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Quantitative Helly-type Theorems and Hypergraph Chains

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Introduction

Helly's hundred years old Theorem is one of the cornerstones of discrete geometry. After much progress in the past fifty years, Helly-type questions are still a very actively researched area, see for example the recent survey of Bárány and Kalai [BK21].

One way of stating Helly's Theorem is that if we have a finite family of convex sets from $\mathbb{R}^d$ such that every $d+1$ member of the family have a point in common, then all the sets from the family have a point in common.

A useful generalisation, the *Fractional Helly Theorem* is about the case, where not every subfamily of size $d+1$ intersect, but a linear fraction of them do. It states the following. If we have a finite family $\mathcal{C}$ of convex sets from $\mathbb{R}^d$ such that at least $\alpha\binom{|\mathcal{C}|}{d+1}$ of the $d+1$ size subfamilies are intersecting, then there is a linear size intersecting subfamily: $\mathcal{C}' \subset C$ with $\bigcap\limits_{C \in \mathcal{C}'} C \neq \emptyset$ and $|\mathcal{C}'| \geq \beta|\mathcal{C}|$, where β depends on α and d, but does not depend on $\mathcal{C}$ [KL79].

Another generalisation, the *Colorful Helly Theorem* concerns more than one family of convex sets. If $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ are families (color classes) of convex sets from $\mathbb{R}^d$ with nonempty intersections of all the heterochromatic $d+1$-tuples, then there is a color class, $\mathcal{C}_j$, such that $\bigcap\limits_{C \in \mathcal{C}_j} C \neq \emptyset$ [Lov74] [Bár82].

The main topic of this thesis is these two generalisation of Helly's Theorem, but in a different setting. Bárány, Katchalski and Pach examined a *Quantitative Helly Theorem*, which states the following. There is a constant $c(d) > 0$ depending on d only such that the following is true for every finite family $\mathcal{C}$ of convex bodies (compact convex sets with nonempty interior) from $\mathbb{R}^d$. If every $2d$ sets from $\mathcal{C}$ have intersection of volume at least 1, then $\bigcap\limits_{C\in\mathcal{C}} C$ has volume at least $c(d)$. In [BKP82], it is proved that one can take $c(d) = d^{-2d^2}$ and conjectured that it should hold with $c(d) = d^{-cd}$ for an absolute constant $c > 0$. It was confirmed with $c(d) \approx d^{-2d}$ by Naszódi [Nas16], whose argument was refined by Brazitikos [Bra17], who showed that one may take $c(d) \approx d^{-3d/2}$.

Our starting points are two recent result. One is a result by Damásdi, Földvári and Naszódi, who showed a Quantitative Colorful Helly Theorem with $3d$ color classes [DFN21]. The other one is a result by Holmsen, who proved, that a certain colorful Helly property for hypergraphs implies a certain fractional helly property [Hol20].

Our two main contributions follow. First, in Theorem 7, we establish a new quantitative variant of the Fractional Helly Theorem of Katchalski and Liu [KL79], where the Helly Number is $3d+1$, improving the previous $O(d^2)$ bound. This part of the work has been published earlier this year in [JN22]. In addition to this result, we prove that a certain Quantitative Colorful Helly Theorem implies a Quantitative Fractional Helly Theorem with a purely combinatorial proof (Theorem 8). The latter, combined with the Quantitative Colorful Helly Theorem of Damásdi, Földvári and Naszódi [DFN21], implies a Quantitative Fractional Helly Theorem with Helly Number $3d$ as is shown in Corollary 5.

The structure of the thesis is the following. In Chapter 1 we state and prove Helly's Theorem and the Fractional and Colorful Helly Theorems. We introduce quantitative Helly-type Theorems in Chapter 2, where we prove Colorful and Fractional versions of the Quantitative Helly Theorem. Section 2.4 contains Theorem 7, our first main result. In Chapter 3 we propose a new combinatorial framework to analyze quantitative Helly-type Theorems, and prove Theorem 8, our second main results. The thesis ends with a discussion of open problems in Chapter 4.

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Contents

Chapter 1

Helly-type Theorems

In this chapter, we state and prove Helly's Theorem and the Colorful and Fractional Helly Theorems.

1.1 Helly's Theorem and Radon's Lemma

Helly found his theorem in 1913, but could not publish it until 1923 due to World War 1 [Hel23]. Radon's proof appeared in 1921 [Rad21].

Theorem 1 ([Hel23])**.** *Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $d+1$ sets of $\mathcal{C}$ have a point in common, then all the sets of $\mathcal{C}$ have a point in common.*

Proof of Theorem 1. We prove Helly's Theorem by induction on the size of $\mathcal{C}$. The first nontrivial case is when $|\mathcal{C}| = d+2$. Let $\mathcal{C} = \{C_1, \ldots, C_{d+2}\}$. The assumption of the Theorem is that for every $i \in [d+2] = \{1, 2, \ldots, d+2\}$ there is an $x_i \in \cap_{j \neq i} C_j$. Let $X = \{x_1, \ldots, x_{d+2}\}$. We will need Radon's Lemma, an easy consequence of affine dependency.

Lemma 1 ([Rad21])**.** *We can partition every $d+2$ point set $X \subset \mathbb{R}^d$ into two disjoint subsets $X = X^+ \cup X^-$ such that* $\operatorname{conv}(X^+) \cap \operatorname{conv}(X^-) \neq \emptyset$*.*

Proof of Lemma 1. Since every $d+2$ vector in $\mathbb{R}^d$ are affinely dependent, there are real numbers $\alpha_1, \dots, \alpha_{d+2}$ not all of them 0 such that $\sum \alpha_i = 0$ and $\sum \alpha_i x_i = 0$. If $I^+ = \{i \in [d+2] : \alpha_i \geq 0\}$ and $I^- = \{i \in [d+2] : \alpha_i < 0\}$, then

$$\sum_{i \in I^+} \alpha_i x_i = -\sum_{i \in I^-} \alpha_i x_i = x.$$

Let $S = \sum_{i \in I^+} \alpha_i$ and $X^+ = \{x_i : i \in I^+\}$, $X^- = \{x_i : i \in I^-\}$. This is a good partition, since $x \in \mathrm{conv}(X^+) \cap \mathrm{conv}(X^-)$, as $x = \sum_{i \in I^+} \frac{\alpha_i}{S} x_i$ is a convex combination and $x = \sum_{i \in I^-} \frac{-\alpha_i}{S} x_i$ is also a convex combination. $\square$

Now we can continue the proof of the case, when $|\mathcal{C}| = d+2$. Let $\mathcal{C}$ and X be as before and let $X^+ \cup X^- = X$ and $x \in \mathrm{conv}(X^+) \cap \mathrm{conv}(X^-)$ as in Radon's Lemma. We claim, that $x \in C_i$ for all $i \in [d+2]$. If $i \in I^+$, then $X^- \subset C_i$ and so $x \in \mathrm{conv}(X^-) \subset C_i$. If $i \in I^-$, then $X^+ \subset C_i$ and $x \in \mathrm{conv}(X^+) \subset C_i$.

For the induction step, let us assume, that Helly's Theorem holds if the size of the family is n, and let $\mathcal{C} = \{C_1, \dots, C_{n+1}\}$ be a family of convex sets from $\mathbb{R}^d$ such that every $d+1$ of them have a point in common. Let $\mathcal{C}' = \{C_1, C_2, \dots, C_{n-1}, C_n \cap C_{n+1}\}$. If the sets from $\mathcal{C}'$ have a point in common, then the sets from $\mathcal{C}$ also have a point in common. We claim that every $d+1$ sets from $\mathcal{C}'$ have a point in common. It is indeed true, since if $C_n \cap C_{n+1}$ is not among the $d+1$ sets, then our assumption about $\mathcal{C}$ guarantees that the $d+1$ sets intersect. On the other hand, if $C_n \cap C_{n+1}$ is among the $d+1$ chosen sets, say $\{C_{i_1}, \dots, C_{i_d}, C_n \cap C_{n+1}\}$, then any $d+1$ sets from $\{C_{i_1}, \dots, C_{i_d}, C_n, C_{n+1}\}$ intersect and Helly's Theorem with $d+2$ sets implies that all of them have a point in common. Now since every $d+1$ sets intersect from the family of n convex sets $\mathcal{C}'$, the induction hypotheses yields a point x in all the members of $\mathcal{C}'$.

$\square$

One might argue that Helly's Theorem is a purely combinatorial consequence of Radon's Lemma. This is a usual case in Helly-type Theorems. We will see more of this phenomenon later. One way of formalizing the notion of "purely combinatorial consequence" will be presented in Section 3.1.

1.2 Colorful and Fractional versions of Helly's Theorem

In this section, we prove two generalisations of Helly's Theorem. As a technical tool, we need the lexicographic ordering of $\mathbb{R}^d$. We say that a point $x \in \mathbb{R}^d$ is lexicographically smaller than a point $y \in \mathbb{R}^d$, in notation, $x <_{\text{lex}} y$, if for the first coordinate j where they differ, $x_j < y_j$. Any compact subset $C \subset \mathbb{R}^d$ has a unique minimum point according to this ordering, we will denote it by $\text{lexmin}(C)$.

Lemma 2 ([Mat02], Lemma 8.1.2)**.** *If $\mathcal{C}$ is a finite family of compact, convex sets of $\mathbb{R}^d$ with nonempty intersection, then there exists a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at most d such that $\text{lexmin}(\mathcal{C}') = \text{lexmin}(\mathcal{C})$.*

Proof of Lemma 2. Let $L = \{x \in \mathbb{R}^d : x <_{\text{lex}} \text{lexmin}(\mathcal{C})\}$. The family $\mathcal{C} \cup \{L\}$ has empty intersection, so by Helly's Theorem, there are at most $d+1$ members, whose intersection alone is nonempty. The set L must be one of the $d+1$, since every subfamily of $\mathcal{C}$ alone has nonempty intersection. The other d convex sets from the $d+1$ can form $\mathcal{C}'$. □

Now we are ready to state and prove the above mentioned two generalisations of Helly's Theorem. The first one is the Colorful Helly Theorem discovered by Lovász [Lov74] (and with the first published proof by Bárány [Bár82]).

Theorem 2 (Colorful Helly Theorem)**.** *If $\mathcal{C}_1, \dots, \mathcal{C}_{d+1}$ are finite families of convex sets from $\mathbb{R}^d$ and for every colorful selection $C_1 \in \mathcal{C}_1, \dots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $\cap_i C_i$ is nonempty, then there is a color class $\mathcal{C}_j$ such that all the sets from $\mathcal{C}_j$ have a point in common.*

Proof of Theorem 2. Note that it is enough to prove the theorem for compact convex sets. Otherwise choose a point from every possible intersection from $\mathcal{C}$, let the set of these points be X, and replace every $C \in \mathcal{C}$ with the convex hull of $X \cap C$.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be finite families of compact convex sets from $\mathbb{R}^d$ and assume that all the colorful intersections are nonempty. Let $C_1^*, \ldots, C_{d+1}^*$ be such that $\mathrm{lexmin}(\cap_i C_i^*)$ is the maximum of the set $\{\mathrm{lexmin}(\cap_i C_i) : C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}\}$. According to Lemma 2, we can rearrange the color classes so that $\mathrm{lexmin}(\cap_{i=1}^{d+1} C_i^*) = \mathrm{lexmin}(\cap_{i=1}^{d} C_i^*) = x^*$. We claim that $x^* \in \cap_{C \in \mathcal{C}_{d+1}} C$. This follows from the stronger statement, that for the highest lexicographic minimum $x^* = \mathrm{lexmin}(\cap_{i=1}^{d} C_i^* \cap C)$ for all $C \in \mathcal{C}_{d+1}$. This is indeed true, since on the one hand $\mathrm{lexmin}(\cap_{i=1}^{d} C_i^* \cap C) \leq_{\mathrm{lex}} \mathrm{lexmin}(\cap_{i=1}^{d} C_i^*)$ by the definition of the C_i^*s and on the other hand $\mathrm{lexmin}(\cap_{i=1}^{d} C_i^* \cap C) \geq_{\mathrm{lex}} \mathrm{lexmin}(\cap_{i=1}^{d} C_i^*)$, because the lexicographic minimum can not decrease if we add one more set to the intersection.

Note that it is enough to prove the theorem for compact convex sets. Choose a point from every possible intersection from $\mathcal{C}$, let the set of these points be X, and replace every $C \in \mathcal{C}$ with the convex hull of $X \cap C$. $\square$

Note that if we take $\mathcal{C}_1 = \ldots = \mathcal{C}_{d+1}$, then we recover Helly's Theorem as a consequence. The second discussed generalisation of Helly's Theorem is the Fractional Helly Theorem of Katchalski and Liu [KL79].

Theorem 3 (Fractional Helly Theorem)**.** *There is a function $\beta[0,1] \to [0,1]$ such that if $\mathcal{C}$ is a finite family of convex sets from $\mathbb{R}^d$ such that among the possible $\binom{|\mathcal{C}|}{d+1}$ subfamilies of size $d+1$ at least $\alpha\binom{|\mathcal{C}|}{d+1}$ are intersecting then there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| \geq \beta(\alpha)|\mathcal{C}|$ such that all the sets from $\mathcal{C}'$ have a point in common.*

One of the many proofs of the Fractional Helly Thoerem is somewhat similar to the proof of the Colorful Helly Theorem. It first appeared in [Mat02].

Proof of Theorem 3. Let $n = |\mathcal{C}|$ and $\mathcal{C} = \{C_1, \dots, C_n\}$. Similarly to the proof of Theorem 2, it is enough to prove the theorem for compact sets. We say that a $d+1$ element index set $I \in \binom{[n]}{d+1}$ is good, if $\cap_{i \in I} C_i \neq \emptyset$, and that a d element index set $S \in \binom{[n]}{d}$ is a seed of a good index set $I \supset S$, if $\operatorname{lexmin}(\cap_{s \in S} C_s) = \operatorname{lexmin}(\cap_{i \in I} C_i)$. By Lemma 2, each good index set has a seed. Since there are $\alpha\binom{n}{d+1}$ good index sets and only $\binom{n}{d}$ possible seeds, there is an index set S which is the seed of at least

$$\frac{\alpha\binom{n}{d+1}}{\binom{n}{d}} = \frac{\alpha(n-d)}{d+1}$$

good index sets. Let $I_1, \dots, I_{\frac{\alpha(n-d)}{d+1}}$ be good index sets whose seed is S. Since for all I_i, there is an index j such that $I_i = S \cup \{j\}$, there are at least $\frac{\alpha(n-d)}{d+1}$ convex sets C_j such that $\operatorname{lexmin}(\cap_{i \in S} C_i) \in C_j$ and $j \notin S$. Together with the sets $C_s : s \in S$, the point $\operatorname{lexmin}(\cap_{s \in S} C_s)$ is contained in at least $\frac{\alpha(n-d)}{d+1} + d \geq \frac{\alpha n}{d+1}$ convex sets from $\mathcal{C}$. $\square$

For the value of β, the above proof gives $\beta(\alpha) = \frac{\alpha}{d+1}$. Helly's Theorem states, that $\beta(1) = 1$. Kalai proved that the optimal β is $\beta(\alpha) = 1 - (1 - \alpha)^{1/(d+1)}$ [Kal84]. One important feature of Kalai's function is that $\beta \to 1$ as $\alpha \to 1$.

Chapter 2

Quantitative Helly-type Theorems

We present quantitative analogues of the three Helly-type Theorem from the previous chapter, where instead of finding points in the intersection of convex sets, we give lower bounds on the volume of the intersection. For such questions, we consider convex bodies, ie. compact convex sets with nonempty interior.

2.1 Quantitative versions of Helly's Theorem

The starting point for Quantitative Helly-type Theorems is an article of Bárány, Katchalski and Pach [BKP82], where they show the following.

Theorem 4. *There is a constant $c(d) > 0$ such that the following holds. Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$. Assume that the intersection of any $2d$ of them is of volume at least 1. Then $\bigcap_{i=1}^{n} C_i$ is of volume at least $c(d)$.*

In [BKP82], it is proved that one can take $c(d) = d^{-2d^2}$ and conjectured that it should hold with $c(d) = d^{-cd}$ for an absolute constant $c > 0$. It was confirmed with $c(d) \approx d^{-2d}$ by Naszódi [Nas16], whose argument was refined by Brazitikos [Bra17], who showed that one may take $c(d) \approx d^{-3d/2}$.

It is common to approximate the volume of a convex body with that of the largest inscribed ellipsoid. As a consequence of John's Theorem [Joh48], for any convex body $C \subset \mathbb{R}^d$, if $E \subset C$ is its largest volume inscribed ellipsoid, then the relation $C \subset dE$ holds if E is origin centered. That implies $\mathrm{vol}\,(E) \leq \mathrm{vol}\,(C) \leq d^d \mathrm{vol}\,(E)$ for the volumes. So the volume of the largest inscribed ellipsoid approximates the volume of a convex body up to a factor of d^d. It might seem a very weak approximation at first sight, but approximating the volume of convex bodies is a hard problem. For example, as a result of Bárány and Füredi asserts, there is no deterministic polynomial time algorithm approximating the volume of convex bodies better than up to a factor of $\left(\frac{cd}{\log d}\right)^d$ [BF87]. So, at least from an algorithmic point of view, the approach is justified.

The proofs in [BKP82], [Nas16] and [Bra17] use ellipsoids and John's Theorem to approximate the volume of convex bodies. In special, Theorem 4 follows from the following result about ellipsoids.

Theorem 5 ([Bra17])**.** *Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$. Assume that the intersection of any $2d$ of them contains an ellipsoid of volume at least one. Then $\bigcap\limits_{i=1}^{n} C_i$ contains an ellipsoid of volume at least $c^d d^{-3d/2}$ with an absolute constant $c > 0$.*

From now on, we will state quantitative Helly-type theorems in terms of volumes of inscribed ellipsoids.

A variant of Theorem 5 is the following observation from [DFN21]. We assume larger subfamilies to have an intersection whose largest inscribed ellipsoid is of volume 1, but in return we get a volume 1 ellipsoid contained in all the convex bodies, not just an ellipsoid of volume $\approx d^{-cd}$. We state it here without proof.

Proposition 1 ([DFN21, Proposition 1.1]). *Let $C_1, \dots, C_n$ be convex bodies in $\mathbb{R}^d$. Assume that the intersection of any $d(d+3)/2$ of them contains an ellipsoid of volume at least one. Then $\bigcap_{i=1}^{n} C_i$ contains an ellipsoid of volume at least one.*

If we are interested in ellipsoids contained in convex bodies, it is a useful variant of the previous Theorem, but in terms of approximating the volume of the intersection of convex bodies it is not very useful, since John ellipsoids approximate the volume only up to a factor of d^d. In other words, Proposition 1 has a stronger assumption than Theorem 5, but we do not obtain a much stronger conclusion in return (at least in terms of volumes).

As an informal notion, we will call quantitative Helly-type theorems where we assume $O(d^2)$-size intersecting subfamilies *theorems with large intersection* and theorems where we assume only $O(d)$-size intersecting subfamilies *theorems with small intersections.* Since for the Colorful and Fractional Helly Theorems only weaker forms of Theorem 5 are avaible, we present also the variants with large intersection.

2.2 Quantitative Colorful and Fractional Helly Theorems with large intersections

When proving quantitative Helly-type theorems with large intersection, one can usually maintain the structure of the proofs of the classical theorems. The "combinatorial part" of the proofs remains the same, but we need new "geometrical tools".

The proof of the Colorful and Fractional Helly Theorems depended on the lexicographic minima of convex sets and on Lemma 2. As an analogue, the following definition and two lemmas introduce the unique lowest ellipsoid of volume one contained in a convex body.

Definition 1. For an ellipsoid E, we define its *height* as the largest value of the orthogonal projection of E on the last coordinate axis.

The proof of the following claim can be found in [DFN21].

Lemma 3 (Uniqueness of Lowest Ellipsoid, [DFN21, Lemma 2.5])**.** *Let C be a convex body, such that it contains an ellipsoid of volume one. Then there is a unique ellipsoid of volume one such that every other ellipsoid of volume one in C has larger height. We call this ellipsoid the* lowest ellipsoid *in C.*

The lowest ellipsoid will play the role of the lexicographic minimum in quantitative theorems with large intersections. The following is an analogue of Lemma 2 for the lowest ellipsoid.

Lemma 4 (Lowest ellipsoid determined by $O(d^2)$ members of an intersection [DFN21, Lemma 3.1])**.** *Let $C_1, \ldots, C_n$ be a finite family of convex bodies in $\mathbb{R}^d$ whose intersection contains an ellipsoid of volume one. Then, there are $d(d+3)/2-1$ indices $i_1, \ldots, i_{d(d+3)/2-1} \in [n]$ such that $\bigcap_{i=1}^{n} C_i$, and $\bigcap_{j=1}^{d(d+3)/2-1} C_{i_j}$ have the same unique lowest ellipsoid.*

For a proof see [DFN21]. The lowest ellipsoid is determined by $\frac{d(d+3)}{2} - 1$ members of the intersection instead of d as in the case of lexicographic minimum and Lemma 2. Now we are ready to state and prove Quantitative Colorful and Fractional Helly Theorems with large intersections. The following is from [Dam17].

Proposition 2 (QCH – large subfamilies [Dam17])**.** *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d(d+3)/2}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for all colorful selections $C_1 \in \mathcal{C}_1, \ldots C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$ the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume one.*

Then, there is a j with $1 \leq j \leq d(d+3)/2$ such that $\bigcap_{C \in \mathcal{C}_j}$ contains an ellipsoid of volume one.

Proof. The following argument, due to Damásdi [Dam17], follows closely the proof of Theorem 2, the only difference is the use of the unique lowest ellipsoid of a convex body (Lemma 3) instead of its lexicographic minimum.

Let $C_1^* \in \mathcal{C}_1, \ldots, C_{d(d+3)/2}^* \in \mathcal{C}_{d(d+3)/2}$ be such, that the lowest ellipsoid of $\bigcap\limits_{i=1}^{d(d+3)/2} C_i^*$ is the highest among all lowest ellipsoids of the colorful intersections. Let E_{max} denote this highest one of the lowest ellipsoids. By Lemma 4 we can rearrange the color classes so that E_{max} is the lowest ellipsoid of $\bigcap\limits_{i=1}^{d(d+3)/2-1} C_i^*$. We claim that $E_{max} \subset \bigcap\limits_{C \in \mathcal{C}_{d(d+3)/2}} C$. This follows from the stronger statement, that E_{max} is the lowest ellipsoid of $C \cap \bigcap\limits_{i=1}^{d(d+3)/2-1} C_i^*$ for all $C \in \mathcal{C}_{d(d+3)/2}$. This is indeed true, since on the one hand their lowest ellipsoid can not be higher than E_{max}, since E_{max} is the highest, and on the other hand it can not be lower than the lowest ellipsoid of $\bigcap\limits_{i=1}^{d(d+3)/2-1} C_i^*$, since we only added one more constraint. $\square$

Observe how much the proof of Proposition 2 is similar to that of Theorem 2. One can do similarly with the Fractional Helly Theorem, as was showed by the author and Naszódi in [JN22].

Proposition 3 (QFH – large subfamilies)**.** *For every dimension $d \geq 1$ and every $\alpha \in (0,1)$ the following holds.*

Let $\mathcal{C}$ be a finite family of convex bodies in $\mathbb{R}^d$. Assume that among all subfamilies of size $\frac{d(d+3)}{2}$, there are at least $\alpha\binom{|\mathcal{C}|}{\frac{d(d+3)}{2}}$ for whom the intersection of the $\frac{d(d+3)}{2}$ members contains an ellipsoid of volume one.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ contains an ellipsoid of volume one, where $\beta = \frac{2\alpha}{d(d+3)}$.

Proof of Proposition 3. The following argument follows closely the proof of Theorem 3, the only difference is the use of the unique lowest ellipsoid of a convex body (Lemma 3) instead of its lexicographic minimum.

Let $\mathcal{C} = \{C_1, \dots, C_n\}$. We call an index set $I \in \binom{[n]}{\frac{d(d+3)}{2}}$ *good*, if the corresponding intersection $\cap_{i \in I} C_i$ contains an ellipsoid of volume at least one. We say that a $\left(\frac{d(d+3)}{2} - 1\right)$-element subset $S \subset I$ of a good index set $I \in \binom{[n]}{\frac{d(d+3)}{2}}$ is a *seed* of I, if $\cap_{i \in I} C_i$ and $\cap_{i \in S} C_i$ have the same lowest ellipsoid. By Lemma 4, all good index sets have a seed.

Since we have $\alpha \binom{n}{\frac{d(d+3)}{2}}$ good index sets and only $\binom{n}{\frac{d(d+3)}{2} - 1}$ possible seeds, there is a $\left(\frac{d(d+3)}{2} - 1\right)$-tuple $S \in \binom{[n]}{\frac{d(d+3)}{2} - 1}$ which is the seed of at least

$$\frac{\alpha \binom{n}{\frac{d(d+3)}{2}}}{\binom{n}{\frac{d(d+3)}{2} - 1}} = \alpha \frac{n - \frac{d(d+3)}{2} + 1}{\frac{d(d+3)}{2}}$$

good index sets. Every such good index set has the form $S \cup \{i\}$ for an i. So we have $\alpha \frac{n - \frac{d(d+3)}{2} + 1}{d(d+3)/2}$ convex bodies containing the lowest ellipsoid of $\cap_{i \in S} C_i$, plus the $(\frac{d(d+3)}{2} - 1)$ convex body from S. Hence, the lowest ellipsoid of $\cap_{i \in S} C_i$ is contained in at least

$$\alpha \frac{n + 1 - d(d+3)/2}{d(d+3)/2} + \frac{d(d+3)}{2} - 1 \geq \frac{2\alpha n}{d(d+3)}$$

convex bodies among the C_i, completing the proof of Proposition 3. $\square$

2.3 Helly-type Theorems for translates of a convex body

Before discussing Quantitative Helly-type Theorems with small intersections, let us state two easy consequence of the Colorful and Fractional Helly Theorems.

Proposition 4. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be finite families of convex bodies, and L a convex body in $\mathbb{R}^d$. Assume that for any colorful subfamily $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ contains a translate of L. Then for some j, the intersection $\bigcap_{C \in \mathcal{C}_j} C$ contains a translate of L.*

Proof. We use the following operation, the *Minkowski difference* of two convex sets A and B:

$$A \sim B := \bigcap_{b \in B} (A - b).$$

It is easy to see that $A \sim B$ is the set of vectors t such that $B + t \subseteq A$.

Now, replace each convex set C in $\cup_i \mathcal{C}_i$ by $C \sim L$, and apply the Colorful Helly Theorem (Theorem 2). □

Proposition 5. *For every dimension $d \geq 1$ and every $\alpha \in (0, 1)$, the following holds.*
Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$ and let L be a convex set in $\mathbb{R}^d$. Assume that among all subfamilies of size $d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{d+1}$ for whom the intersection of the $d+1$ members contains a translate of L.
Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\frac{\alpha}{d+1}|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ contains a translate of L.

Proof. Replace each convex set C in $\mathcal{C}$ by $C \sim L$, and apply the Fractional Helly Theorem (Theorem 3). □

2.4 Quantitative Colorful and Fractional Helly Theorems with small intersections

In quantitative Helly-type theorems, where the assumption is about linear-sized subfamilies, one usually needs more than a clever translation of the proofs of the classical theorems. In this section, we present "linear-sized" quantitative variants of the Colorful and Fractional Helly Theorems. Both proof uses the propositions of the previous section and the following technical Lemma.

Lemma 5 ([DFN21, Lemma 3.2]). *Assume that the origin centered Euclidean unit ball,* $\mathbf{B}^d$ *is the largest volume ellipsoid contained in the convex set* C *in* $\mathbb{R}^d$*. Let* E *be another ellipsoid in* C *of volume at least* $\delta \operatorname{vol}\left(\mathbf{B}^d\right)$ *with* $0 < \delta < 1$*. Then there is a translate of* $\frac{\delta}{d^{d-1}}\mathbf{B}^d$ *which is contained in* E*.*

Proof of Lemma 5. If the length of all d semi-axes $a_1, \ldots, a_d$ of E are at least λ for some $\lambda > 0$, then clearly, $\lambda\mathbf{B}^d + c \subset E$, where c denotes the center of E. We will show that all the semi-axes are long enough.

By John's theorem, $E \subset C \subset d\mathbf{B}^d$. Therefore, $a_i \leq d$ for every $i = 1, \ldots, d$. Since the volume of E divided by the volume of $\mathbf{B}^d$ is $a_1 \cdot \ldots \cdot a_d \geq \delta$, we have $a_i \geq \frac{\delta}{d^{d-1}}$ for every $i = 1, \ldots, d$, completing the proof of Lemma 5. □

The following Quantitative Colorful Helly Theorem is shown in [DFN21].

Theorem 6 (Quantitative Colorful Helly Theorem). *] Let* $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ *be finite families of convex sets in* $\mathbb{R}^d$*. Assume that for any colorful choice of* $2d$ *sets,* $C_{i_k} \in \mathcal{C}_{i_k}$ *for each* $1 \leq k \leq 2d$ *with* $1 \leq i_1 < \ldots < i_{2d} \leq 3d$*, the intersection* $\bigcap_{k=1}^{2d} C_{i_k}$ *contains an ellipsoid of volume one.*
Then, there is an i *with* $1 \leq i \leq 3d$ *such that the intersection* $\bigcap_{C \in \mathcal{C}_i} C$ *contains an ellipsoid of volume* $c^{d^2} d^{-5d^2/2+d}$*, where* c *is the universal constant from Theorem 5.*

The Quantitative Colorful Helly Theorem, Theorem 6 and its proof are completely from [DFN21], they are presented here in order to help the comparison of the proofs of Theorem 6 and Theorem 7. The latter is one of the main results of the thesis.

Proof of Theorem 6. In this proof we will replace volume by normalized volume, which is the volume divided by the volume of the ball $\mathbf{B}^d$ of unit radius centered at the origin.

To prove Theorem 6, we will assume that all colorful choices of $2d$ sets contain an ellipsoid of normalized volume at least one.

Consider the lowest ellipsoid of normalized volume 1 in all colorful choices of $2d-1$ sets. We may assume that the highest one of these ellipsoids is $\mathbf{B}^d$. By possibly changing the indices of the families, we may assume that the choice is $C_1 \in \mathcal{C}_1, \ldots, C_{2d-1} \in \mathcal{C}_{2d-1}$. We call $\mathcal{C}_{2d}, \mathcal{C}_{2d+1}, \ldots, \mathcal{C}_{3d}$ the *remaining families*.

Consider the half-space $H_1 \supset \mathbf{B}^d$ with outer normal e_d, bounded by a supporting hyperplane of $\mathbf{B}^d$. Clearly, $\mathbf{B}^d$ is the largest volume ellipsoid contained in $M := C_1 \cap \ldots \cap C_{2d-1} \cap H_1$.

Next, take an arbitrary colorful choice $C_{2d} \in \mathcal{C}_{2d}, C_{2d+1} \in \mathcal{C}_{2d+1}, \ldots, C_{3d} \in \mathcal{C}_{3d}$ of the remaining $d+1$ families. We claim that the intersection of any $2d$ sets of

$$C_1, \ldots, C_{2d-1}, H_1, C_{2d}, \ldots, C_{3d}$$

contains an ellipsoid of normalized volume at least 1. Indeed, if H_1 is not among those $2d$ sets, then our assumption ensures this. If H_1 is among them, then by the choice of H_1, the claim holds.

Therefore, by Theorem 5, the intersection

$$\bigcap_{i=1}^{3d} C_i \cap H_1$$

contains an ellipsoid E of normalized volume at least $\delta := c^d d^{-3d/2}$. Clearly, $E \subset M$.

Since $\mathbf{B}^d$ is the maximum volume ellipsoid contained in M, by Lemma 5, we have that there is a translate of $\frac{\delta}{d^{d-1}}\mathbf{B}^d$ which is contained in E and thus in $\bigcap\limits_{i=2d}^{3d} C_i$.

Thus, we have shown that any colorful choice $C_{2d} \in \mathcal{C}_{2d}, \ldots, C_{3d} \in \mathcal{C}_{3d}$ of the remaining $d+1$ families, $\bigcap_{i=2d}^{3d} C_i$ contains a translate of the same convex body $c^d d^{-5d/2+1} \mathbf{B}^d$. It follows from Corollary 4 that there is an index $2d \leq i \leq 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains a translate of $c^d d^{-5d/2+1} \mathbf{B}^d$, which is an ellipsoid of normalized volume $c^{d^2} d^{-5d^2/2+d}$, finishing the proof of Theorem 6. $\square$

A Quantitative Fractional Helly Theorem was shown by the author and Naszódi in [JN22]. The proof of Theorem 7 uses the same tools as that of Theorem 6, but has a different structure. It is the first main result of the thesis.

Theorem 7 (Quantitative Fractional Helly Theorem)**.** *For every dimension $d \geq 1$ and every $\alpha \in (0,1)$, there is a $\beta \in (0,1)$ such that the following holds. Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $3d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{3d+1}$ for whom the intersection of the $3d+1$ members contains an ellipsoid of volume one.*
Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ contains an ellipsoid of volume at least $c^{d^2} d^{-5d^2/2+d}$, where c is the universal constant from Theorem 5.

Proof of Theorem 7. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$. We call an index set $I \in \binom{[n]}{3d+1}$ *good*, if the corresponding intersection $\cap_{i \in I} C_i$ contains an ellipsoid of volume at least one. We say that a $2d$-element subset $S \subset I$ of a good index set $I \in \binom{[n]}{3d+1}$ is a *seed* of I, if the volume of the John ellipsoid of $\cap_{i \in S} C_i$ is at most $c^{-d} d^{3d/2}$ times the volume of the John ellipsoid of $\cap_{i \in I} C_i$, where c is the absolute constant from Theorem 5. By Theorem 5, all good index sets have a seed.

Since we have $\alpha\binom{n}{3d+1}$ good index sets and only $\binom{n}{2d}$ possible seeds, there is a $(2d)$-tuple $S \in \binom{[n]}{2d}$ which is the seed of at least

$$\frac{\alpha\binom{n}{3d+1}}{\binom{n}{2d}} \geq \gamma\binom{n}{d+1}$$

good index sets. Here γ depends on α and d, but not on n.

Let $I_1, \ldots, I_{\gamma\binom{n}{d+1}}$ be good index sets whose seed is S. Denote the John ellipsoid of the intersection $\cap_{i\in S} C_i$ by $\mathcal{E}$ and the John ellipsoid of $\cap_{i\in I_j} C_i$ by $\mathcal{E}_j$. By Lemma 5, for every j, there is a $v_j \in \mathbb{R}^d$ such that $c^d d^{-5d/2}\mathcal{E} + v_j \subseteq \mathcal{E}_j$.

Thus, we have shown that at least $\gamma\binom{n}{d+1}$ of the $(d+1)$-wise intersections contain a translate of $c^d d^{-5d/2+1}\mathcal{E}$. We can apply Proposition 5 with $L = c^d d^{-5d/2+1}\mathcal{E}$, which implies that there are $\frac{\gamma}{d+1}n$ such C_i, that their intersection contains a translate of $c^d d^{-5d/2+1}\mathcal{E}$. And, since $\mathcal{E}$ has volume at least one, this ellipsoid has volume at least $c^{d^2} d^{-5d^2/2+d}$, completing the proof of Theorem 7.

$\square$

Note, that since the largest inscribed ellipsoid approximates the volume of a convex body up to a factor of d^d, we get the following corollaries of Theorems 6 and 7.

Corollary 1 (Corollary of Theorem 6)**.** *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ be finite families of convex sets in $\mathbb{R}^d$. Assume that for any colorful choice of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \le k \le 2d$ with $1 \le i_1 < \ldots < i_{2d} \le 3d$, the intersection $\bigcap\limits_{k=1}^{2d} C_{i_k}$ is of volume at least one.*
Then, there is an i with $1 \le i \le 3d$ such that $\operatorname{vol}\left(\bigcap\limits_{C\in\mathcal{C}_i} C\right) \ge c^{d^2} d^{-5d^2/2}$ with a universal constant c.

Corollary 2 (Corollary of Theorem 7)**.** *For every dimension $d \ge 1$ and every $\alpha \in (0,1)$, there is a $\beta \in (0,1)$ such that the following holds.*
Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $3d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{3d+1}$ for whom the intersection of the $3d+1$ members is of volume one.
Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ is of volume at least $c^{d^2} d^{-5d^2/2}$ with a universal constant c.

Chapter 3

Quantitative Helly-type Hypergraph Chains

In this chapter we propose a new combinatorial framework in which quantitative Helly-type questions can be analyzed. But first, we show how the classical Helly-type Theorems can be stated as properties of certain hypergraphs.

3.1 Helly-type Hypergraphs

We describe Helly's Theorem and the Fractional and Colorful Helly Theorems in the language of hypergraphs. Let V be a (possibly infinite) set. A *hypergraph* on the base set V is any family of its subsets, in notation $\mathcal{H} \subset 2^V$. A hypergraph is downwards closed, if $H \in \mathcal{H}$ and $G \subset H$ implies $G \in \mathcal{H}$. A downwards closed hypergraph $\mathcal{H}$ has *Helly Number* h, if for every finite subset $S \subset V$ the relation $\binom{S}{h} \subset \mathcal{H}$ implies $S \in \mathcal{H}$. Now let us denote the family of convex sets of $\mathbb{R}^d$ by $\mathrm{Cvx}(d)$ and the hypergraph which contains the subfamilies of convex sets with nonempty intersection by $\mathcal{K}_d = \{\mathcal{C} \subset \mathrm{Cvx}(d) : \cap_{C \in \mathcal{C}} C \neq \emptyset\}$. Helly's Theorem states that $\mathcal{K}_d$ has Helly-number $d + 1$.

A downwards closed hypergraph $\mathcal{H}$ over a base set V has *Fractional Helly Number* k, if there exists a function $\beta : (0,1) \to (0,1)$ such that whenever $S \subset V$ is a finite subset such that $\left|\mathcal{H} \cap \binom{S}{k}\right|$, the number of edges of $\mathcal{H}$ of size k in S is at least $\alpha\binom{|S|}{k}$ with an $\alpha \in (0,1)$, then there exists a subset $S' \subset S$ of size at least $\beta|S|$ such that $S' \in \mathcal{H}$. The Fractional Helly Theorem states that $\mathcal{K}_d$ has Fractinal Helly Number $d+1$.

We turn to phrasing the Colorful Helly Theorem in an abstract setting. Let $S_1, \dots, S_k$ be (not necessarily disjoint) sets, which we will call color classes, and let $S = \cup_{i=1}^{k} S_i$ be their union. We will call elements of $S_1 \times \cdots \times S_k$ *colorful selections*. With slight abuse of notation, if $s \in S_1 \times \cdots \times S_k$ is a colorful selection and $\mathcal{H} \subset 2^V$ is a hypergraph on $V \supset S$, we will write $s \in \mathcal{H}$, if s, considered as an unordered subset of S, is in $\mathcal{H}$. With this convention, we will use the notation $\mathcal{H} \cap (S_1 \times \cdots \times S_k)$ and $(S_1 \times \cdots \times S_k) \subseteq \mathcal{H}$.

A downwards closed hypergraph $\mathcal{H}$ over a base set V has *Colorful Helly Number* k, if for every k finite subset $S_1, \dots, S_k \subset V$ such that $(S_1 \times \cdots \times S_k) \subseteq \mathcal{H}$, there exists a color class S_j with $S_j \in \mathcal{H}$. The Colorful Helly Theorem states that $\mathcal{K}_d$ has Colorful Helly Number $d+1$.

Can we phrase our quantitative results in the language of hypergraphs? Let $\mathcal{E}_d(v)$ be the hypergraph whose vertices are convex bodies from $\mathbb{R}^d$ and whose edges are those sets of convex bodies from $\mathbb{R}^d$, whose intersection contains an ellipsoid of volume v. Propositions 1, 2 and 3 states that $\mathcal{E}_d(v)$ has Helly Number, Colorful Helly Number and Fractional Helly Number $\frac{d(d+3)}{2}$ for every $v > 0$. What about Theorems 5, 6 and 7? Their assumptions and conclusions are about two different kinds of intersection, so their statements can not be translated into a property of only one hypergraph.

Alon, Kalai, Matoušek and Meshulam [AKMM02] considered Helly-type results in the abstract setting. They showed, that if a hypergraph has bounded Fractional Helly Number, then it also has the so called (p,q) property (see the definition in [AKMM02]). Holmsen [Hol20], in a recent breakthrough, showed that if a hypergraph has Colorful Helly Number k, then it has Fractional Helly Number at most k. In this sense, the Fractional Helly Theorem can be deduced from the Colorful Helly Theorem with a purely combinatorial proof. Similarly, Proposition 2 implies Proposition 3. But note that Holmsen's result does not immediately imply a similar relationship between Theorem 6 and Theorem 7. Our goal is to lay the ground for such results.

3.2 Helly-type Hypergraph Chains

When one considers quantitative geometric Helly-type results, the translation to the abstract theory is not straightforward. We propose to consider *hypergraph chains* as follows.

Definition 2. Let V be a (possibly infinite) base set. The infinite sequence $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ of hypergraphs over the base set V is a *hypergraph chain*, if every $\mathcal{H}_\ell$ is downwards closed and for all $\ell \in \mathbb{Z}$, $\mathcal{H}_\ell \subset \mathcal{H}_{\ell+1}$.

Hypergraph chains are generalisations of downwards closed hypergraphs in the sense, that if $\mathcal{H}$ is a hypergraph, then $(\mathcal{H})_{\ell\in\mathbb{Z}}$ is a hypergraph chain. In this sense, $(\mathcal{K}_d)_{\ell\in\mathbb{Z}}$ and $(\mathcal{E}_d(v))_{\ell\in\mathbb{Z}}$ for any $v \in \mathbb{R}_+$ are hypergraph chains.A more interesting example is $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ where $v \in (0,1)$ is a real number and for an $\ell \in \mathbb{Z}$, a family of convex bodies from $\mathbb{R}^d$ is an edge in $\mathcal{E}_d\left(v^\ell\right)$, if and only if their intersection is conatins an ellipsoid of volume at least v^ℓ. A similar, but different hypergraph is when a family of convex bodies is an edge, if and only if their intersection is of volume at least v^ℓ. Let us denote this hypergraph by $\mathcal{Q}_d\left(v^\ell\right)$. In the rest of the chapter we will use $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ to demonstrate results about hypergraph chains, but similar results hold also for $\left(\mathcal{Q}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$.

Now we turn to phrase quantitative Helly-type results in the language of hypergraph chains.

Definition 3. A hypergraph chain $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ over a base set V has *Helly Number h*, if for every $S \subseteq V$, $\binom{S}{h} \subset \mathcal{H}_\ell$ implies $S \in \mathcal{H}_{\ell+1}$.

According to Helly's Theorem, $(\mathcal{K}_d)_{\ell\in\mathbb{Z}}$ has Helly Number $d+1$. Propsition 1 implies that $(\mathcal{E}_d(v))_{\ell\in\mathbb{Z}}$ for any $v \in \mathbb{R}_+$ has Helly Number $\frac{d(d+3)}{2}$.

More interestingly, Theorem 5 states that if $v \approx d^{-3d/2}$, then $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ has Helly Number $2d$.

Definition 4. A hypergraph chain $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ over a base set V has *Colorful Helly Number k*, if whenever $S_1, \ldots, S_k$ are finite subsets (color classes) of V and $S_1 \times \ldots \times S_k \subset \mathcal{H}_\ell$, then there is a color class S_j with $S_j \in \mathcal{H}_{\ell+1}$.

Note that by taking $S_1 = S_2 = ... = S_k = S$, a hypergraph chain with Colorful Helly Number k has Helly Number $h \leq k$.

According to the definition, $(\mathcal{K}_d)_{\ell\in\mathbb{Z}}$ has Colorful Helly Number $d+1$.

More interestingly, the Quantitative Colorful Helly Theorem, Theorem 6 may be stated as follows. If $v \approx d^{-cd^2}$ from Theorem 6, then $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ has Colorful Helly Number $3d$.

Definition 5. A hypergraph chain $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ over a base set V has *Fractional Helly Number k*, if there exists a function $\beta : (0,1) \to (0,1)$ such that for every finite set $S \subset V$, if $|\mathcal{H}_\ell \cap \binom{S}{k}| \geq \alpha\binom{|S|}{k}$ with some $\alpha \in (0,1)$, then there exists an $S' \subset S$ with $|S'| \geq \beta(\alpha)|S|$ and $S' \in \mathcal{H}_{\ell+1}$.

As in the previous two cases, $(\mathcal{K}_d)_{\ell\in\mathbb{Z}}$ has Fractional Helly Number $d+1$ and Theorem 7 states, that if $v \approx d^{-cd^2}$ from Theorem 7, then $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ has Fractional Helly Number $3d+1$.

Now we are ready to state our main result, which is a quantitative analogue of Theorem 3 from [Hol20].

Theorem 8. *If the hypergraph chain $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ has Colorful Helly Number k, then $(\mathcal{H}_{(k+1)l})_{\ell\in\mathbb{Z}}$ has Fractional Helly Number k.*

Here, the obtained Fractional Helly Number is the same as the assumed Colorful Helly Number, but not for the exact same hypergraph chain: we can only take every $(k+1)$st element from the original chain. Can the Fractional Helly number be less than the Colorful Helly number? If for a hypergraph chain, the Helly Number is smaller than the Colorful Helly Number, the answer is a partial yes. We can show a stability version of the Helly property under the addition assumption of the colorful Helly property.

Theorem 9. *If the hypergraph chain $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ has Helly Number h and Colorful Helly Number $k \geq h$, then there exists a function $\beta : (0,1) \to [0,1)$ with $\lim_{\alpha\to 1}\beta(\alpha) = 1$ such that for every finite set $S \subset V$, if $|\mathcal{H}_\ell \cap \binom{S}{h}| \geq \alpha\binom{|S|}{h}$ with some $\alpha \in (0,1)$, then there exists an $S' \subset S$ with $|S'| \geq \beta(\alpha)|S|$ and $S' \in \mathcal{H}_{\ell+2}$.*

As far as we know, the best possible β here might assign 0 to a large fraction of αs from $(0,1)$, this is the difference from hypergraph chains with Fractional Helly Number $2d$, where this is not possible. But at least, if α is very close to 1, then $\beta(\alpha)$ is also close to 1.

3.3 Proof of Theorems 8 and 9

Let us begin with an analogue of Lemma 3.1 from [Hol20]. We denote by $\omega_h(\mathcal{H}_\ell|_S)$ the size of the largest h-clique of S, ie. the size of the largest subset $K \subset S$ such that $\binom{K}{h} \subset \mathcal{H}_\ell$.

Lemma 6. *If $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ is a hypergraph chain with Colorful Helly Number k over the base set V, then for every finite subset $S \subset V$ and every positive integer $h \leq k$ the inequality*

$$\left|\binom{S}{h} \setminus \mathcal{H}_\ell\right| \geq \binom{k}{h}^{-1}\binom{\frac{1}{h}(|S| - \omega_h(\mathcal{H}_{\ell+1}|_S))}{h}$$

holds.

Proof of Lemma 6. Let $\ell \in \mathbb{Z}$ be fixed and assume $\omega_h(\mathcal{H}_{\ell+1}|_S) \geq h$, otherwise there is nothing to prove. Let $M_1, \dots, M_t \in \binom{S}{h} \setminus \mathcal{H}_{\ell+1}$ be a maximum number of disjoint missing edges from $\mathcal{H}_{\ell+1}$, each of size h. Since $\binom{S \setminus (M_1 \cup \ldots \cup M_t)}{h} \subset \mathcal{H}_{\ell+1}$, we have $\omega_h(\mathcal{H}_{\ell+1}|_S) \geq |S \setminus (M_1 \cup \ldots \cup M_t)| = |S| - th$ or, equivalently, $t \geq \frac{1}{h}((|S| - \omega_h(\mathcal{H}_{\ell+1}|_S)))$.

Since $(\mathcal{H}_\ell)_{\ell \in \mathbb{Z}}$ has Colorful Helly Number k, for every $I \in \binom{[t]}{k}$, there is a $J \in \binom{I}{h}$ and an $M \in \binom{S}{h} \setminus \mathcal{H}_\ell$ such that $|M \cap M_j| = 1$ for every $j \in J$. One particular M can appear at most $\binom{t-h}{k-h}$ times in this way, so there are at least $\binom{t}{k} / \binom{t-h}{k-h} = \binom{t}{h} / \binom{k}{h}$ missing edges $M \in \binom{S}{h} \setminus \mathcal{H}_\ell$. □

This Lemma alone is enough to prove Theorem 9.

Proof of Theorem 9. Assume for an $\ell \in \mathbb{Z}$, that the largest edge of $\mathcal{H}_{\ell+2}$ in S is of size at most $(1-\varepsilon)|S|$ for some $\varepsilon > 0$. Since $(\mathcal{H}_\ell)_{\ell \in \mathbb{Z}}$ has Helly Number h, this implies $\omega_h(\mathcal{H}_{\ell+1}|_S) \leq (1-\varepsilon)|S|$. Now Lemma 6 implies $\left|\binom{S}{h} \setminus \mathcal{H}_\ell\right| \geq \binom{k}{h}^{-1} \binom{\varepsilon |S|}{h} \geq \delta \binom{|S|}{h}$ with some $\delta(\varepsilon, k, h) > 0$, proving Theorem 9. □

Before proving Theorem 8, we need the following technical Lemma, which is an analogue of Lemma 3.2 from [Hol20] and can be proved using Lemma 6.

Lemma 7. *Let $(\mathcal{H}_\ell)_{\ell \in \mathbb{Z}}$ be a hypergraph chain with Colorful Helly Number k over a base set V and $S \subset V$ a finite subset with $|S| = n$ large enough. If for an $\ell \in \mathbb{Z}$ and $c \in (0,1)$ the inequality $\omega_h(\mathcal{H}_{\ell+1}|_S) \leq cn/2$ holds, then given any $i \in [k]$ and a family $\mathcal{F}_i \subset \binom{S}{i}$ with $|\mathcal{F}_i| \geq c\binom{n}{i}$ there exists another family $\mathcal{F}_{i-1} \subset \binom{S}{i-1}$ and an $M \in \binom{S}{h} \setminus \mathcal{H}_\ell$ such that $|\mathcal{F}_{i-1}| \geq \left(\frac{c}{12kh}\right)^h \binom{n}{n-1}$ and $A \cup \{v\} \in \mathcal{F}_i$ for all $A \in \mathcal{F}_{i-1}$ and $v \in M$.*

Proof of Lemma 7. For every $A \in \binom{S}{i-1}$ let $\Gamma_A = \{v \in S : A \cup \{v\} \in F_i\}$ and let

$$\mathcal{P} = \{(A, M) : A \in \binom{S}{i-1}, M \in \binom{\Gamma_A}{h} \setminus \mathcal{H}_\ell\}.$$

We want to lower bound $|\mathcal{P}|$. By Lemma 6, for a fixed $A \in \binom{S}{i-1}$ there are at least $\binom{k}{h}^{-1} \binom{\frac{1}{h}(|\Gamma_A| - (c/2)n)}{h}$ distinct $M \in \binom{\Gamma_A}{h} \setminus \mathcal{H}_\ell$ such that $(A, M) \in \mathcal{P}$. Jensen's inequality gives

$$
\begin{aligned}
|\mathcal{P}| &\geq \binom{k}{h}^{-1} \sum_{A\in\binom{S}{i-1}} \binom{\frac{1}{h}\left(|\Gamma_A| - (c/2)n\right)}{h} \\
&\geq \binom{k}{h}^{-1} \binom{n}{i-1} \binom{\binom{n}{i-1}^{-1}\frac{1}{h}\sum_{A\in\binom{S}{i-1}}\left(|\Gamma_A| - (c/2)n\right)}{h}.
\end{aligned}
$$

Since

$$
\sum_{A\in\binom{S}{i-1}} |\Gamma_A| = i|\mathcal{F}_i| \geq ic\binom{n}{i} > (n-i)c\binom{n}{i-1},
$$

we get

$$
\sum_{A\in\binom{S}{i-1}} \left(|\Gamma_A| - (c/2)n\right) > (n-i)c\binom{n}{i-1} - (c/2)n\binom{n}{i-1},
$$

and thus

$$
|\mathcal{P}| \geq \binom{k}{h}^{-1} \binom{n}{i-1} \binom{\frac{nc}{2h} - \frac{ci}{h}}{h}.
$$

If n is large enough compared to i, k and h, then

$$
|\mathcal{P}| \geq \left(\frac{c}{12hk}\right)^h \binom{n}{i-1}\binom{n}{h}.
$$

Since there are $\binom{n}{h}$ possible $M \in \binom{S}{h}$, there is an M, with at least $\left(\frac{c}{12hk}\right)^h \binom{n}{i-1}$ different $A \in \binom{S}{i-1}$ such that $(A, M) \in \mathcal{P}$. These As will form $\mathcal{F}_{i-1}$. $\square$

Now we are ready to prove Theorem 8.

Proof of Theorem 8. Let $f(x) = \left(\frac{x}{12hk}\right)^h$, $\alpha_0 = \alpha$, $\alpha_{i+1} = f(\alpha_i)$ and $\beta = \alpha_{k-1}$. Fix an ℓ and suppose for contradiction, that $\left|\mathcal{H}_\ell \cap \binom{S}{k}\right| \geq \alpha\binom{n}{k}$, but $\mathcal{H}_{\ell+k+1}$ has no edge of size at least βn inside S. Since $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ has Colorful

Helly Number k, it has Helly Number $h \leq k$, so $\mathcal{H}_{\ell+k+1}$ having no edge of size at least βn implies $\omega_h(\mathcal{H}_{\ell+k}|_S) < \beta n$.

Let $\mathcal{F}_k = \mathcal{H}_\ell \cap \binom{S}{k}$. Since $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ is a hypergraph chain, $\mathcal{F}_k \subset \mathcal{H}_{\ell+i}$ for all $i \geq 0$, in particular, $\mathcal{F}_k \subset \mathcal{H}_{\ell+k}$. We have $|\mathcal{F}_k| \geq \alpha\binom{n}{k}$ and $\omega_h(\mathcal{H}_{\ell+k}|_S) < \beta n \leq (\alpha/2)n$, so we can apply Lemma 7 with $c = \alpha$ to obtain an $\mathcal{F}_{k-1} \subset \binom{S}{k-1}$ with $|\mathcal{F}_{k-1}| \geq \alpha_1\binom{n}{k-1}$ and an $M_1 \in \binom{S}{h}\setminus\mathcal{H}_{\ell+k-1}$ such that $A\cup\{v\} \in \mathcal{F}_k$ for all $A \in \mathcal{F}_{k-1}$ and $v \in M_1$. Now we have $|\mathcal{F}_{k-1}| \geq \alpha_1\binom{n}{k-1}$ and $\omega_h(\mathcal{H}_{\ell+k-1}|_S) \leq \omega_h(\mathcal{H}_{\ell+k}|_S) < \beta n \leq (\alpha_1/2)n$ and we can apply Lemma 7 again, this time with $c = \alpha_1$, to obtain an $\mathcal{F}_{k-2} \subset \binom{S}{k-2}$ with $|\mathcal{F}_{k-2}| \geq \alpha_2\binom{n}{k-2}$ and an $M_2 \in \binom{S}{h} \setminus \mathcal{H}_{\ell+k-2}$ such that $A \cup \{v\} \in \mathcal{F}_{k-1}$ for all $A \in \mathcal{F}_{k-2}$ and $v \in M_2$. Note that $A \cup \{v_1, v_2\} \in \mathcal{F}_k = \mathcal{H}_\ell \cap \binom{S}{k}$ for all $A \in \mathcal{F}_{k-2}$, $v_1 \in M_1$, $v_2 \in M_2$.

After repeating this process $k-1$ times, we get an $\mathcal{F}_1 \subset \binom{S}{1}$ with $|\mathcal{F}_1| \geq \alpha_{k-1}n = \beta n$ and $M_1, \ldots, M_{k-1} \in \binom{S}{k} \setminus \mathcal{H}_{\ell+1}$ such that $A \cup \{v_1, \ldots, v_{k-1}\} \in \mathcal{H}_\ell \cap \binom{S}{k}$ for all $A \in \mathcal{F}_1$, $v_1 \in M_1, \ldots, v_{k-1} \in M_{k-1}$. Since $\omega_h(\mathcal{H}_{\ell+2}|_S) < \beta n$, the set of vertices of $\mathcal{F}_1$, $V(\mathcal{F}_1)$ is not an h-clique in $\mathcal{H}_{\ell+2}$ and there must be an $M_k \in \binom{V(\mathcal{F}_1)}{h} \setminus \mathcal{H}_{\ell+1}$ by $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ having Helly Number h. But regarding $M_1, \ldots, M_k$ as color classes, $(\mathcal{H}_\ell)_{\ell\in\mathbb{Z}}$ having Colorful Helly-number k yields a contradiction, since $M_1 \times \ldots \times M_k \subset \mathcal{H}_\ell$, but there is no color class $M_i \in \mathcal{H}_{\ell+1}$. $\square$

3.4 Geometric Consequences

If $v \approx d^{-cd^2}$ from Theorem 6, then $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ has Colorful Helly Number $3d$ by Theorem 6, so the following Corollary follows from Theorem 8.

Corollary 3. *For every dimension $d \geq 1$ and every $\alpha \in (0,1)$, there is a $\beta \in (0,1)$ such that the following holds.*
Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $3d$, there are at least $\alpha\binom{|\mathcal{C}|}{3d}$ for whom the intersection of the $3d$ members contains an ellipsoid of volume at least one.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that $\bigcap\limits_{C \in \mathcal{C}'} C$ contains an ellipsoid of volume at least d^{-cd^3} with a universal constant $c > 0$.

Proof. The above claim is equvivalent to saying that $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ has Fractional Helly Number $3d$, if $v = d^{-cd^3}$ with a universal constant c. Theorem 6 states that $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ has Colorful Helly Number $3d$, if $v = d^{-c'd^2}$ as in Theorem 6. By applying Theorem 8 to the latter Hypergraph Chain, we can conclude, that $\left(\mathcal{E}_d\left(v^{(3d+1)\ell}\right)\right)_{\ell\in\mathbb{Z}}$ has Fractional Helly Number $3d$ and $v = d^{-c'd^2}$. But this is equvivalent to $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$ having Fractional Helly Number $3d$ if $v = d^{-cd^3}$ with a constant c. $\square$

This is a slight improvement on the Fractional Helly Number, which was $3d+1$ in Theorem 7. Can we go below $3d$? Theorem 9 implies at least a stability version of the Quantitative Helly Theorem with Helly Number $2d$ as follows.

Corollary 4. *For every positive integer d there exists a function $\beta : (0,1) \to [0,1)$ with $\lim_{\alpha\to 1}\beta(\alpha) = 1$ such that the following holds.*
Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $2d$, there are at least $\alpha\binom{|\mathcal{C}|}{2d}$ for whom the intersection of the $2d$ members contains an ellipsoid of volume at least one.
Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that $\bigcap\limits_{C \in \mathcal{C}'} C$ contains an ellipsoid of volume at least d^{-cd^2} with a universal constant $c > 0$.

Proof. Since $\left(\mathcal{E}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$, with $v = d^{-cd^2}$ from Theorem 6, has Helly Number $2d$ by Theorem 5 and Colorful Helly Number $3d$ by Theorem 6, we can apply Theorem 9. The assumption of Corollary 4 states that for a finite subset of convex sets $\mathcal{C}$, the inequality $\left|\mathcal{E}_d(v^0) \cap \binom{\mathcal{C}}{2d}\right| \geq \alpha\binom{|\mathcal{C}|}{2d}$ holds with some $\alpha \in (0,1)$, where v can be $v = d^{-cd^2}$ from Theorem 6. Theorem 9 yields a subfamily $\mathcal{C}' \subset \mathcal{C}$ with $\mathcal{C}' \in \mathcal{E}_d(v^2)$ and $|\mathcal{C}'| \geq \beta(\alpha)|\mathcal{C}|$, where β is the function from Theorem 9. For $\mathcal{C}'$, the inequality $\operatorname{vol}\left(\bigcap\limits_{C\in\mathcal{C}'} C\right) \geq \left(d^{-cd^2}\right)^2 = d^{-2cd^2}$ holds. $\square$

The above two Corollaries has analogues in the case where we use Theorems 8 and 9 for the hypergraph chain $\left(\mathcal{Q}_d\left(v^\ell\right)\right)_{\ell\in\mathbb{Z}}$. We state them here for the readers convenience.

Corollary 5. *For every dimension $d \geq 1$ and every $\alpha \in (0,1)$, there is a $\beta \in (0,1)$ such that the following holds.*
Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $3d$, there are at least $\alpha\binom{|\mathcal{C}|}{3d}$ for whom the intersection of the $3d$ members is of volume at least one.
Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that $\operatorname{vol}\left(\bigcap_{C\in\mathcal{C}'} C\right) \geq d^{-9d^3}$.

Corollary 6. *For every positive integer d there exists a function $\beta : (0,1) \to [0,1)$ with $\lim_{\alpha\to 1}\beta(\alpha) = 1$ such that the following holds.*
Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $2d$, there are at least $\alpha\binom{|\mathcal{C}|}{2d}$ for whom the intersection of the $2d$ members is of volume at least one.
Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that $\operatorname{vol}\left(\bigcap_{C\in\mathcal{C}'} C\right) \geq d^{-cd^2}$ with a universal constant $c > 0$.

Chapter 4

Discussion

One important question about quantitative Helly type Theorems is whether we can achieve Helly Number $2d$ in Quantitative Colorful and Fractional Helly Theorems.

Conjecture 1. *There is a constant $c(d) > 0$ such that whenever $\mathcal{C}_1, \ldots, \mathcal{C}_{2d}$ are finite families of convex bodies from $\mathbb{R}^d$ with every colorful intersection having volume at least 1, then there is a family $\mathcal{C}_j$, such that $\bigcap_{C \in \mathcal{C}_j} C$ is of volume at least $c(d)$?*

Conjecture 2. *There is a constant $c(d)$ and a function $\beta : (0,1) \to (0,1)$ such that the following is true.*

Whenever $\mathcal{C}$ is a finite family of convex bodies from $\mathbb{R}^d$ such that at least $\alpha\binom{|\mathcal{C}|}{2d}$ of the $2d$-wise intersections from $\mathcal{C}$ is of volume at least 1 with $\alpha \in (0,1)$, then there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ such that $|\mathcal{C}'| \geq \beta(\alpha)|\mathcal{C}|$ and $\bigcap_{C \in \mathcal{C}'} C$ is of volume at least $c(d)$.

Our whole thesis can be viewed as a collection of attempts trying to solve these conjectures. Corollaries 1 and 5 are the current records with Helly Number $3d$ instead of the conjectured $2d$ for both questions. Our Theorem 8 shows, that proving Conjecture 1 would also confirm Conjecture 2. In the end, our Corollary 6 is a partial result towards Conjecture 2 by showing the existence of the function β at least in a small neighbourhood of 1.

Bibliography

[AKMM02] Noga Alon, Gil Kalai, Jiri Matousek, and Roy Meshulam. Transversal numbers for hypergraphs arising in geometry. *Advances in Applied Mathematics*, 29(1):79–101, 2002.

[Bár82] Imre Bárány. A generalization of Carathéodory's theorem. *Discrete Math.*, 40(2-3):141–152, 1982.

[BF87] Imre Bárány and Zoltán Füredi. Computing the volume is difficult. *Discrete & Computational Geometry*, 2(4):319–326, 1987.

[BK21] Imre Bárány and Gil Kalai. Helly-type problems. *arXiv preprint arXiv:2108.08804*, 2021.

[BKP82] Imre Bárány, Meir Katchalski, and János Pach. Quantitative Helly-type theorems. *Proc. Amer. Math. Soc.*, 86(1):109–114, 1982.

[Bra17] Silouanos Brazitikos. Brascamp-Lieb inequality and quantitative versions of Helly's theorem. *Mathematika*, 63(1):272–291, 2017.

[Dam17] Gábor Damásdi. *Some problems in combinatorial geometry*. PhD thesis, Master's thesis. Budapest, 2017. link: https://web. cs. elte. hu/blobs . . . , 2017.

[DFN21] Gábor Damásdi, Viktória Földvári, and Márton Naszódi. Colorful helly-type theorems for the volume of intersections of convex bodies. *Journal of Combinatorial Theory, Series A*, 178:105361, 2021.

[Hel23] Eduard Helly. Über mengen konvexer körper mit gemeinschaftlichen punkten. *Jber. Deutsch. Math. Verein*, 32:175–176, 1923.

[Hol20] Andreas F Holmsen. Large cliques in hypergraphs with forbidden substructures. *Combinatorica*, 40(4):527–537, 2020.

[JN22] Attila Jung and Márton Naszódi. Quantitative fractional helly and (p, q)-theorems. *European Journal of Combinatorics*, 99:103424, 2022.

[Joh48] Fritz John. Extremum problems with inequalities as subsidiary conditions, studies and essays presented to r. courant on his 60th birthday, january 8, 1948, 1948.

[Kal84] Gil Kalai. Intersection patterns of convex sets. *Israel Journal of Mathematics*, 48(2):161–174, 1984.

[KL79] M. Katchalski and A. Liu. A problem of geometry in $\mathbf{R}^n$. *Proc. Amer. Math. Soc.*, 75(2):284–288, 1979.

[Lov74] László Lovász. Exercise 206. *Mat. Lapok*, 25(3-4):p. 370, 1974. (in Hungarian).

[Mat02] Jiri Matousek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[Nas16] Márton Naszódi. Proof of a conjecture of Bárány, Katchalski and Pach. *Discrete Comput. Geom.*, 55(1):243–248, 2016.

[Rad21] Johann Radon. Mengen konvexer körper, die einen gemeinsamen punkt enthalten. *Mathematische Annalen*, 83(1-2):113–115, 1921.