt{\lambda}d & b \\ -\sqrt{\lambda}b & d \end{pmatrix} \right\}.$$

Then

$$\det\left((M, I, I)\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix}\right) = \det\begin{pmatrix} as + ct & -\lambda bs - \lambda dt \\ bs + dt & as + ct \end{pmatrix} = (\lambda d^2 + \lambda b^2)s^2 + (\lambda b^2 + \lambda d^2)t^2,$$

so it is a scalar multiple of $s^2 + t^2$, so the Kronecker normal form of $(M, I, I)\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix}$ can only be $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$. □

Proposition 6.3.2. Let $\lambda, \mu \in \mathbb{R}$ with $\lambda, \mu > 0$. Then the matrix pencils

$$T = \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & -\lambda t \\ 0 & 0 & t & s \end{pmatrix} \text{ and}$$

$$S = \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & -\mu t \\ 0 & 0 & t & s \end{pmatrix}$$

are on the same $\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(4, \mathbb{R}) \times \mathrm{GL}(4, \mathbb{R})$-orbit if and only if either $\lambda = \mu$, or $\lambda = \frac{1}{\mu}$.

Proof. First we will prove the "if" part. If $\lambda = \mu$, then $T = S$, so we might assume that $\lambda \neq \mu$, which is to say $\mu = \frac{1}{\lambda}$. Then

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\lambda} \end{pmatrix}\right) T = S.$$

Now let us prove the "only if" part and assume $T \approx S$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ such that $(M,I,I)T \approxeq S$. $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$, $\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix}$ and $\begin{pmatrix} s & -\mu t \\ t & s \end{pmatrix}$ are all indecomposable matrix pencils (as their determinant is irreducible), so from Theorem 2.2.7 it follows that there are two cases.

(1) The first case is when

$$(M,I,I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix} = \begin{pmatrix} as+ct & -bs-dt \\ bs+dt & as+ct \end{pmatrix} \approxeq \begin{pmatrix} s & -t \\ t & s \end{pmatrix},$$

and

$$(M,I,I)\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix} \approxeq \begin{pmatrix} s & -\mu t \\ t & s \end{pmatrix}.$$

Then from Lemma 6.3.1

$$M \in \left\{ \begin{pmatrix} -d & b \\ b & d \end{pmatrix}, \begin{pmatrix} d & b \\ -b & d \end{pmatrix} \right\}.$$

Then also

$$\det\left((M,I,I)\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix}\right) = (d^2+\lambda b^2)s^2 + (\pm 2bd \mp 2\lambda bd)st + (b^2+\lambda d^2)t^2$$

and

$$\det\begin{pmatrix} s & -\mu t \\ t & s \end{pmatrix} = s^2 + \mu t^2$$

are scalar multiples of each other. Then

$$b^2 + \lambda d^2 = \mu(d^2 + \lambda b^2)$$

and

$$\pm 2bd(1-\lambda) = 0.$$

From the first equation we get that if $\lambda = 1$ then so is μ (since $b^2 + d^2 > 0$ from $M \in \mathrm{GL}(2,\mathbb{R})$) so we can assume that $\lambda \neq 1$, from which it follows that $bd = 0$. If $d = 0$ then from the first equation $b^2 = \mu\lambda b^2$, so $\mu = \frac{1}{\lambda}$, and if $b = 0$, then $\lambda d^2 = \mu d^2$, so $\lambda = \mu$.

(2) The second case is when

$$(M,I,I)\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix} = \begin{pmatrix} as+ct & -bs-dt \\ bs+dt & as+ct \end{pmatrix} \approxeq \begin{pmatrix} s & -t \\ t & s \end{pmatrix},$$

and

$$(M,I,I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \approxeq \begin{pmatrix} s & -\mu t \\ t & s \end{pmatrix}.$$

Then from Lemma 6.3.1

$$M \in \left\{ \begin{pmatrix} -\sqrt{\lambda}d & b \\ \sqrt{\lambda}b & d \end{pmatrix}, \begin{pmatrix} \sqrt{\lambda}d & b \\ -\sqrt{\lambda}b & d \end{pmatrix} \right\},$$

and the polynomials

$$\det\left((M,I,I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix}\right) = (\lambda d^2+b^2)s^2 + (\pm 2\lambda bd \mp 2bd)st + (\lambda b^2+d^2)t^2$$

and

$$\det\begin{pmatrix} s & -\mu t \\ t & s \end{pmatrix} = s^2+\mu t^2$$

are scalar multiples of each other, so we are done, because the polynomials are the same as in the first case except that b and d are switched.

□

Proposition 6.3.3. Let $[\lambda_i:\mu_i]\in\mathbb{RP}^1$ and

$$T_i = \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & \lambda_i s+\mu_i t \end{pmatrix}$$

for $i=1,2$. Then T_1 and T_2 are on the same $\mathrm{GL}(2,\mathbb{R})\times\mathrm{GL}(4,\mathbb{R})\times\mathrm{GL}(4,\mathbb{R})$-orbit if and only if either $[\lambda_1:\mu_1]=[\lambda_2:\mu_2]$, or $[\lambda_1:\mu_1]=[\lambda_2:-\mu_2]$ in $\mathbb{RP}^1$.

Proof. As for the "if" part, if $[\lambda_1:\mu_1]=[\lambda_2:\mu_2]$ then $T_1\approx T_2$ by multiplying the last column by an appropriate scalar, and if $[\lambda_1:\mu_1]=[\lambda_2:-\mu_2]$, then

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \vartheta \end{pmatrix}\right) T_1 = T_2,$$

where ϑ is an appropriate scalar.

For the "only if" part assume that $T_1\approx T_2$ and let $M\in\mathrm{GL}(2,\mathbb{R})$ such that $(M,I,I)T_1\approxeq T_2$. Using the fact that $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ is indecomposable, since its determinant is irreducible, it follows from Theorem 2.2.7 that $(M,I,I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix}\approxeq\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$. From Lemma 6.3.1 it follows that for some $b,d\in\mathbb{R}$

$$M\in\left\{\begin{pmatrix} -d & b \\ b & d \end{pmatrix}, \begin{pmatrix} d & b \\ -b & d \end{pmatrix}\right\}.$$

From the fact that

$$(M,I,I)\begin{pmatrix} s & 0 \\ 0 & \lambda_1 s+\mu_1 t \end{pmatrix} = \begin{pmatrix} \pm ds\mp bt & 0 \\ 0 & (\pm\lambda_1 d+\mu_1 b)s+(\mp\lambda_1 b+\mu_1 d)t \end{pmatrix} \approxeq \begin{pmatrix} s & 0 \\ 0 & \lambda_2 s+\mu_2 t \end{pmatrix}$$

and from Theorem 2.2.7 it follows that either $b=0$ and

$$\underbrace{\lambda_2(\mp\lambda_1 b+\mu_1 d)}_{=\lambda_2\mu_1 d} = \underbrace{\mu_2(\pm\lambda_1 d+\mu_1 b)}_{=\pm\mu_2\lambda_1 d},$$

in which case $\lambda_2\mu_1 = \pm\lambda_1\mu_2$, so $[\lambda_1 : \mu_1] \in \{[\lambda_2 : \mu_2], [\lambda_2 : -\mu_2]\}$, or $\mp\lambda_1 b + \mu_1 d = 0$ and $\mu_2 \cdot (\pm d) = \lambda_2 \cdot (\mp b)$. These conditions are equivalent to $\lambda_1 b \pm \mu_1 d = 0$ and $\lambda_2 b + \mu_2 d = 0$, respectively. Then both vectors $(\lambda_1, \pm\mu_1)$ and (λ_2, μ_2) are orthogonal to the nonzero vector (b, d), so they are linearly dependent, which means that $[\lambda_2 : \mu_2] = [\lambda_1 : \pm\mu_1]$. ☐

Lemma 6.3.4. Using Notation 4.3.1 and Definition 4.3.10, every quadratic binary form f over $\mathbb{R}$ such that $f(s, 1)$ is quadratic and irreducible is on the same $\mathrm{GL}(2, \mathbb{R})$-orbit as $s^2 + t^2$ in $\mathcal{J}$.

Proof. Let $as^2 + bst + ct^2$ be such a binary form. Then $b^2 - 4ac < 0$, and since

$$a\left(s - \frac{b}{2a}t\right)^2 + b\left(s - \frac{b}{2a}t\right)t + ct^2 = as^2 + \left(c - \frac{b^2}{4a}\right)t^2,$$

we can say that $as^2+bst+ct^2$ is on the same $\mathrm{GL}(2, \mathbb{R})$-orbit as $\lambda s^2+\mu t^2$ where $\lambda, \mu \in \mathbb{R}\backslash\{0\}$ and their signs are the same. We can assume that they are both positive since we can multiply by scalars. Then dividing s by $\sqrt{\lambda}$ and t by $\sqrt{\mu}$ we are done. ☐

Proposition 6.3.5. In $\mathbb{R}^2 \otimes \mathbb{R}^4 \otimes \mathbb{R}^4$ the regular orbits are classified by the tensors listed in Proposition 5.1.10 (with real parameter value a for the last type in the list) together with the following tensors and families of tensors parametrized by $0 < \lambda \leq 1$ and $0 \leq \mu$:

$$\begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & t & s & -t \\ 0 & 0 & t & s \end{pmatrix}, \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & -\lambda t \\ 0 & 0 & t & s \end{pmatrix}, \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s+\mu t \end{pmatrix},$$

$$\begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & t & s \end{pmatrix}.$$

Proof. If we switch out the tensors above to the clearly $\mathrm{GL}_{2,4,4}$-equivalent

$$\begin{pmatrix} s & t & 0 & 0 \\ -t & s & 0 & 0 \\ 0 & -t & s & t \\ 0 & 0 & -t & s \end{pmatrix}, \begin{pmatrix} s & t & 0 & 0 \\ -t & s & 0 & 0 \\ 0 & 0 & s & \lambda t \\ 0 & 0 & -t & s \end{pmatrix}, \begin{pmatrix} s & t & 0 & 0 \\ -t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s-\mu t \end{pmatrix},$$

$$\begin{pmatrix} s & t & 0 & 0 \\ -t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, \begin{pmatrix} s & t & 0 & 0 \\ -t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & -t & s \end{pmatrix},$$

then Proposition 4.3.14, together with Propositions 6.3.2 and 6.3.3, yields that all of the proposed tensors are on different orbits.

Let $T \in \mathbb{R}^2 \otimes \mathbb{R}^4 \otimes \mathbb{R}^4$ be a regular matrix pencil, then from Proposition 4.2.8 it follows that $T \approx sI+tJ$ with J in generalized Jordan normal form. Also of course $sI+tJ \approx sI-tJ$.

We can assume from Remark 6.0.2 that J is not in regular Jordan normal form. Then J is either a 4×4 generalized Jordan block, or it is block diagonal either with two 2×2 block, or one 2×2 and two 1×1 blocks.

If J is a 4×4 generalized Jordan block, then, using the notation from Definition 4.3.12, $sI - tJ = \psi([(f,2)])$, where f is a binary quadratic form for which $f(s,1)$ is irreducible. But f is on the same $\mathrm{GL}(2,\mathbb{R})$-orbit as s^2+t^2 from Lemma 6.3.4, so

$$sI - tJ \approx \psi([(s^2+t^2,2)]) = \begin{pmatrix} s & t & 0 & 0 \\ -t & s & 0 & 0 \\ 0 & -t & s & t \\ 0 & 0 & -t & s \end{pmatrix} \approx \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & t & s & -t \\ 0 & 0 & t & s \end{pmatrix}.$$

If J has a 2×2 block and two 1×1 blocks, then grouping them we have a 3×3 and a 1×1 block. We know that the 3×3 block is not on the same $\mathrm{GL}_{2,3,3}$-orbit with any of the ones listed in Proposition 5.1.10 so from Proposition 6.2.1 it follows that the 3×3 block is on the same $\mathrm{GL}_{2,3,3}$-orbit as

$$\begin{pmatrix} s & -t & 0 \\ t & s & 0 \\ 0 & 0 & s \end{pmatrix},$$

so

$$sI + tJ \approx \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & as+bt \end{pmatrix},$$

and we can divide the last column by a if $a \neq 0$, otherwise by b. If $a \neq 0$ then we can assume $\frac{b}{a} \geq 0$ from Proposition 6.3.3.

In the rest of the proof we can assume that J has two 2×2 blocks. From the fact that J is not in Jordan canonical form and from Proposition 6.1.1 we can assume that the first block is $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$. The second block is either a regular or a generalized Jordan block. Let us first assume that it is the former, then

$$sI + tJ = \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s+at & 0 \\ 0 & 0 & t & s+at \end{pmatrix}.$$

If $M = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$, then from Lemma 6.3.1 we know that $(M,I,I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \approx \begin{pmatrix} s & -t \\ t & s \end{pmatrix}$, and

$$\det\left((M,I,I)\begin{pmatrix} s+at & 0 \\ t & s+at \end{pmatrix}\right) = (1+a^2)^2 s^2,$$

so since $(M,I,I)\begin{pmatrix} s+at & 0 \\ t & s+at \end{pmatrix}$ is indecomposable from Proposition 4.2.3, its Kronecker normal form can only be $\begin{pmatrix} s & 0 \\ t & s \end{pmatrix}$, which means that from Lemma 4.2.2,

$$sI + tJ \approx \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & t & s \end{pmatrix}.$$

We can now assume that the second block in J is a generalized Jordan block, then

$$sI + tJ = \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & -c_0t \\ 0 & 0 & t & s - c_1t \end{pmatrix},$$

where $c_1^2 - 4c_0 < 0$. Let $M = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}$ (without knowing what b is at this point), then $(M, I, I)\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \approx \begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ from Lemma 6.3.1, and

$\det\left((M, I, I)\begin{pmatrix} s & -c_0t \\ t & s - c_1t \end{pmatrix}\right) = (c_0b^2 + c_1b + 1)s^2 + (c_1b^2 + (2-2c_0)b - c_1)st + (b^2 - c_1b + c_0)t^2,$

so here it is easy to see that the coefficients of s^2 and t^2 are nonzero, but $(c_0 - 1)^2 + c_1^2 > 0$ and with

$$b = \frac{c_0 - 1 \pm \sqrt{(c_0 - 1)^2 + c_1^2}}{c_1},$$

it holds that

$$\det\left((M, I, I)\begin{pmatrix} s & -c_0t \\ t & s - c_1t \end{pmatrix}\right) = s^2 + \lambda t^2$$

for $\lambda = \frac{b^2 - c_1b + c_0}{c_0b^2 + c_1b + 1} \neq 0$. The polynomial $s^2 + \lambda t^2$ is not a square, so if it were reducible, then that would mean that the Kronecker normal form of $(M, I, I)\begin{pmatrix} s & -c_0t \\ t & s - c_1t \end{pmatrix}$ was a diagonal matrix pencil. This cannot be the case because of Proposition 4.2.3, so $s^2 + \lambda t^2$ is irreducible, which means $\lambda > 0$. The Kronecker normal form of $(M, I, I)\begin{pmatrix} s & -c_0t \\ t & s - c_1t \end{pmatrix}$ can therefore only be $\begin{pmatrix} s & -\lambda t \\ t & s \end{pmatrix}$, so from Lemma 4.2.2,

$$sI + tJ \approx \begin{pmatrix} s & -t & 0 & 0 \\ t & s & 0 & 0 \\ 0 & 0 & s & -\lambda t \\ 0 & 0 & t & s \end{pmatrix},$$

and from Proposition 6.3.2 we can assume that $\lambda \leq 1$. □

6.4 Parfenov's theorem over $\mathbb{R}$

We have thus proved the following result, which follows from Remark 3.1.2, Propositions 4.2.11, 5.1.10, 6.1.1, 6.2.1 and 6.3.5 and Corollaries 3.2.8, 5.2.7 and 5.3.5. This result can be found (with a somewhat different proof) in [6].

Theorem 6.4.1 ([6]). There are a finite number of $\mathrm{GL}(k_1, \mathbb{R}) \times ... \times \mathrm{GL}(k_r, \mathbb{R})$-orbits in the space $\mathbb{R}^{k_1} \otimes ... \otimes \mathbb{R}^{k_r}$ if and only if the r-tuple $(k_1, ..., k_r)$ is one of the following: $(n), (m, n), (2, 2, n), (2, 3, n)$. Moreover, in each of these cases a complete irredundant list of representatives of the orbits can be easily deduced from Propositions 4.2.11, 5.1.10, 6.1.1, 6.2.1.

Appendices

A SageMath code for computing values of invariants in $K^3 \otimes K^3 \otimes K^3$

In the code we will be using Notation 5.2.6.

```
def zero_triple():
        return matrix([[0,0,0],[0,0,0],[0,0,0]])

def S(triple):
        return triple.subs(s=1,t=0,u=0)

def T(triple):
        return triple.subs(s=0,t=1,u=0)

def U(triple):
        return triple.subs(s=0,t=0,u=1)

def f(triple, i1, i2, i3):
        poly = det(triple)
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(s, sparse = False)) > i1:
                poly = poly.coefficients(s, sparse = False)[i1]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t, sparse = False)) > i2:
                poly = poly.coefficients(t, sparse = False)[i2]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(u, sparse = False)) > i3:
                return poly.coefficients(u, sparse = False)[i3]
        else:
                return 0

def h(triple):
        poly = det(
            block_matrix(
                [
                    [t_2 * T(triple), t_1 * S(triple)],
                    [t_1 * S(triple), t_3 * U(triple)]
                ]
            )
        ).full_simplify()
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_1, sparse = False)) > 2:
                poly = poly.coefficients(t_1, sparse = False)[2]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_2, sparse = False)) > 2:
                poly = poly.coefficients(t_2, sparse = False)[2]
```

```
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_3, sparse = False)) > 2:
                return poly.coefficients(t_3, sparse = False)[2]
        else:
                return 0

def q(triple):
        poly = det(
            block_matrix(
                [
                    [zero_triple(),t_1*S(triple),t_2*T(triple)],
                    [t_4*S(triple),zero_triple(),t_3*U(triple)],
                    [t_5*T(triple),t_6*U(triple),zero_triple()]
                ]
            )
        )
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_1, sparse = False)) > 2:
                poly = poly.coefficients(t_1, sparse = False)[2]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_2, sparse = False)) > 1:
                poly = poly.coefficients(t_2, sparse = False)[1]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_3, sparse = False)) > 2:
                poly = poly.coefficients(t_3, sparse = False)[2]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_4, sparse = False)) > 1:
                poly = poly.coefficients(t_4, sparse = False)[1]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_5, sparse = False)) > 2:
                poly = poly.coefficients(t_5, sparse = False)[2]
        else:
                return 0
        if type(poly) != sage.rings.integer.Integer and len(poly.
    coefficients(t_6, sparse = False)) > 1:
                return poly.coefficients(t_6, sparse = False)[1]
        else:
                return 0

def H(triple):
        return (h(triple) - (1/3)*f(triple,2,1,0)*f(triple,0,1,2) - (1/3)*f(
    triple,2,0,1)*f(triple,0,2,1) + (2/3)*f(triple,1,2,0)*f(triple,1,0,2) +
    (1/12)*f(triple,1,1,1)^2).full_simplify()

def Q(triple):
```

```
        return (q(triple) - (1/2)*h(triple)*f(triple,1,1,1) + (3/2)*f(triple
    ,3,0,0)*f(triple,0,3,0)*f(triple,0,0,3) - (1/2)*f(triple,3,0,0)*f(triple
    ,0,2,1)*f(triple,0,1,2) - (1/2)*f(triple,0,3,0)*f(triple,2,0,1)*f(triple
    ,1,0,2) - (1/2)*f(triple,0,0,3)*f(triple,2,1,0)*f(triple,1,2,0) - (1/2)*
    f(triple,1,1,1)*f(triple,1,2,0)*f(triple,1,0,2) + (1/2)*f(triple,2,1,0)*
    f(triple,1,0,2)*f(triple,0,2,1) + (1/2)*f(triple,1,2,0)*f(triple,2,0,1)*
    f(triple,0,1,2)).full_simplify()

def invariant(triple):
        return (H(triple)^3/Q(triple)^2).full_simplify()
```

References

[1] URL: `https://math.stackexchange.com/questions/1905543/find-the-rank-of-the-tensor`. Visited on 29 October 2022.

[2] Boris Alexeev, Michael A. Forbes **and** Jacob Tsimerman. "Tensor Rank: Some Lower and Upper Bounds". **in** *2011 IEEE 26th Annual Conference on Computational Complexity*: 2011, **pages** 283–291. DOI: `10.1109/CCC.2011.28`.

[3] I. Assem, D. Simson **and** A. Skowronski. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*. Elements of the Representation Theory of Associative Algebras. Cambridge University Press, 2006. ISBN: 9780521584234.

[4] Peter Bürgisser, Michael Clausen **and** Amin Shokrollahi. *Algebraic Complexity Theory*. **volume** 315. **january** 1997. ISBN: 978-3-642-08228-3. DOI: `10.1007/978-3-662-03338-8`.

[5] Kris Coolsaet. "On the classification of nonsingular $2\times2\times2\times2$ hypercubes". **in** *Designs, Codes and Cryptography*: 68 (**july** 2013). DOI: `10.1007/s10623-012-9737-8`.

[6] Dragomir Z. Djokovic **and** Peter W. Tingley. "Natural group actions on tensor products of three real vector spaces with finitely many orbits". **in** *Electronic Journal of Linear Algebra*: 8 (**may** 2001). DOI: `10.13001/1081-3810.1061`.

[7] Mátyás Domokos **and** Vesselin Drensky. "Defining relation for semi-invariants of three by three matrix triples". **in** *Journal of Pure and Applied Algebra*: 216.10 (2012), **pages** 2098–2105. ISSN: 0022-4049. DOI: `https://doi.org/10.1016/j.jpaa.2012.01.017`. URL: `https://www.sciencedirect.com/science/article/pii/S0022404912000345`.

[8] Richard Ehrenborg. "Canonical Forms of Two by Two by Two Matrices". **in** *Journal of Algebra*: 213 (1999), **pages** 195–224.

[9] Moussong Gábor. *Geometria*. Typotex, 2014. ISBN: 978-963-2792-57-6.

[10] F.R. Gantmacher. *The Theory of Matrices (Vol. I & II)*. Chelsea Publishing Company, 1959.

[11] Johan Håstad. "Tensor rank is NP-complete". **in** *Automata, Languages and Programming*: **byeditor** Giorgio Ausiello, Mariangiola Dezani-Ciancaglini **and** Simonetta Ronchi Della Rocca. Berlin, Heidelberg: Springer Berlin Heidelberg, 1989, **pages** 451–460. ISBN: 978-3-540-46201-9.

[12] Thomas D. Howell **and** Jean Lafon. *The Complexity of the Quaternion Product*. techreport. USA, 1975.

[13] V.G. Kac. "Some remarks on nilpotent orbits". **in** *Journal of Algebra*: 64.1 (1980), **pages** 190–213. ISSN: 0021-8693. DOI: `https://doi.org/10.1016/0021-8693(80)90141-6`.

[14] L. Kronecker. "Algebraische Reduction der Schaaren bilinearer Formen". **in** *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*: (1890).

[15] J. M. Landsberg. *Geometry and Complexity Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017. DOI: `10.1017/9781108183192`.

[16] J.M. Landsberg. *Tensors: Geometry and Applications*. Graduate studies in mathematics. American Mathematical Society, 2011. ISBN: 9780821869079.

[17] S.M. Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, 1998. ISBN: 9780387984032.

[18] M. Michałek **and** B. Sturmfels. *Invitation to Nonlinear Algebra*. Graduate Studies in Mathematics. American Mathematical Society, 2021. ISBN: 9781470453671.

[19] Peter J. Olver. *Classical Invariant Theory*. London Mathematical Society Student Texts. Cambridge University Press, 1999. DOI: 10.1017/CBO9780511623660.

[20] S. R. Nagpaul P. B. Bhattacharya S. K. Jain. *Basic Abstract Algebra*. 2 **edition**. Cambridge University Press, 1994. ISBN: 0521466296.

[21] Pyotr Glebovich Parfenov. "Orbits and their closures in the spaces $\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_r}$". **in** *Sbornik: Mathematics*: 192.1 (2001), **page** 89.

[22] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.5)*. https://www.sagemath.org. 2022.

[23] Vin de Silva **and** Lek-Heng Lim. "Tensor Rank and the Ill-Posedness of the Best Low-Rank Approximation Problem". **in** *SIAM Journal on Matrix Analysis and Applications*: 30.3 (2008), **pages** 1084–1127. DOI: 10.1137/06066518X. URL: https://doi.org/10.1137/06066518X.