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Abstract

The aim of this thesis is to explore the notions of measure and Baire category in different settings and to survey dualities between them. The main sources of this work are John C. Oxtoby's book *Measure and Category*, which focuses on the real line, and Alexander S. Kechris' book *Classical Descriptive Set Theory*. Although we follow the structure of Oxtoby's book, we present several results in a more general setting.

We begin by introducing basic notions and theorems on Baire category and measurability. We introduce sets that do not have the Baire property (BP) and are non-measurable.

We study the following classical theorems from measure theory together with their categorical duals: Lusin's theorem, Egoroff's theorem, Fubini's theorem and the Poincaré recurrence theorem. We present a generalised version of the Sierpiński–Erdős duality theorem and the Duality Principle. We also introduce the density topology in $\mathbb{R}^p$, which is interesting on its own: in this topological space, a set is Lebesgue measurable if and only if it has the BP.

In the last chapter, we build on the theory of infinite games and the density topology to prove that in a Polish space every analytic set is universally measurable and has the BP.

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Chapter 1

Introduction

This thesis provides a glimpse into my studies in a few branches of mathematical analysis. I am fortunate to have had the guidance and support of my supervisor throughout the course of my work. He suggested two main topics that could shape the trajectory of my thesis: the exploration of dualities between measure theory and Baire category, and the study of descriptive set theory. While initially separate, these two topics naturally converged as I read John C. Oxtoby's book Measure and Category [3] and learnt about descriptive set theory from my supervisor.

In the second chapter, we lay the foundation by introducing some basic notions and propositions from descriptive set theory, measure theory and Baire category, most of which will reoccur frequently throughout this thesis. These include Polish spaces and their completely metrisable subspaces, the regularity of finite Borel measures on Polish spaces, Radon measures and their properties in Polish spaces, the σ-ideal of meagre sets, the σ-algebra of sets that have the Baire property and the Baire Category Theorem.

In the third chapter, we extend Oxtoby's [3] investigations of measure theory and Baire category from the real line with the Lebesgue measure to Polish spaces with continuous Radon measures. That is, we preserve the most important properties of the space and the measure: separability, complete metrisability, and the inner regularity, local finiteness and continuity of the measure. As we will see, the majority of the theorems presented in this chapter rely on these properties. In this setting, we explore classical theorems in measure theory and their Baire category analogues, such as Lusin's theorem and its dual, which characterise μ-measurable and Baire measurable functions based on how well they can be approximated by continuous functions. Another example is Egoroff's theorem, which guarantees that pointwise convergence of a series of functions can be strengthened to uniform convergence on an arbitrarily large set. This theorem does not have a Baire category analogue. We also study the well-known theorem of Fubini and its Baire category counterpart, the Kuratowski–Ulam theorem. Both of these theorems offer profound insights by establishing equivalent statements about the sections of sets that possess certain properties, namely measurability or the Baire property.

Then we prove the Poincaré recurrence theorem, which states that in certain dynamical systems, almost every point is recurrent, with the exception of a meagre set of measure zero. Next, we present a generalised version of the Sierpiński–Erdős duality theorem, which says that, assuming the continuum hypothesis, in an uncountable Polish space X with a continuous Radon measure there is an involution $f\colon X \to X$ such that $f(E)$ is meagre if and only if E is a nullset. Therefore, if a proposition P involves only the notions of measure zero, meagreness and purely set-theoretic notions, the proposition P^* obtained by interchanging the terms "meagre" and "of measure zero" holds if and only if P holds. Furthermore, we introduce the density topology. In this topology, a set is meagre if and only if it is a Lebesgue nullset, and it has the Baire property if and only if it is Lebesgue measurable. We build on these results in Chapter 4.

In the last chapter of this thesis, we study descriptive set theory, specifically focusing on analytic sets. We are inspired by Kechris' work [1] and use the dynamic framework of infinite games to explore the properties of these sets. We examine well-known games like Choquet and strong Choquet, as well as different versions of the Banach–Mazur game. Our main objective is to prove the following two results: analytic sets have the Baire property and they are universally measurable.

Chapter 2

Preliminaries

This chapter presents definitions of essential notions in descriptive set theory, measure theory and Baire category. Additionally, it serves as a practical toolbox, providing a collection of useful short lemmas and propositions that we repeatedly utilise in the thesis.

2.1 Notations

In this section, we provide a list of common notations that are used throughout the thesis.

Notation 2.1. Let X be an arbitrary set and $A \subseteq X$. We denote the complement of A with respect to X by $X \setminus A = A^c$.

Notation 2.2. The symmetric difference of the sets A and B is $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Notation 2.3. We denote the disjoint union of sets by $\dot{\cup}$.

Notation 2.4. Let (X, d) be a metric space, let $x \in X$ and $r > 0$. Then we denote the open ball of centre x and radius r by $B(x, r)$.

Notation 2.5. We denote the cardinality of the continuum by $\mathfrak{c}$.

Notation 2.6. Let X and Y be arbitrary sets and $A \subseteq X \times Y$. The projection $\mathrm{pr}_X(A)$ of A on X is $\{x \in X : \exists y \in Y \ \ (x, y) \in A\}$. We denote the vertical x-section of A by $A_x = \{y \in Y : (x, y) \in A\}$, and the horizontal y-section by $A^y = \{x \in X : (x, y) \in A\}$.

Notation 2.7. We denote the Lebesgue measure on $\mathbb{R}^p$ by λ and the outer Lebesgue measure by $\overline{\lambda}$.

2.2 Polish spaces

In this section, we present a collection of well-known theorems and lemmas from descriptive set theory. While these results are widely recognised in the field, we omit their proofs since the development of basic notions of descriptive set theory is not the main topic of this thesis.

Definition 2.8. [1, Def 3.1] A topological space (X, τ) is **completely metrisable** if there exists a complete metric d on X that induces the topology τ.

Definition 2.9. [1, Def 3.1] A topological space X is a **Polish space** if it is separable and completely metrisable.

Definition 2.10. [1, Ch 3.] The **Cantor space** is $\mathcal{C} = 2^{\mathbb{N}}$, the product of infinitely many copies of $\{0, 1\}$ with the discrete topology.

Definition 2.11. [1, Ch 3.] The **Baire space** is $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, the product of infinitely many copies of $\mathbb{N}$ with the discrete topology.

Lemma 2.12. *[1, Thm 3.11] If X is a metric space and $Y \subseteq X$ is completely metrisable, then Y is G_δ. Conversely, if X is completely metrisable and $Y \subseteq X$ is G_δ, then Y is completely metrisable. In particular, if X is a Polish space, then $Y \subseteq X$ is Polish $\iff Y \subseteq X$ is G_δ.*

Lemma 2.13. *[1, Thm 17.10, Thm 17.11] Every finite Borel measure μ on a Polish space X is regular, that is, for any μ-measurable set $B \subseteq X$ we have*

$$\mu(B) = \inf\{\mu(U) \colon U \supseteq B, U \text{ open}\} = \sup\{\mu(K) \colon K \subseteq B, K \text{ compact}\}.$$

Lemma 2.14. *[1, Thm 6.2] If X is a nonempty perfect Polish space, there is an embedding of the Cantor space into X.*

Theorem 2.15 (Cantor–Bendixson)**.** *[1, Thm 6.4] Let X be a Polish space. Then for any closed set $F \subseteq X$ there exist a perfect set P and a countable set M such that $F = P \dot{\cup} M$.*

Corollary 2.16. *[1, Cor 6.5] Any uncountable Polish space contains a homeomorphic copy of $\mathcal{C}$ and in particular has cardinality continuum.*

Proposition 2.17. *The Cantor space is homeomorphic to the product of two copies of itself: $\mathcal{C} \cong \mathcal{C} \times \mathcal{C}$.*

2.3 Measure theory

In this section, we focus on presenting the key definitions in measure theory that are crucial for the thesis, particularly those that might not be covered in our BSc programme. We prioritise explaining the concepts that are essential in the following chapters, such as the properties of a Radon measure.

Definition 2.18. Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $A \subseteq X$ has measure zero and is called a **nullset** if and only if there exists $B \in \mathcal{M}$ such that $A \subseteq B$ and $\mu(B) = 0$. (For notational simplicity, we write $\mu(A) = 0$.)

Definition 2.19. Let X be an arbitrary set and let $\mathcal{N}$ be a family of subsets of X. Then $\mathcal{N}$ is a **σ-ideal** if the following hold:

- $\emptyset \in \mathcal{N}$.
- If $A \in \mathcal{N}$ and $B \subseteq A$, then $B \in \mathcal{N}$.
- If (A_n) is a sequence of sets in $\mathcal{N}$, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{N}$.

The following proposition follows immediately from the definitions.

Proposition 2.20. *In a measure space $(X, \mathcal{M}, \mu)$ the sets of measure zero form a σ-ideal.*

In general, σ-ideals serve as a way to capture notions of smallness in mathematical analysis.

Remark 2.21. We will need the following identity: for any sets A, B, C the following holds: $A\Delta B = C \iff A\Delta C = B$.

Proposition 2.22. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{N}$ be the σ-ideal of the nullsets. Then the family of sets $\{M\Delta N \colon M \in \mathcal{M}, N \in \mathcal{N}\}$ is a σ-algebra and it is generated by $\mathcal{M}\cup\mathcal{N}$. In fact, the following holds: $\sigma(\mathcal{M}\cup\mathcal{N}) = \{M\cup N \colon M \in \mathcal{M}, N \in \mathcal{N}\}$.*

Proof. Let $\mathcal{A} = \{M\Delta N \colon M \in \mathcal{M}, N \in \mathcal{N}\}$. Since $\mathcal{A}$ contains both $\mathcal{M}$ and $\mathcal{N}$, and $\mathcal{A} \subseteq \sigma(\mathcal{M}\cup\mathcal{N})$, it suffices to show that $\mathcal{A}$ is a σ-algebra. The empty set is in both $\mathcal{M}$ and $\mathcal{N}$, so $\emptyset = \emptyset\Delta\emptyset \in \mathcal{A}$. Now let $M\Delta N \in \mathcal{A}$ with $M \in \mathcal{M}, N \in \mathcal{N}$, and let us show that its complement is also in $\mathcal{A}$. We have $(M\Delta N)^c = M^c\Delta N \in \mathcal{A}$ since $M^c \in \mathcal{M}$. For the countable union, let (M_n) be a sequence of sets in $\mathcal{M}$ and let (N_n) be a sequence in $\mathcal{N}$. We need to check whether $\bigcup\limits_{n\in\mathbb{N}} (M_n\Delta N_n)$ is in $\mathcal{A}$. Since $\bigcup\limits_{n\in\mathbb{N}} M_n \in \mathcal{M}$, by Remark 2.21, it suffices to show that $\left(\bigcup\limits_{n\in\mathbb{N}} (M_n\Delta N_n)\right) \Delta \left(\bigcup\limits_{n\in\mathbb{N}} M_n\right) \in \mathcal{N}$. We have that $\left(\bigcup\limits_{n\in\mathbb{N}} M_n\right) \setminus \left(\bigcup\limits_{n\in\mathbb{N}} (M_n\Delta N_n)\right) \subseteq \bigcup\limits_{n\in\mathbb{N}} N_n \in \mathcal{N}$ and $\left(\bigcup\limits_{n\in\mathbb{N}} (M_n\Delta N_n)\right) \setminus \left(\bigcup\limits_{n\in\mathbb{N}} M_n\right) \subseteq \bigcup\limits_{n\in\mathbb{N}} N_n \in \mathcal{N}$, hence by Proposition 2.20, their union is in $\mathcal{N}$.

To prove the second part, let $M\Delta N \in \sigma(\mathcal{M}\cup\mathcal{N})$ with $M \in \mathcal{M}, N \in \mathcal{N}$ and let $N' \in \mathcal{M}\cap\mathcal{N}$ be such that $N \subseteq N'$. Then we have that $M \setminus N = (M \setminus N') \cup ((N' \cap M) \setminus N)$, so $M\Delta N = (M \setminus N') \cup ((N' \cap M) \setminus N) \cup (N \setminus M)$, where $(M \setminus N') \in \mathcal{M}$ and $(((N' \cap M) \setminus N) \cup (N \setminus M)) \in \mathcal{N}$, which concludes the proof. $\square$

Definition 2.23. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{N}$ be the σ-ideal of nullsets. We define the σ-algebra of **measurable sets** as $\mathcal{M}' = \{M \cup N \colon M \in \mathcal{M}, N \in \mathcal{N}\} = \sigma(\mathcal{M} \cup \mathcal{N})$ and μ extends to $\mathcal{M}'$: for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$ we have $\mu(M \cup N) = \mu(M)$.

Definition 2.24. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{N}$ be the σ-ideal of nullsets. Then for sets A and B in X let $A \sim B \iff A \Delta B \in \mathcal{N}$. This is clearly an equivalence relation.

Remark 2.25. Note that in a measure space $(X, \mathcal{M}, \mu)$, a set A is measurable if and only if there exists a set $B \in \mathcal{M}$ such that $A \sim B$.

Definition 2.26. Let X be a Polish space and $A \subseteq X$. Then A is **universally measurable** if it is μ-measurable for any σ-finite Borel measure μ on X.

If we have a Borel measure, then the measurable sets differ from Borel sets only by a "noise" of measure zero. They capture the idea of being almost indistinguishable from Borel sets while still allowing for a negligible amount of noise in terms of measure.

Definition 2.27. Let $(X, \mathcal{M})$ and $(Y, \mathcal{A})$ be two measurable spaces, $f \colon X \to Y$ a measurable map and μ a measure on X. Then the **pushforward measure** $f_*(\mu)$ on Y is defined by $f_*(\mu)(B) = \mu(f^{-1}(B))$ for all $B \in \mathcal{A}$.

Definition 2.28. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mathcal{M}$ contains every singleton in X. Then μ is a **continuous measure** if $\mu(\{x\}) = 0$ for every $x \in X$.

Definition 2.29. Let $(X, \mathcal{M})$ be a measurable space and $x \in X$ a point. The **Dirac measure** δ_x is defined as $\delta_x(A) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$ for all $A \in \mathcal{M}$.

Definition 2.30. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then μ is a **discrete measure** if there exist a sequence (x_n) in X and a sequence (α_n) in $\mathbb{R}$ such that for all $A \in \mathcal{M}$ we have $\mu(A) = \sum\limits_{n \in \mathbb{N}} \alpha_n \delta_{x_n}(A)$.

Definition 2.31. Let X be a topological space and let μ be a measure on the Borel sets of X.

- The measure μ is **inner regular** if for any open set $G \subseteq X$ we have $\mu(G) = \sup\{\mu(K) \colon K \subseteq G, K \text{ is compact}\}$.
- The measure μ is **outer regular** if for any Borel set $B \subseteq X$ we have $\mu(B) = \inf\{\mu(G) \colon B \subseteq G, G \text{ is open}\}$.
- The measure μ is **locally finite** if every point has a neighbourhood U for which $\mu(U) < \infty$.
- In a Hausdorff space X, a measure μ on the Borel sets of X is a **Radon measure** if it is inner regular and locally finite.

Lemma 2.32. *A Radon measure μ on a separable metric space X is σ-finite. Moreover, there exists a partition of X into countably many G_δ sets with finite measure.*

Proof. Since the measure is locally finite, every point $x \in X$ has a neighbourhood U_x with finite measure, and by the Lindelöf property, we can select open sets $U_{x_1}, U_{x_2}, \ldots$ such that $\bigcup_{n \in \mathbb{N}^+} U_{x_n} = X$ and $\mu(U_{x_n}) < \infty$.

Now let $G_n = U_{x_n} \setminus \bigcup_{i=1}^{n-1} U_{x_i}$ for all $n \in \mathbb{N}^+$. These are G_δ sets with finite measure. □

Lemma 2.33. *Let μ be a Radon measure on a separable metric space X. Then μ is outer regular, moreover for any measurable set M and for all $\varepsilon > 0$ there exists an open set $G \supseteq M$ such that $\mu(G \setminus M) < \varepsilon$.*

Proof. First, let B be a Borel subset of X and fix any $\varepsilon > 0$. As constructed in the proof of Lemma 2.32, let (U_n) be a sequence of open sets such that $\bigcup_{n \in \mathbb{N}} U_n = X$ and $\mu(U_n) < \infty$ for all $n \in \mathbb{N}$. Now lemmas 2.12 and 2.13 imply that there exist open sets $G_n \subseteq U_n$ such that $B \cap U_n \subseteq G_n$ and $\mu(G_n \setminus (B \cap U_n)) < \frac{\varepsilon}{2^{n+1}}$. Now set $G = \bigcup_{n \in \mathbb{N}} G_n$. Consequently, $\mu(G \setminus B) < \varepsilon$.

Since any measurable set M is contained in a Borel set B such that $\mu(B \setminus M) = 0$, the statement follows for measurable sets as well. □

Remark 2.34. Let μ be a Radon measure on a separable metric space X. Then for any measurable set M and for all $\varepsilon > 0$ there exists a closed set $F \subseteq M$ such that $\mu(M \setminus F) < \varepsilon$.

Proposition 2.35. *Let μ be a Radon measure on the Borel sets of a Polish space X and let N be a nullset. Then there is a G_δ nullset G such that $N \subseteq G$.*

Proof. By definition, there is a Borel set B such that $N \subseteq B$ and $\mu(B) = 0$. By Lemma 2.33, there exists a sequence of open sets (G_n) such that $B \subseteq G_n$ and $\mu(G_n \setminus B) < \frac{1}{n+1}$. Now let $G = \bigcap_{n \in \mathbb{N}} G_n$, which is clearly a G_δ nullset and $N \subseteq G$. □

2.4 Baire category

In this section, we provide a comprehensive exposition of every definition and proof, recognising the importance of establishing a thorough familiarity with these classes of sets. This is essential in order to effectively compare and contrast them with their measure-theoretic counterparts.

Definition 2.36. [1, Ch 8.] In a topological space X, there are certain types of sets in the sense of Baire category:

- A set $A \subseteq X$ is **nowhere dense** if its closure has empty interior: $\operatorname{int} \overline{A} = \emptyset$.

- A set $A \subseteq X$ is **meagre** if it is a countable union of nowhere dense sets: $\exists A_n, n \in \mathbb{N}$ nowhere dense sets such that $A = \bigcup_{n\in\mathbb{N}} A_n$.

- A set is **comeagre** if it is the complement of a meagre set.

Remark 2.37. [1, Ch 8.] A is nowhere dense $\iff$ $\overline{A}$ is nowhere dense $\iff$ $\overline{A}^c$ is dense $\iff$ for every nonempty open set $V \subseteq X$ there is a nonempty open set $V' \subseteq V$ such that $\overline{A} \cap V' = \emptyset$.

A set is comeagre if and only if it contains the intersection of a countable family of dense open sets.

Proposition 2.38. *[3, Thm 1.2] Let X be a topological space and $A \subseteq X$ be a nowhere dense set. Then any subset B of A is also nowhere dense.*

Proof. The proof is simple: since $\overline{B} \subseteq \overline{A}$ and $\operatorname{int} \overline{B} \subseteq \operatorname{int} \overline{A} = \emptyset$, the set B is nowhere dense. □

Proposition 2.39. *If X is a topological space, $\emptyset \neq U \subseteq X$ open, $A \subseteq U$. Then:*

1. *The set A is nowhere dense in U if and only if A is nowhere dense in X.*
2. *The set A is meagre in U if and only if A is meagre in X.*

Proof. Suppose that $A \subseteq U$ is nowhere dense in X. By Remark 2.37, for every nonempty open set $U \cap V$ there is a nonempty open $V' \subseteq V \cap U$ such that $V' \cap (\overline{A} \cap U) \subseteq V' \cap \overline{A} = \emptyset$, so A is nowhere dense in U. To prove the converse, let $A \subseteq U$ be nowhere dense in U. Let $V \subseteq X$ be a nonempty open set. If $V \cap U = \emptyset$, then it is a suitable subset. Otherwise, $V \cap U$ is nonempty and relatively open in U, so there is a nonempty open subset $V' \subseteq V$ such that $V' \cap A = \emptyset$. □

The following proposition follows immediately from the definitions.

Proposition 2.40. *[1, Ch 8.] In a topological space X, the meagre sets form a σ-ideal.*

Meagre sets, like sets of measure zero, also represent a notion of smallness, but in a different sense. Intuitively, a meagre set can be thought of as a set that is "thin" or "sparse" in some sense. Now we introduce the dual notion of measurability.

Definition 2.41. [1, Def 8.21] A subset A of a topological space X has **the Baire property** if there exist an open set G and a meagre set P such that $A = G \Delta P$.

Thus, sets with the Baire property are similar to open sets, differing from them only by a small, negligible subset, a meagre set.

Notation 2.42. We will abbreviate the term Baire property by BP.

Lemma 2.43. *Let X be a topological space, let $G \subseteq X$ be an open set and let $F \subseteq X$ be a closed set. Then the sets $\overline{G} \setminus G$ and $F \setminus \operatorname{int} F$ are nowhere dense.*

Proof. It suffices to show that $\overline{G} \setminus G$ is nowhere dense. It is closed since $\overline{G} \setminus G = \overline{G} \cap G^c$, so we have to show that it has empty interior. Let U be an open set in $\partial G = \overline{G} \setminus G$. Then every $x \in U$ has a neighbourhood that is disjoint from G, which is impossible by $x \in \partial G$. Therefore, U is empty. □

Definition 2.44. Let X be a topological space. For sets A and B in X let $A \approx B \iff A \Delta B$ is meagre. This is clearly an equivalence relation.

Remark 2.45. Note that in a topological space X, a set A has the BP if and only if there exists an open set in the $\approx$-equivalence class of A.

Proposition 2.46. *[3, Thm 4.1] A subset A of a topological space X has the BP if and only if there exist a closed set F and a meagre set Q such that $A = F \Delta Q$.*

Proof. Follows immediately from Lemma 2.43 and Remark 2.45. □

Proposition 2.47. *[1, Prop 8.22] In any topological space, sets with the BP form a σ-algebra, which is generated by the open sets and the meagre sets.*

Proof. Let X be a topological space. First, we need to show that if a set A has the BP, then so does A^c. The set A can be written as $A = G \Delta P$, where G is open and P is meagre, so this means that $A^c = (G \Delta P)^c = G^c \Delta P$. The set G^c is closed, therefore, A^c has the BP by Proposition 2.46. Now let (A_n) be a sequence of sets with the BP, and let $A_n = G_n \Delta P_n$ with G_n open and P_n meagre for all $n \in \mathbb{N}$. Let $G = \bigcup_{n \in \mathbb{N}} G_n, P = \bigcup_{n \in \mathbb{N}} P_n$ and $A = \bigcup_{n \in \mathbb{N}} A_n$. This way, G is open, P is meagre and $G \setminus P \subseteq A \subseteq G \cup P$, from which we get that $G \Delta A \subseteq P$ is meagre and $A \approx G$, therefore A has the BP. Since the empty set has the BP, we have proven that this is a σ-algebra, and it is clear from the definition that it is the smallest one that contains all open and meagre sets. □

Corollary 2.48. *In a topological space X, every Borel set has the BP.*

Definition 2.49. [1, Def 8.37] Let X and Y be topological spaces. A function $f\colon X \to Y$ is **Baire measurable** if $f^{-1}(G)$ has the BP for every open subset G of Y.

Proposition 2.50. *[1, Prop 8.23] A subset of a topological space X has the BP if and only if it is of the form $G \dot\cup P$, where G is a G_δ set and P is a meagre set (or of the form $F \setminus Q$, where F is an F_σ set and Q is a meagre set).*

Proof. If a set A has the BP, it is of the form $A = U \Delta M$, where U is open and M is meagre. Any meagre set is contained in an F_σ meagre set by Remark 2.37, so let F be a meagre F_σ set with $A \Delta U \subseteq F$. Then the set $G = U \setminus F$ is G_δ and $G \subseteq A$ and the set $P = A \setminus G \subseteq F$ is meagre. □

Proposition 2.51. *Let X be a Polish space with a countable basis $\{V_n\}_{n\in\mathbb{N}}$ and let A be a subset of X. Then A has the BP if and only if for every basic open set V_n either A is meagre in V_n or there exists a nonempty basic open $V_n' \subseteq V_n$ such that A is comeagre in V_n'.*

Proof. First, let us assume that A has the BP and therefore can be represented as $A = U\Delta M$, where U is open and M is meagre. Let V_n be a nonempty basic open set. If $V_n \cap A \subseteq M$, then A is meagre in V_n. If $V_n \cap A \cap U \neq \emptyset$, then there exists a basic open set $V_n' \subseteq V_n \cap U$. This way, A is comeagre in V_n' since $V_n' \setminus A \subseteq M$, which is meagre.

To prove the converse, consider

$$U = \bigcup\{V_n \colon V_n \text{ is basic open, } A \text{ is comeagre in } V_n\}.$$

Observe that A is comeagre in U. Since $\overline{U} \setminus U$ is nowhere dense, it suffices to show that $A \setminus \overline{U}$ is meagre. Let $V_n \subseteq \overline{U}^c$ be a basic open set. Then $A \cap V_n$ is meagre in V_n, since otherwise there would be a nonempty basic open set $V_n' \subseteq V_n \subseteq \overline{U}^c$ such that A is comeagre in V_n', but this is a contradiction since $V_n' \cap U = \emptyset$. Hence $A \setminus \overline{U}$ is a countable union of meagre sets, therefore it is meagre, which concludes the proof. □

Proposition 2.52. *[1, Prop 8.1] In a topological space X, the following statements are equivalent:*

(i) Every nonempty open set is non-meagre.

(ii) Every comeagre set is dense.

(iii) The intersection of countably many dense open sets is dense.

Proof. By Remark 2.37, the statements (ii) and (iii) are clearly equivalent. If A is a nonempty meagre open set, then its complement is not dense, thus $(ii) \implies (i)$. If a comeagre set does not meet a nonempty open set, then that open set is meagre, so $(i) \implies (ii)$, which concludes the proof. □

Definition 2.53. [1, Def 8.2] A topological space X is a **Baire space** if the equivalent statements of Proposition 2.52 hold in X.

Proposition 2.54. *[1, Prop 8.3] Any open subset U of a Baire space X is also a Baire space.*

Proof. We will use Proposition 2.52 (iii) to prove this claim. Let (U_n) be a sequence of dense open sets in U, thus they are open in X as well. Then for all $n \in \mathbb{N}$ the set $U_n \cup \overline{U}^c$ is dense in X and since X is a Baire space, $\bigcap\limits_{n\in\mathbb{N}} (U_n \cup \overline{U}^c) = (\bigcap\limits_{n\in\mathbb{N}} U_n) \cup \overline{U}^c$ is also dense in X. Consequently, $\bigcap\limits_{n\in\mathbb{N}} U_n$ is dense in U. □

Proposition 2.55. *[3, Thm 9.2] A subset E of a Baire space X is comeagre if and only if it contains a dense G_δ subset of X.*

Proof. The proof follows from Remark 2.37 and Proposition 2.52. $\square$

Theorem 2.56 (Baire Category Theorem). *[1, Thm 8.4] Every completely metrisable space is Baire.*

Proof. Let (X, d) be a complete metric space and (U_n) a sequence of dense open sets in X. Fix any nonempty open set U. Since $U \cap U_0$ is nonempty open, we can set an open ball B_0 of radius < 1 such that $\overline{B}_0 \subseteq U \cap U_0$. In the nth step, where $n \geq 1$, let B_n be an open ball of radius $< \frac{1}{n+1}$ such that $\overline{B}_n \subseteq B_{n-1} \cap U_n$. Let x_n be the centre of B_n for all $n \in \mathbb{N}$. Then (x_n) is a Cauchy sequence and since (X, d) is complete, there exists $x \in X$ such that $x_n \to x \in \bigcap_{n\in\mathbb{N}} \overline{B}_n = \bigcap_{n\in\mathbb{N}} B_n \subseteq (\bigcap_{n\in\mathbb{N}} U_n) \cap U$, which concludes the proof. $\square$

Remark 2.57. By Lemma 2.12 and the Baire Category Theorem, a G_δ subspace of a completely metrisable space is also a Baire space.

The Baire Category Theorem guarantees that a nonempty Polish space is nonmeagre in itself. Therefore, meagre sets form a nontrivial σ-ideal, hence we obtain a nontrivial notion of smallness, which can be used as an alternative to the measure theoretic notion of nullsets. In fact, it works in many cases when there is no natural measure on a space. The Baire Category Theorem provides a powerful tool in non-constructive proofs: if X is a Polish space of objects and P is a property of these objects, then it is often easier to prove that objects lacking the property P form a meagre set than to explicitly construct an object of property P. This can be viewed as the topological dual of the random method in discrete mathematics.

Although they are very similar, the notions of nullsets and meagre sets do not coincide, as the following theorem illustrates.

Theorem 2.58. *Let X be a Polish space and μ a continuous Radon measure on X. Then X can be decomposed into two sets A and B such that $B = A^c$, A is meagre and B is of measure zero.*

Proof. Since X is Polish, it is separable. Let $\{a_1, a_2, \ldots\}$ be a countable dense subset of X. Since μ is continuous and locally finite, for every $i, j \in \mathbb{N}^+$ we can find an open ball centred at a_i with $\mu(B_{i,j}) < \frac{1}{2^{i+j}}$. Now put $G_j = \bigcup_{i\in\mathbb{N}^+} B_{i,j}$ and $B = \bigcap_{j\in\mathbb{N}^+} G_j$. To show that B has measure zero we fix $\varepsilon > 0$ and prove that $\mu(B) < \varepsilon$. There exists $j \in \mathbb{N}^+$ such that $\frac{1}{2^j} < \varepsilon$ and $\mu(B) \leq \mu(G_j) \leq \sum_{i=1}^{\infty} \mu(B_{i,j}) \leq \sum_{i=1}^{\infty} \frac{1}{2^{i+j}} = \frac{1}{2^j} < \varepsilon$, hence B has measure zero. By Remark 2.37, B is comeagre, so $A = B^c$ is meagre, which completes the proof. $\square$

Theorem 2.56 ensures that the partition into sets of measure zero and meagre sets in Theorem 2.58 is not trivial when the entire space does not have measure zero.

Corollary 2.59. *Let X be a Polish space and μ a continuous Radon measure on the Borel sets of X. Then any subset $E \subseteq X$ can be decomposed into two sets A and B such that A is meagre and B is of measure zero.*

Chapter 3

Dual theorems

In this chapter, we study analogies between measure theory and Baire category by exploring dual theorems, drawing inspiration from Oxtoby's [3] work. However, we go beyond the traditional scope of these theorems on the real line and extend our exploration to Polish spaces. By doing so, we gain a deeper understanding of these theorems and their applicability.

3.1 Bernstein sets

After introducing the notions of measurable sets and sets with the BP, a natural question arises: are there sets in a given measure space that are not measurable or lack the BP? In Oxtoby's book [3], two examples are presented in the context of the real line, both relying on the axiom of choice. The first example is the Vitali set [3, Ch 5.], constructed by selecting one element from each coset of the rational numbers. The proof of the Vitali set's non-measurability and absence of the BP relies on the translation invariance of Lebesgue measure and the continuity of addition.

In Oxtoby's book, the other example is the so-called Bernstein set. In this section, we construct a Bernstein set in an arbitrary Polish space X, and we show that such a set lacks the BP and is non-measurable with respect to any continuous Radon measure on X. Let us start with a lemma concerning the cardinality of the family of uncountable closed sets.

Lemma 3.1. *In an uncountable Polish space X, the class of closed subsets of cardinality c has cardinality c.*

Proof. The space has a countable basis, so there are at most c open sets since every open set can be written as a countable union of basis elements. Every closed set is the complement of an open set, so there are at most c closed sets. By Corollary 2.16, there is a subspace $C \subseteq X$ homeomorphic to $\mathcal{C}$. Then, by Proposition 2.17, we have that $C \cong C \times C = \bigcup_{x \in C} (\{x\} \times C)$. Hence we obtained c many disjoint closed sets of cardinality c in X. $\square$

The fact that both measurable sets and sets with the BP form σ-algebras suggests that the desired set is likely to bear a striking resemblance to its complement.

Theorem 3.2 (Bernstein set)**.** *In an uncountable Polish space X, there exists a set B such that both B and B^c meet every closed subset of X that has cardinality continuum.*

Proof. Let $\{P_\alpha \colon \alpha < c\}$ be a transfinite enumeration of all closed subsets of X with cardinality continuum. By transfinite recursion, we can pick distinct points $a_\alpha, b_\alpha \in P_\alpha$ such that $a_\alpha, b_\alpha \notin \{a_\beta \colon \beta < \alpha\} \cup \{b_\beta \colon \beta < \alpha\}$. This can be done, since $2|\alpha| \leq \max\{|\alpha|, \aleph_0\} < c$, while $|P_\alpha| = c$.

Let $B = \{a_\alpha \colon \alpha < c\}$. Now there is no $\alpha < c$ such that $P_\alpha \subseteq B$ or $P_\alpha \subseteq B^c$. $\square$

Theorem 3.3. *Let X be a perfect Polish space, let μ be a continuous Radon measure on X and let B be a Bernstein set. Then every measurable subset of B or B^c is of measure zero, and similarly, every subset of B or B^c that has the BP is meagre. In particular, B is non-measurable and lacks the BP.*

Proof. Fix any measurable subset A of B. Every closed set contained in A is countable and therefore, of measure zero. Remark 2.34 implies that $\mu(A) = 0$.

Now let A be a subset of B that has the BP. By Proposition 2.50, it can be written as $A = G \dot{\cup} P$, where G is a G_δ-set and P is meagre. Then G is countable since otherwise, by Lemma 2.14, we could find a Cantor set in G, which would contradict the construction of B. Consequently, A is meagre. $\square$

The following result is not particularly difficult to obtain, but it is rather surprising. Although initially it was not easy to find sets that are non-measurable or lack the BP, it turns out that there are actually many such sets.

Proposition 3.4. *In a perfect Polish space X with a continuous Radon measure μ, any measurable subset of positive measure contains a non-measurable subset, and any non-meagre set contains a subset that lacks the BP.*

Proof. Let B be a Bernstein set and let A be a measurable set with $\mu(A) > 0$. Since at most one of the sets $A \cap B$ and $A \cap B^c$ can have measure zero, by Theorem 3.3, at least one of them is non-measurable. Similarly, if A is non-meagre, at most one of the sets $A \cap B$ and $A \cap B^c$ can be meagre, so at least one of them lacks the BP. $\square$

Corollary 3.5. *Let X be a perfect Polish space and let μ be a nonzero continuous Radon measure on X. Then there exists a set $E \subseteq X$ that is μ-measurable but lacks the BP and there exists a set $F \subseteq X$ that has the BP but is non-measurable.*

Proof. By Theorem 2.58 there is a nonmeagre nullset $E' \subseteq X$ and a meagre set $F' \subseteq X$ with positive measure. Now by Proposition 3.4 there are $E \subseteq E'$ and $F \subseteq F'$ such that E is a nullset that lacks the BP and F is a non-measurable meagre set. □

3.2 Lusin's theorem

In this section, our objective is to approximate μ-measurable and Baire measurable functions with continuous functions. Lusin's theorem and its Baire category counterpart provide a characterisation of μ-measurable and Baire measurable functions based on the size of the set where the approximating functions remain continuous.

Theorem 3.6 (Lusin). *Let X be a Polish space with a Radon measure μ and let Y be a second countable space. A function $f\colon X \to Y$ is μ-measurable if and only if for every $\varepsilon > 0$ there exists a closed set $F \subseteq X$ with $\mu(F^c) < \varepsilon$ such that f is continuous on F.*

Proof. Let $U_1, U_2, \dots$ be a countable basis of Y, and since f is measurable, we can apply Lemma 2.33 and Remark 2.34 to the preimages of the basis elements: there exist open sets $G_1, G_2, \dots$ and closed sets $F_1, F_2, \dots$ such that $F_n \subseteq f^{-1}(U_n) \subseteq G_n$ and $\mu(G_n \setminus F_n) < \frac{\varepsilon}{2^n}$ for all $n \in \mathbb{N}^+$. Let $F = X \setminus \bigcup_{n\in\mathbb{N}^+} (G_n \setminus F_n)$, so we have $\mu(F^c) < \varepsilon$ and F is closed. To see that f is continuous on F note that $f^{-1}(U_n) \cap F = G_n \cap F$ is indeed relatively open for all $n \in \mathbb{N}^+$.

To prove the converse, let $F_1, F_2, \dots$ be a sequence of closed sets such that $\mu(F_n^c) < \frac{1}{n}$, and the restriction $f_n = f|_{F_n}$ is continuous for all $n \in \mathbb{N}^+$. We have to show that for an arbitrary open set G the preimage $f^{-1}(G)$ is μ-measurable. For each n there exists an open set $G_n \subseteq X$ such that $f_n^{-1}(G) = G_n \cap F_n$. Let $F = \bigcup_{n\in\mathbb{N}^+} F_n$. Since $f^{-1}(G) = (f^{-1}(G) \cap F) \cup (f^{-1}(G) \setminus F)$, it suffices to show that these two sets are measurable. On the one hand, $\mu(f^{-1}(G) \setminus F) = 0$, so it is μ-measurable. On the other hand, $f^{-1}(G) \cap F = \bigcup_{n\in\mathbb{N}^+} (f^{-1}(G) \cap F_n) = \bigcup_{n\in\mathbb{N}^+} f_n^{-1}(G) = \bigcup_{n\in\mathbb{N}^+} (G_n \cap F_n)$, so it is also measurable. The proof is complete. □

The following theorem is the category analogue of Lusin's theorem, offering a more compelling result. While Lusin's theorem allowed for an approximation of a measurable function with an arbitrarily small ε error, the dual theorem is stronger.

Theorem 3.7. *Let X be a topological space and Y be a second countable space. A function $f\colon X \to Y$ is Baire measurable if and only if there exists a comeagre set A such that f is continuous on A.*

Proof. First, let us assume that f is Baire measurable. Let $U_1, U_2, \dots$ be a countable basis of Y. Then there exist open sets $G_1, G_2, \dots$ and meagre sets $P_1, P_2, \dots$ such that $f^{-1}(U_n) = G_n \Delta P_n$ for each $n \in \mathbb{N}^+$. Let $A = X \setminus \bigcup_{n \in \mathbb{N}^+} P_n$, which is comeagre. Now $f|_A$ is continuous because we have $f^{-1}(U_n) \cap A = (G_n \Delta P_n) \cap A = G_n \cap A$, which is relatively open.

To prove the converse, let us assume that f is continuous on a comeagre set A, which means that for any open set U in Y, there exists an open set G in X such that $f|_A^{-1}(U) = G \cap A$. Since $G \setminus A^c = f|_A^{-1}(U) \subseteq f^{-1}(U) \subseteq f|_A^{-1}(U) \cup A^c = G \cup A^c$, we can conclude that $f^{-1}(U)$ has the BP, so f Baire measurable. □

Remark 3.8. The natural dual of Theorem 3.7 fails even on $[0, 1]$. The characteristic function of a Cantor set of positive measure is measurable but there is no set of measure 1 on which it is continuous.

3.3 Egoroff's theorem

The following result highlights the idea that pointwise convergence can be strengthened to uniform convergence on a large portion of the space, demonstrating the interplay between measure theory and function convergence properties.

Theorem 3.9 (Egoroff). *Let (X, μ) be a finite measure space and (Y, d) be a metric space. If a sequence of functions $f_n \colon X \to Y$ converges μ-almost everywhere to a limit function f, then for every $\varepsilon > 0$, there is a measurable subset B of X such that (f_n) converges to f uniformly on B and $\mu(B^c) < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Let us define measurable sets $A_{n,k} = \bigcup_{m \geq n} \{x \in X \colon d(f_m(x), f(x)) \geq \frac{1}{k}\}$ for all $n, k \in \mathbb{N}^+$. This way $A_{n,k} \supseteq A_{n+1,k}$, and by the definition of pointwise convergence, we know that for any $k \in \mathbb{N}^+$ we have $\mu(\bigcap_{n \in \mathbb{N}^+} A_{n,k}) = 0$. Since $\mu(X) < \infty$, we have continuity from above: for all $k \in \mathbb{N}^+$, there is an index $n(k)$ such that $\mu(A_{n(k),k}) < \frac{\varepsilon}{2^k}$. Let $B = X \setminus \bigcup_{k \in \mathbb{N}^+} A_{n(k),k}$. Then $\mu(B^c) < \varepsilon$ and for every $k \in \mathbb{N}^+$ we have $d(f_m(x), f(x)) < \frac{1}{k}$ for all $m \geq n(k)$ and all $x \in B$. Therefore f_n converges to f uniformly on B. □

Unlike other theorems we have explored, this particular theorem does not have an analogous result in Baire category theory. In fact, Oxtoby's book provides a counterexample [3, Page 38], even in the case of $\mathbb{R}$, to illustrate this distinction. We would expect that if a sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$ converges to f pointwise, then there is a comeagre set $B \subseteq \mathbb{R}$ such that (f_n) converges to f uniformly on B.

We will present a sequence of functions that is pointwise convergent on $\mathbb{R}$ but any set on which it converges uniformly is nowhere dense. This is much stronger than what we expect from a counterexample for this statement. Let

$$\varphi(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2 - 2x, & x \in (\frac{1}{2}, 1] \\ 0, & x \in \mathbb{R} \setminus [0, 1] \end{cases}.$$

Then $\lim_{n\to\infty} \varphi(2^n x) = 0$ for all $x \in \mathbb{R}$. Let $\{r_i : i \in \mathbb{N}^+\}$ be a dense sequence in $\mathbb{R}$ and let $f_n(x) = \sum_{i=1}^{\infty} 2^{-i}\varphi(2^n(x - r_i))$ for all $n \in \mathbb{N}^+$. Then f_n is continuous on $\mathbb{R}$, since it is the sum of a uniformly convergent series of continuous functions, and for every $x \in \mathbb{R}$ we have $\lim_{n\to\infty} f_n(x) = 0$. Now let (a, b) be an open interval. For some i we have $r_i \in (a, b)$, and then $\sup_{x\in(a,b)} f_n(x) \geq \frac{1}{2^i}$ for all sufficiently large n. Thus f_n does not converge uniformly on (a, b). Now let B be a set on which f_n converges uniformly. Then, by continuity, f_n also converges uniformly on $\overline{B}$. From what we have shown, the set $\overline{B}$ cannot contain an interval and therefore it is nowhere dense.

3.4 Fubini's theorem

The following theorem is an important special case of Fubini's theorem.

Theorem 3.10 (Fubini). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces and A be a $(\mu \times \nu)$-measurable subset of $X \times Y$. Then if $X \times Y$ is σ-finite or $(\mu \times \nu)(A) < \infty$, the following hold:*

(i) The sections A_x are ν-measurable for μ-almost every $x \in X$ (similarly for Y).

(ii) $(\mu \times \nu)(A) = 0 \iff \nu(A_x) = 0$ for μ-almost every $x \in X \iff \mu(A^y) = 0$ for ν-almost every $y \in Y$.

Theorem 3.11 (Kuratowski–Ulam). *[1, Thm 8.41] Let X and Y be second countable topological spaces and let $A \subseteq X \times Y$ be a set with the BP. The following hold:*

(i) The set $\{x \in X : A_x \text{ has the BP in } Y\}$ is comeagre in X and similarly for Y.

(ii) The set A is meagre in $X \times Y \iff$ the set $\{x \in X : A_x \text{ is meagre in } Y\}$ is comeagre in $X \iff$ the set $\{y \in Y : A^y \text{ is meagre in } X\}$ is comeagre in Y.

Proof. To prove the category analogue of Fubini's theorem we need two lemmas.

Lemma 3.12. *[1, Lemma 8.42] Let F be a subset of the product space $X \times Y$, where X and Y are topological spaces, and Y is second countable. If F is nowhere dense in $X \times Y$, then the set $\{x \in X \mid F_x \text{ is nowhere dense in } Y\}$ is comeagre in X.*

Proof. We may assume that F is closed. Let $U = (X \times Y) \setminus F$, which is open, therefore it suffices to show that U_x is dense for all but a meagre set of $x \in X$. Let $V_1, V_2, \ldots$ be a countable basis of Y. Since for any $G \subseteq X$ nonempty open we have $U \cap (G \times V_n) \neq \emptyset$, the projection $U_n = \mathrm{pr}_X(U \cap (X \times V_n))$ is dense open in X. If $x \in \bigcap_{n \in \mathbb{N}^+} U_n$, then $U_x \cap V_n \neq \emptyset$ for all $n \in \mathbb{N}^+$, thus U_x is dense. $\square$

By taking a countable union, we get that if $A \subseteq X \times Y$ is meagre, then A_x is meagre for all but a meagre set of $x \in X$, which proves ($\Longrightarrow$) of (ii).

To prove (i) we write A as $A = U \Delta M$, where U is open and M is meagre. Since $A_x = U_x \Delta M_x$ for every $x \in X$, (i) follows from ($\Longrightarrow$) of (ii).

Lemma 3.13. *[1, Lemma 8.43] Let $A \subseteq X$ and $B \subseteq Y$, where X and Y are second countable topological spaces. Then $A \times B$ is meagre if and only if at least one of A and B is meagre.*

Proof. By ($\Longrightarrow$) of (ii), if $A \times B$ is meagre but A is not, there exists $x \in A$ such that $(A \times B)_x = B$ is meagre.

Now let us assume that A is meagre, so it can be written as $A = \bigcup_{n \in \mathbb{N}^+} F_n$, where F_n is nowhere dense for all n. Then $A \times B = \bigcup_{n \in \mathbb{N}^+} (F_n \times B)$, and it suffices to show that $F_n \times B$ is nowhere dense. This is true because if $G \subseteq X$ is a dense open set, then $G \times Y$ is dense open in $X \times Y$, hence $F_n \times B \subseteq F_n \times Y$ is nowhere dense in $X \times Y$. $\square$

Finally, we prove ($\Longleftarrow$) of (ii). Let $A \subseteq X \times Y$ be a set with the BP for which the set $\{x \in X \colon A_x \text{ is meagre in } Y\}$ is comeagre in X but A is non-meagre. Now A can be written as $A = U \Delta M$ with U open, M meagre. Since U cannot be meagre, there exist open sets $G \subseteq X$, $H \subseteq Y$ with $G \times H \subseteq U$ and $G \times H$ not meagre. By Lemma 3.13, both G and H are non-meagre. Now, by our assumption and ($\Longrightarrow$) of (ii), there is $x \in G$ such that A_x and M_x are meagre, hence $H \subseteq U_x \subseteq A_x \cup M_x$ is meagre, a contradiction. $\square$

Remark 3.14. [1, Ch 8] Theorem 3.11 fails in the absence of the BP. In fact, there exists a non-meagre subset $A \subseteq [0,1]^2$ with no 3 collinear points. This result relies on the Axiom of Choice.

3.5 Poincaré recurrence theorem

Poincaré's remarks on recurrence were made before the formal development of measure theory and Baire category, yet they continue to hold in these frameworks. His theorem on recurrence has significant implications in dynamics, across disciplines such as physics, astronomy, and chaos theory, shedding light on the long-term behaviour of various systems.

We begin by exploring Poincaré's theorem within a broad context, and then proceed to apply the findings to our specific space.

Definition 3.15. [3, Ch 17.] Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathcal{I} \subseteq \mathcal{S}$ be a σ-ideal. Let T be a map of X into X.

- The map $T\colon X \to X$ is **$\mathcal{S}$-measurable,** if $T^{-1}E \in \mathcal{S}$ for all $E \in \mathcal{S}$.
- The map T is **measure-preserving** if it is $\mathcal{S}$-measurable and for every $S \in \mathcal{S}$ we have $\mu(S) = \mu(T^{-1}(S))$.
- The **positive semiorbit** of a point x is the sequence $x, Tx, T^2x, \ldots$
- A point x in a set G is **recurrent with respect to** G if $T^ix \in G$ for infinitely many positive integers i.
- A subset E of X is a **wandering set** if $E, T^{-1}E, T^{-2}E, \ldots$ are disjoint.
- The map T is **dissipative** if there is a wandering set in $\mathcal{S} \setminus \mathcal{I}$, otherwise it is nondissipative.
- For $E \subseteq X$, let $D(E) = \{x \in E\colon T^ix \in E$ for at most finitely many $i \in \mathbb{N}^+\}$.
- The map T has the **recurrence property** if $D(E) \in \mathcal{I}$ for all $E \in \mathcal{S}$. Note that if T is $\mathcal{S}$-measurable and $E \in \mathcal{S}$, then $D(E) \in \mathcal{S}$ since $D(E) = X \setminus \bigcap\limits_{n\in\mathbb{N}} \bigcup\limits_{i\geq n} T^{-i}E$.
- If X is a topological space, then a point $x \in X$ is **recurrent under** T if it is recurrent with respect to every neighbourhood of itself.

Theorem 3.16. *[3, Thm 17.2] Let $T\colon X \to X$ be an $\mathcal{S}$-measurable map. It has the recurrence property if and only if it is nondissipative.*

Proof. If T has the recurrence property, it has to be nondissipative since otherwise there would be a wandering set $E \in \mathcal{S} \setminus \mathcal{I}$ for which $D(E) = E$ contradicting the recurrence property.

Now suppose that T is nondissipative and fix any $E \in \mathcal{S}$. Consider $F = E \setminus \bigcup\limits_{k\in\mathbb{N}^+} T^{-k}E \in \mathcal{S}$. Observe that for every $k \in \mathbb{N}$ the set $T^{-k}F$ is wandering: indeed, for any $0 \leq i < j$ we have $T^{-j}F \cap T^{-i}F \subseteq T^{-j}E \setminus \bigcup\limits_{k\in\mathbb{N}^+} T^{-i-k}E = \emptyset$. Since T is nondissipative, $T^{-k}F$ set belongs to $\mathcal{I}$ for all $k \geq 0$, and so do $\bigcup\limits_{k\in\mathbb{N}} T^{-k}F$ and $H = E \cap \bigcup\limits_{k\in\mathbb{N}} T^{-k}F$ because $\mathcal{I}$ is a σ-ideal. On the other hand, $H = D(E)$ because for each k we have $T^{-k}F = \{x \in X\colon T^kx \in E,\ \ \forall i > k\ \ (T^ix \notin E)\}$. We conclude that T has the recurrence property. $\square$

Theorem 3.17. *Let X be a Polish space with a finite Borel measure μ such that $\mu(U) > 0$ for every nonempty open $U \subseteq X$. Let T be a measure-preserving homeomorphism of X onto itself. Then for an open subset $G \subseteq X$ every point in G is recurrent with respect to G except for a meagre set of measure zero.*

Proof. Fix a nonempty open set $G \subseteq X$. First, let $\mathcal{S}$ be the σ-algebra of all μ-measurable subsets of X and $\mathcal{I}$ be the σ-ideal of nullsets. Then T is $\mathcal{S}$-measurable and nondissipative since wandering sets are nullsets because $\mu(X)$ is finite. Thus, by Theorem 3.16, T has the recurrence property. We conclude that almost every point of G is recurrent with respect to G.

Second, let $\mathcal{S}$ be the σ-algebra of subsets of X that have the BP, and let $\mathcal{I}$ be the σ-ideal of meagre sets. Again, T is $\mathcal{S}$-measurable. Note that there is no non-empty open wandering set since they are not nullsets. Let $E = U \Delta P$ be a wandering set with the BP, where U is open and P is meagre. For any integers $0 \leq i < j$ we have $T^{-i}E \cap T^{-j}E = \emptyset$, so $T^{-i}U \cap T^{-j}U \subseteq T^{-i}P \cup T^{-j}P$, hence $T^{-i}U \cap T^{-j}U$ is a meagre open set, thus, by Theorem 2.56, it is empty. Now U is an open wandering set and therefore empty, hence E is meagre. Thus, T is nondissipative, so it has the recurrence property. We conclude that comeagre many points of G are recurrent with respect to G. $\square$

The Poincaré recurrence theorem tells us that in certain dynamical systems the set of non-recurrent points can be considered "small" in terms of both measure and Baire category.

Theorem 3.18 (Poincaré recurrence theorem)**.** *Let X be a Polish space with a finite measure μ, for which $\mu(U) > 0$ for any open set U. Let $T\colon X \to X$ be a measure-preserving homeomorphism. Then μ-almost every and comeagre many points in X are recurrent under T.*

Proof. Let $U_1, U_2, \dots$ be a countable basis of X, and let $E_k \subseteq U_k$ be the set of points that are not recurrent with respect to U_k. By Theorem 3.17, E_k is a meagre set of measure zero for each $k \geq 1$, so the set $E = \bigcup_{k \in \mathbb{N}^+} E_k$ is also a meagre set of measure zero. We have to show that any $x \in E^c$ is recurrent under T. Let U be a neighbourhood of x. Then $x \in U_k \subseteq U$ for some k and $x \notin E_k$, therefore $T^i x \in U_k \subseteq U$ for infinitely many positive integers i, thus x is recurrent under T. $\square$

3.6 Sierpiński–Erdős duality theorem

In this section we are going to prove the following metatheorem.

Theorem 3.19 (Duality Principle)**.** *[3, Thm 19.4] (CH) Let P be any proposition involving solely the notions of measure zero, meagreness and notions of pure set theory. Let P^* be the proposition obtained from P by interchanging the terms "set of measure zero" and "meagre set" wherever they appear. Then each of the propositions P and P^* implies the other.*

The Duality Principle is implied by the following theorem, which serves as a sufficient condition to establish the duality between measure theory and Baire category theory.

Theorem 3.20 (Sierpiński–Erdős). *[3, Thm 19.3] (CH) Let X be an uncountable Polish space with a continuous Radon measure μ. There exists an involution $f\colon X \to X$ such that for any $E \subseteq X$ the image $f(E)$ is meagre if and only if $\mu(E) = 0$, and $\mu(f(E)) = 0$ if and only if E is meagre.*

The proof of this theorem relies on two purely set-theoretic theorems.

Theorem 3.21. *[3, Thm 19.5] Let X be a set of cardinality $\aleph_1$ and let $\mathcal{K}$ be a family of subsets of X that satisfies the following:*

(i) $\mathcal{K}$ is a σ-ideal.

(ii) The union of $\mathcal{K}$ is X.

(iii) $\mathcal{K}$ has a subclass $\mathcal{G}$ of cardinality $\leq \aleph_1$ such that each member of $\mathcal{K}$ is contained in some member of $\mathcal{G}$.

(iv) The complement of each set in $\mathcal{K}$ contains a set of cardinality $\aleph_1$ that belongs to $\mathcal{G}$.

Then there exists a decomposition of X into $\aleph_1$ disjoint sets X_α, each of cardinality $\aleph_1$, such that a subset E of X belongs to $\mathcal{K}$ if and only if it is contained in a countable union of the sets X_α.

Proof. Let $\mathcal{G} = \{G_\alpha \colon \alpha < \omega_1\}$ and $H_\alpha = \bigcup_{\beta\leq\alpha} G_\beta$ and $K_\alpha = H_\alpha \setminus \bigcup_{\beta<\alpha} H_\beta$. Let $B = \{\alpha < \omega_1 \colon K_\alpha \text{ is uncountable}\}$. By properties (i) and (iv), the set B is unbounded. Hence, there exists an order-preserving bijection $\phi\colon \omega_1 \to B$. For every $\alpha < \omega_1$ let $X_\alpha = H_{\phi(\alpha)} \setminus \bigcup_{\beta<\alpha} H_{\phi(\beta)}$. Then, by construction, the sets X_α are disjoint and they belong to $\mathcal{K}$. Each of the sets X_α has cardinality $\aleph_1$ because $X_\alpha \supseteq K_{\phi(\alpha)}$ and $K_{\phi(\alpha)}$ is uncountable. By property (iii), for any set $E \in \mathcal{K}$ there exists an ordinal $\beta < \omega_1$ such that $E \subseteq G_\beta$. Then there exists $\alpha < \omega_1$ such that $\beta < \phi(\alpha)$, and therefore $E \subseteq G_\beta \subseteq H_\beta \subseteq H_{\phi(\alpha)} = \bigcup_{\gamma\leq\alpha} X_\gamma$. Observe that by property (ii), we have $X = \bigcup \mathcal{K} \subseteq \bigcup_{\alpha<\omega_1} X_\alpha$. Consequently, $\{X_\alpha \colon \alpha < \omega_1\}$ is a decomposition of X with the required properties. □

Theorem 3.22. *[3, Thm 19.6] Let X be a set of cardinality $\aleph_1$. Let $\mathcal{K}$ and $\mathcal{L}$ be two families of subsets of X that satisfy the properties $(i) - (iv)$ from Theorem 3.21. Suppose that there exist complementary sets M and N such that $M \in \mathcal{K}$ and $N \in \mathcal{L}$. Then there exists a one-to-one map f of X onto itself such that $f = f^{-1}$ and such that $f(E) \in \mathcal{L}$ if and only if $E \in \mathcal{K}$.*

Proof. Let $\{X_\alpha \colon \alpha < \omega_1\}$ be the decomposition of X corresponding to $\mathcal{K}$ as in the proof of Theorem 3.21. We may assume that M is part of the generating class $\mathcal{G}$ and that $G_0 = M$. Then $X_0 = M$ because M cannot be countable since the complement of N contains a set of cardinality $\aleph_1$ that belongs to $\mathcal{L}$. Similarly, let $\{Y_\alpha \colon \alpha < \omega_1\}$ be the decomposition of X corresponding to $\mathcal{L}$ with $Y_0 = N$. Then $M = \bigcup_{0<\alpha<\omega_1} Y_\alpha$

and $N = \bigcup\limits_{0<\alpha<\omega_1} X_\alpha$. Since $N = M^c$, the sets X_α and Y_α for $0 < \alpha < \omega_1$ constitute a decomposition of X into sets of cardinality $\aleph_1$. Let $f_\alpha \colon X_\alpha \to Y_\alpha$ be a bijection for every $0 < \alpha < \omega_1$. Now let f equal to f_α on X_α and equal to f_α^{-1} on Y_α for $0 < \alpha < \omega_1$. Thus, f is a one-to-one map of X onto itself, $f = f^{-1}$, and since $X_0 = \bigcup\limits_{0<\alpha<\omega_1} Y_\alpha$ and $Y_0 = \bigcup\limits_{0<\alpha<\omega_1} X_\alpha$, we have that $f(X_\alpha) = Y_\alpha$ for all $0 \le \alpha < \omega_1$. Theorem 3.21 implies that $f(E) \in \mathcal{L}$ if and only if $E \in \mathcal{K}$. □

Lemma 3.23. *[3, Thm 19.1] Let X be an uncountable Polish space with a continuous Radon measure μ. The complement of any set of measure zero in X contains a set of measure zero of cardinality c. The complement of any meagre set in X contains a nowhere dense set of cardinality c.*

Proof. Let $A \subseteq X$ be a nullset. Then there exists a Borel set $B \supseteq A$ such that $\mu(B) = 0$. By Remark 2.34, the set B^c contains an uncountable compact set, which, by Lemma 2.12 and Corollary 2.16, contains a subspace C homeomorphic to $\mathcal{C}$. Now Proposition 2.17 says that $C \cong C \times C = \bigcup\limits_{x \in C} (\{x\} \times C)$. Now we have to show that there is $x \in C$ for which $\mu(\{x\} \times C) = 0$. Suppose the contrary. By Lemma 2.32, there exists a sequence of G_δ sets $G_n \subseteq X$ such that $\bigcup\limits_{n \in \mathbb{N}} G_n = X$ and $\mu(G_n) < \infty$ for all $n \in \mathbb{N}$. Now let $C_{n,k} = \{x \in C \colon \mu(G_n \cap (\{x\} \times C)) \ge \frac{1}{k}\}$ for all $n, k \in \mathbb{N}, k \ge 1$. We have that $\bigcup\limits_{n \in \mathbb{N}, k \in \mathbb{N}^+} C_{n,k} = C$, which is of cardinality continuum, so there exist $n \in \mathbb{N}$ and $k \in \mathbb{N}^+$ such that $C_{n,k}$ contains infinitely many points. Now, using the fact that sets of the form $\{x\} \times C$ are pairwise disjoint, we have that $\mu(G_n) \ge \sum\limits_{x \in C_{n,k}} \mu(G_n \cap (\{x\} \times C)) \ge \sum\limits_{x \in C_{n,k}} \frac{1}{k} = \infty$, which is a contradiction. Consequently, there exists $x \in C$ such that $\mu(\{x\} \times C) = 0$.

Similarly, let $A \subseteq X$ be a meagre set. then it is contained in an F_σ meagre set B, so B^c is an uncountable G_δ set and by Corollary 2.16, it contains a subspace C homeomorphic to $\mathcal{C}$. By Proposition 2.17 we have that $C \cong C \times C = \bigcup\limits_{x \in C} (\{x\} \times C)$. Now we have to show that there is $x \in C$ for which $\{x\} \times C$ is nowhere dense. Suppose the contrary. These are closed sets, so we have that $\mathrm{int}(\{x\} \times C) \neq \emptyset$ for all $x \in C$. But then we obtained continuum many disjoint nonempty open sets in X, which contradicts the fact that X is separable. Thus, there is $x \in C$ such that $\{x\} \times C$ is nowhere dense. □

Proof of Theorem 3.20. By Theorem 3.22 and Theorem 2.58, it suffices to verify that both the family of meagre sets and the family of nullsets satisfy properties $(i) - (iv)$ of Theorem 3.21. Let $\mathcal{K}$ denote the class of meagre sets and $\mathcal{L}$ denote the class of sets of measure zero. To verify the conditions stated in Theorem 3.21, we focus on the following properties. The class $\mathcal{K}$ is generated by the F_σ meagre sets, while $\mathcal{L}$ is generated by the G_δ sets of measure zero by Proposition 2.35. Notably, both of these classes have cardinality continuum. By assuming the continuum hypothesis, we satisfy the third condition, and the fourth condition follows from Lemma 3.23. The first two conditions are clear.

3.7 Baire meets Lebesgue

In this section, whenever we write "measurable", we mean Lebesgue measurable.

The density topology offers a new way to understand Lebesgue measurability by linking it directly to the Baire property in a different topology. This perspective allows us to explore the relationship between Lebesgue measurability and the underlying structure of the density topology, revealing the connections between measure theory and Baire category theory in a fresh and insightful manner.

Definition 3.24. [2, Ch 7.] Let $A \subseteq \mathbb{R}^p$ be an arbitrary set, $x \in \mathbb{R}^p$ a point. Then

$$\overline{d}(x, A) = \limsup_{r \to 0} \frac{\overline{\lambda}(B(x, r) \cap A)}{\lambda(B(x, r))}$$

is the **upper density** of A at x, and

$$\underline{d}(x, A) = \liminf_{r \to 0} \frac{\overline{\lambda}(B(x, r) \cap A)}{\lambda(B(x, r))}$$

is the **lower density** of A at x. If these are equal, then $d(x, A) = \overline{d}(x, A) = \underline{d}(x, A)$ is the **density** of A at x. The point x is a **density point** of A if $d(x, A) = 1$.

Lemma 3.25. *[2, Lemma 7.1] For any $A \subseteq \mathbb{R}^p$ and $x \in \mathbb{R}^p$ we have*

$$\overline{d}(x, A) + \underline{d}(x, A^c) \geq 1,$$

and if $d(x, A) = 0$, then $d(x, A^c) = 1$. If A is measurable, then $\overline{d}(x, A) + \underline{d}(x, A^c) = 1$, and $d(x, A) = 1$ implies $d(x, A^c) = 0$.

Proof. Since $\overline{\lambda}$ is subadditive, we have $\overline{\lambda}(A \cap B(x, r)) + \overline{\lambda}(B(x, r) \setminus A) \geq \lambda(B(x, r))$. By dividing and taking the limsup of both sides we get

$$\overline{d}(x, A) \geq \limsup_{r \to 0} \left(1 - \frac{\overline{\lambda}(B(x, r) \setminus A)}{\lambda(B(x, r))}\right) = 1 - \liminf_{r \to 0} \frac{\overline{\lambda}(B(x, r) \setminus A)}{\lambda(B(x, r))} = 1 - \underline{d}(x, A^c)$$

This implies the other statements since if A is measurable, then

$$\lambda(A \cap B(x, r)) + \lambda(B(x, r) \setminus A) = \lambda(B(x, r)).$$

□

Theorem 3.26 (Lebesgue's density theorem)**.** *[2, Thm 7.2] For any Lebesgue measurable set $A \subseteq \mathbb{R}^p$, the density of A is 1 at almost every point in A.*

Definition 3.27. [2, Ch 7.] Let $A \subseteq \mathbb{R}^p$. A set $M \subseteq \mathbb{R}^p$ is a **measurable hull** of A if M is measurable, $A \subseteq M$ and if N is a measurable set containing A, then $\lambda(M \setminus N) = 0$.

It is easy to prove the following well-known lemma.

Lemma 3.28. *[2] Any set $A \subseteq \mathbb{R}^p$ has a measurable hull M for which $\overline{\lambda}(A) = \lambda(M)$.*

Lemma 3.29. *[2, Lemma 7.3] Let $A \subseteq \mathbb{R}^p$ and M be its measurable hull. Then for any measurable set B, we have $\overline{\lambda}(A \cap B) = \lambda(M \cap B)$ and for all $x \in \mathbb{R}^p$ we have $\underline{d}(x, A) = \underline{d}(x, M)$ and $\overline{d}(x, A) = \overline{d}(x, M)$.*

Proof. Let N be a measurable hull of $A \cap B$ that is a subset of the measurable set $M \cap B$. Let $C = (M \setminus B) \cup N$ be another measurable set that contains A, thus $\lambda(M \setminus C) = 0$ and $\lambda((M \cap B) \setminus N) = 0$ because $(M \cap B) \setminus N \subseteq M \setminus C$. Consequently, $\overline{\lambda}(A \cap B) = \lambda(N) = \lambda(M \cap B)$. The other two statements follow immediately. $\square$

Theorem 3.30. *[2, Lemma 7.4] Let $A \subseteq \mathbb{R}^p$. Then $d(x, A) = 1$ at almost every point in A. The set A is measurable if and only if $d(x, A) = 0$ at almost every point in A^c.*

Proof. The first part of the theorem follows from Lemma 3.29 and Theorem 3.26. Also by Lebesgue's density theorem and Lemma 3.25, we only need to prove that if $d(x, A) = 0$ at almost every point in A^c, then A is Lebesgue measurable. Let M be a measurable hull of A. Then at almost every point x in $M \setminus A$ we have $d(x, A) = d(x, M) = 1$. On the other hand, at almost every point x in $M \setminus A$ we have $d(x, A) = 0$ by assumption. Hence $\lambda(M \setminus A) = 0$ and A is measurable. $\square$

Definition 3.31. [2, Ch 8] A subset A of $\mathbb{R}^p$ is called **d-open**, if it is Lebesgue measurable and for all $x \in A$ we have $d(x, A) = 1$.

Theorem 3.32. *[2, Thm 8.1] The d-open sets form a topology on $\mathbb{R}^p$.*

Proof. It is clear that $\emptyset$ and $\mathbb{R}^p$ are d-open. Now let A and B be d-open sets. For arbitrary $x \in A \cap B$ we have $d(x, A) = d(x, B) = 1$, so by Lemma 3.25 and the measurability of A and B we get that $d(x, A^c) = d(x, B^c) = 0$. By taking their union, $d(x, A^c \cup B^c) = 0$ and by Lemma 3.25, $d(x, A \cap B) = 1$. Thus $A \cap B$ is d-open.

Now we have to check if an arbitrary union of d-open sets is also d-open. Let A_i be d-open sets for all $i \in I$ and $A = \bigcup_{i \in I} A_i$. If $x \in A_i$ for some $i \in I$, we have $\underline{d}(x, A) \geq \underline{d}(x, A_i) = 1$, so $d(x, A) = 1$. It remains to prove that A is Lebesgue measurable. Fix a point $x \in A$, then there exists $i \in I$ such that $x \in A_i$. Then $d(x, A_i) = 1$ and since A_i is measurable, by Lemma 3.25, we have $d(x, A_i^c) = 0$. Now $d(x, A^c) = 0$ follows from $A^c \subseteq A_i^c$. Thus, by Theorem 3.30, we conclude that A is Lebesgue measurable. $\square$

Definition 3.33. [2, Ch 8.] The topology of d-open sets is called the **density topology**. We denote it by τ_d.

Remark 3.34. Observe that the density topology on $\mathbb{R}^p$ is finer than the Euclidean topology.

Notation 3.35. [2, Ch 8.] Let $\text{d-int}\, A$ (resp. $\text{d-cl}\, A$) denote the interior (resp. closure) of the set A with respect to the density topology.

Theorem 3.36. *[2, Thm 8.2] If $A \subseteq \mathbb{R}^p$ is Lebesgue measurable, then*

$$\text{d-int}\, A = \{x \in A \colon d(x, A) = 1\}.$$

For any $B \subseteq \mathbb{R}^p$,

$$\text{d-cl}\, B = B \cup \{x \in \mathbb{R}^p \colon \overline{d}(x, B) > 0\}.$$

Proof. Let $A_1 = \{x \in A \colon d(x, A) = 1\}$. Since $\text{d-int}\, A$ is d-open and $\text{d-int}\, A \subseteq A$, for all $x \in \text{d-int}\, A$ we have $d(x, A) = d(x, \text{d-int}\, A) = 1$. Therefore $\text{d-int}\, A \subseteq A_1$. Since A is measurable and $\lambda(A \setminus A_1) = 0$ by Theorem 3.30, A_1 is measurable as well. Furthermore, for all $x \in A_1$ we have $d(x, A_1) = d(x, A) = 1$. Thus A_1 is d-open and $A_1 \subseteq \text{d-int}\, A$. The proof of the first claim is complete.

Now let B be an arbitrary subset of $\mathbb{R}^p$. Suppose that $x \notin B$ and $d(x, B) = 0$. Let M be a measurable hull of B. Then Lemma 3.29 implies that $d(x, M) = 0$ and by Lemma 3.25, we have that $d(x, M^c) = 1$. Set $D = \text{d-int}\,(M^c \cup \{x\})$. Then by the first part of this theorem, $x \in D$. Since $D \cap B = \emptyset$ and D is d-open, we obtain that $x \notin \text{d-cl}\, B$. Hence $\text{d-cl}\, B \subseteq B \cup \{x \in \mathbb{R}^p \colon \overline{d}(x, B) > 0\}$.

Now suppose that $x \notin \text{d-cl}\, B$. Then x has a neighbourhood in τ_d that does not meet B, that is, there exists a d-open set D such that $x \in D$ and $D \cap B = \emptyset$. Then

$$d(x, D) = 1 \overset{\text{L. 3.25}}{\Longrightarrow} d(x, D^c) = 0 \overset{B \subseteq D^c}{\Longrightarrow} d(x, B) = 0 \implies x \notin B \cup \{x \in \mathbb{R}^p \colon \overline{d}(x, B) > 0\},$$

which concludes the proof. $\square$

It is immediate from Theorem 3.36 that every set of measure zero is d-closed. In particular, every countable set is d-closed and we obtain further theorems.

Theorem 3.37. *[2, Thm 8.4] For any set $A \subseteq \mathbb{R}^p$, the following are equivalent:*

(i) The set A is nowhere dense in the density topology.

(ii) The set A is meagre in the density topology.

(iii) The set A is of measure zero.

Proof. The implications $(iii) \implies (i) \implies (ii)$ are clear. By Theorem 3.26 and Theorem 3.36, if a measurable set is nowhere dense, it has measure zero since otherwise its d-interior would not be empty. If a set A is nowhere dense in the density topology, $\text{d-cl}\, A$ is nowhere dense and measurable, therefore $\overline{\lambda}(A) \leq \lambda(\text{d-cl}\, A) = 0$. This proves $(i) \implies (iii)$ and consequently, by taking a countable union, it proves $(ii) \implies (iii)$ as well. $\square$

Theorem 3.38. *[2, Thm 8.10] For any set $A \subseteq \mathbb{R}^p$, the following are equivalent:*

(i) The set A is F_σ in the density topology.

(ii) The set A is G_δ in the density topology.

(iii) The set A is Borel in the density topology.

(iv) The set A has the BP in the density topology.

(v) The set A is Lebesgue measurable.

Proof. The implications $(i) \implies (iii)$, $(ii) \implies (iii)$ and $(iii) \implies (iv)$ are clear by Corollary 2.48. By Theorem 3.37, (iv) implies (v). It suffices to prove $(v) \implies (i)$ and $(v) \implies (ii)$. If A is Lebesgue measurable, it can be written as $A = F \cup N$ where F is F_σ in the euclidean topology and $\lambda(N) = 0$. Since τ_d is finer than the Euclidean topology, F is F_σ in the density topology as well and N is d-closed by Theorem 3.36. We conclude that A is F_σ in the density topology. Now $(v) \implies (ii)$ follows by taking complements. $\square$

Chapter 4

Analytic sets are nice

We have seen that Borel sets have the BP (Corollary 2.48) and are universally measurable (Definition 2.26). It is a naturally arising question whether this holds for a larger family of sets. In this chapter, we introduce the family of analytic sets, which is significantly broader than the σ-algebra of Borel sets, and we prove that every analytic set has the BP and is universally measurable. To achieve this, we draw upon the beautiful theory of infinite games following Kechris' work [1].

The motivation behind the definition and study of analytic sets was when Lusin noticed that the following seemingly true proposition is actually false: *For every Borel set $B \subseteq \mathbb{R}^2$ the projection of B on the x-axis is Borel.* The key to the often overlooked error is that the identity $f(A \cap B) = f(A) \cap f(B)$ is not true in general, it may fail even if $A, B \subseteq \mathbb{R}^2$ are Borel and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is the projection on the x-axis. In fact, projections of Borel sets form a much larger family. We call them analytic.

Definition 4.1. [1, Def 14.1] Let X be a Polish space. A set $A \subseteq X$ is **analytic** if it is a continuous image of a Polish space.

Definition 4.2. Let X be a Polish space. A set $B \subseteq X$ is **co-analytic** if B^c is analytic.

The following examples illustrate that many nicely definable sets are analytic, so it is indeed useful to explore the properties of this class of sets.

Example 4.3. Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$ with the supremum norm. Then the set

$$A = \{f \in C[0,1] \colon \exists x_0 \in [0,1] \ \ f \text{ is not differentiable at } x_0\}$$

is analytic but not Borel.

Example 4.4. [1, Thm 27.5] Let X be a Polish space and let $K(X)$ be the hyperspace of compact sets in X, which is also a Polish space with the Hausdorff metric. The set

$$\{K \in K(X) \colon K \text{ is uncountable}\}$$

is analytic. If X is uncountable, then it is not Borel.

Example 4.5. [1, Ex 27.15] Let X be a Polish space. The following set is analytic in $X^{\mathbb{N}}$:

$$\{(x_n) \in X^{\mathbb{N}} \colon (x_n) \text{ has a convergent subsequence}\}.$$

If X is not K_σ, then it is not Borel.

The development of even the basic theory of analytic sets lies beyond the scope of this thesis. Here we list only the most important facts:

Theorem 4.6. *[1, Ex 14.3, Thm 14.2, Prop 14.4] Let X be a Polish space. The following hold:*

- *Let $A \subseteq X$. Then A is analytic $\iff$ A is the projection of a closed set in $X \times \mathcal{N}$.*
- *The image of an analytic set under a Borel isomorphism between Polish spaces is analytic.*
- *Every Borel set is analytic, but if X is uncountable, then the converse does not hold.*
- *Let (A_n) be a sequence of analytic sets. Then the sets $\bigcap\limits_{n\in\mathbb{N}} A_n$ and $\bigcup\limits_{n\in\mathbb{N}} A_n$ are also analytic.*

4.1 Infinite games

In order to study certain properties of analytic sets, such as measurability and whether or not they have the BP, we will strongly rely on the theory of infinite games. We will start with the basic concepts.

Definition 4.7. [1, Ch 2.] Let A be a nonempty open set. We denote by A^n the set of finite sequences of length n from A:

$$A^n = \{(a_0, \dots, a_{n-1}) \colon \forall i < n \ \ (a_i \in A)\}.$$

If $n = 0$, we have $A^0 = \{\emptyset\}$. The empty sequence is $\emptyset$.

The **set of all finite sequences from** A is

$$A^{<\mathbb{N}} = \{(a_0, a_1, \dots, a_{n-1}) \colon n \in \mathbb{N}, \forall i < n \ \ (a_i \in A)\} = \bigcup_{n\in\mathbb{N}} A^n.$$

If $s, t \in A^{<\mathbb{N}}$, we say that s is an **initial segment** of t and t is an extension of s if there exists $m \leq \operatorname{length}(t)$ such that $s = t|m$. We write $s \subseteq t$.

The **concatenation** of two sequences $s = (s_0, \dots, s_{n-1}), t = (t_0, \dots, t_{m-1}) \in A^{<\mathbb{N}}$ is the sequence $(s_0, \dots, s_{n-1}, t_0, \dots, t_{m-1})$ and we denote it by $s\hat{\ }t$.

Definition 4.8. [1, Def 2.1] A **tree** on a set A is a subset T of $A^{<\mathbb{N}}$ that is closed under initial segments: $(a_0, \ldots, a_{n-1}) \in T,\ k \leq n \implies (a_0, \ldots, a_{k-1}) \in T$. The elements of T are the **nodes** of T.

The **body** of T is the set of all infinite branches in T:

$$[T] = \{(a_0, a_1, \ldots) \in A^{\mathbb{N}} \colon \forall n \in \mathbb{N}\ (a_0, \ldots, a_{n-1}) \in T\}.$$

A tree T is **pruned** if every node has a proper extension in T, that is, $(a_0, \ldots, a_{n-1}) \in T \implies \exists a_n \in A\ (a_0, \ldots, a_{n-1}, a_n) \in T$.

Equip the set A with the discrete topology, and then we can view $A^{\mathbb{N}}$ as the product space of infinitely many copies of A.

Lemma 4.9. *[1, Ch 2.] In the space $A^{\mathbb{N}}$ the following sets form a basis:*

$$N_s = \{x \in A^{\mathbb{N}} \colon s \subseteq x\},$$

where $s \in A^{<\mathbb{N}}$.

The map $T \mapsto [T]$ is a bijection between pruned trees on A and closed subsets of $A^{\mathbb{N}}$.

We define two-person infinite games in a general setting, so that later in the case of specific games the notions of strategy and winning strategy will be familiar.

Definition 4.10. [1, Ch 20.] Let $L \neq \emptyset$ be an arbitrary set. During the game, I and II take turns in playing elements of L.

Let $T \subseteq L^{<\mathbb{N}}$ be a pruned tree. This is the set of **legal positions** in the game. Players I and II take turns in playing

$$\begin{array}{llllll} \text{I} & l_0 & & l_2 & & \ldots \\ \text{II} & & l_1 & & l_3 & \end{array}$$

with $l_k \in L$ and $(l_0, \ldots, l_{k-1}) \in T$ for all $k \in \mathbb{N}$.

Let $N_I \subseteq [T]$ be a set. I wins a run $(l_0, l_1, \ldots)$ of the game if $(l_0, l_1, \ldots) \in N_I$, otherwise II wins. We call N_I the winning set of player I. (It is also called the payoff set.)

We denote this game by $G(T, N_I)$

Definition 4.11. [1, Ch 20.] A nonempty pruned subtree $\sigma \subseteq T$ is a **strategy** for player I if the following hold:

(i) if $(l_0, l_1, \ldots, l_{2j}) \in \sigma$, then for all $l_{2j+1} \in L,\ (l_0, l_1, \ldots, l_{2j}, l_{2j+1}) \in \sigma$;

(ii) if $(l_0, l_1, \ldots, l_{2j-1}) \in \sigma$, then for a unique $l_{2j} \in L,\ (l_0, l_1, \ldots, l_{2j-1}, l_{2j}) \in \sigma$.

A strategy σ for I is **winning** in $G(T, N_I)$ if $[\sigma] \subseteq N_I$. That is, if player I follows σ, then she wins.

We can define strategy and winning strategy for player II if we swap the parities. A strategy τ is winning for II if $[\tau] \cap N_I = \emptyset$.

Remark 4.12. I and II cannot have winning strategies simultaneously.

Definition 4.13. [1, Ch 20.] A game is **determined** if one of the players has a winning strategy.

The following theorem is a fundamental tool for deciding whether a particular game is determined.

Theorem 4.14 (Gale–Stewart)**.** *[1, Thm 20.1] Let T be a nonempty pruned tree on L. Let $N_I \subseteq [T]$ be closed or open in $[T]$. Then $G(T, N_I)$ is determined.*

Let us start our journey through infinite games with the Choquet game, which characterises Baire spaces.

Definition 4.15. [1, Def 8.10] Let X be a nonempty topologcial space. We define the **Choquet game** G_X of X as follows: player I and II take turns in playing nonempty open subsets of X

$$\begin{array}{llllll} \mathrm{I} & U_0 & & U_1 & & \dots \\ \mathrm{II} & & V_0 & & V_1 & \end{array}$$

so that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$. Player II wins this run of the game if $\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$, otherwise I wins.

Theorem 4.16 (Oxtoby)**.** *[1, Thm 8.11] Let X be a nonempty topological space. Then X is a Baire space $\iff$ player I has no winning strategy in the Choquet game G_X.*

Proof. ($\Longleftarrow$) Suppose that X is not a Baire space, so there exists a nonempty open set U_0 and a sequence (G_n) of dense open sets such that $\bigcap_{n\in\mathbb{N}} G_n \cap U_0 = \emptyset$. We will describe a winning strategy for player I that starts with the set U_0. Then, if II plays $V_0 \subseteq U_0$, player I can follow by $U_1 = V_0 \cap G_0 \neq \emptyset$ since G_0 is dense. Player I will continue this strategy: if II plays V_n, then we have $V_n \cap G_n \neq \emptyset$, and I can play $U_{n+1} = V_n \cap G_n$ for any $n \in \mathbb{N}$. Therefore, $\bigcap_{n\in\mathbb{N}} U_n \subseteq \bigcap_{n\in\mathbb{N}} G_n \cap U_0 = \emptyset$, and this is indeed a winning strategy for player I.

($\Longrightarrow$) Now suppose that I has a winning strategy σ and let U_0 be the first move of player I according to σ. By Proposition 2.54, it suffices to prove that U_0 is not Baire. We will construct a nonempty pruned subtree $S \subseteq \sigma$ such that for any $p = (U_0, V_0, \dots, U_n) \in S$, the set $\mathcal{U}_p = \{U_{n+1} \colon (U_0, V_0, \dots, U_n, V_n, U_{n+1}) \in S\}$ consists of pairwise disjoint open sets and $\bigcup \mathcal{U}_p$ is dense in U_n. We will construct S inductively by selecting which sequences of length n from σ we put in S. First, $\emptyset \in S$ and if

$(U_0, V_0, \ldots, U_{n-1}, V_{n-1}) \in S$, then $(U_0, V_0, \ldots, U_{n-1}, V_{n-1}, U_n) \in S$ for the unique U_n response of player I for which $(U_0, V_0, \ldots, U_{n-1}, V_{n-1}, U_n) \in \sigma$. The harder part is selecting which V_n responses of player II to add to $p = (U_0, V_0, \ldots, U_n) \in S$. For any $V_n \subseteq U_n$, if $V_n^* = U_{n+1}$ is what σ requires I to play next, then U_{n+1} is a nonempty open subset of V_n. By Zorn's Lemma, there exists a maximal collection $\mathcal{V}_p$ of nonempty open subsets $V_n \subseteq U_n$ such that the sets $\{V_n^* \colon V_n \in \mathcal{V}_p\}$ are pairwise disjoint. Put in S all $(U_0, V_0, \ldots, U_n, V_n)$ with $V_n \in \mathcal{V}_p$. Then $\mathcal{U}_p = \{U_{n+1} \colon (U_0, V_0, \ldots, U_n, V_n, U_{n+1}) \in S\} = \{V_n^* \colon V_n \in \mathcal{V}_p\}$ consists of pairwise disjoint sets and $\bigcup \mathcal{U}_p$ is dense in U_n, since if there was a nonempty open set $\tilde{V}_n \subseteq U_n$ that is disjoint from $\bigcup \mathcal{U}_p$, then $\mathcal{V}_p \cup \{\tilde{V}_n\}$ would violate the maximality of $\mathcal{V}_p$.

Now let $W_n = \bigcup\{U_n \colon (U_0, V_1, \ldots, U_n) \in S\}$. It follows by induction that W_n is open and dense in U_0 for all $n \in \mathbb{N}$. We claim that (W_n) is a sequence of dense open sets in U_0 with empty intersection. Otherwise, if $x \in \bigcap_{n\in\mathbb{N}} W_n$, there exists a unique $(U_0, V_0, U_1, V_1, \ldots) \in [S]$ such that $x \in U_n$ for all $n \in \mathbb{N}$, so $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$, contradicting the fact that $(U_0, V_0, \ldots) \in [\sigma]$ and σ is a winning strategy for I. Thus, U_0 is not a Baire space and the proof is complete. □

Definition 4.17. [1, Def 8.12] A nonempty topological space X is a **Choquet space** if player II has a winning strategy in G_X.

Remark 4.18. [1, Ch 8.] Every Choquet space is a Baire space since both players cannot have winning strategies.

Remark 4.19. [1, Ex 8.15, 8.16] Let X be a nonempty topological space. The following implications hold: X is Polish $\implies$ X is a completely metrisable space $\implies$ X is Choquet $\implies$ X is Baire.

Proof. The only implication we need to prove is that completely metrisable spaces are Choquet. In the Choquet game on X, player II selects a nonempty open set V_n such that $\overline{V}_n \subseteq U_n$ and $\operatorname{diam}(V_n) < \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Since the space is complete, $\bigcap_{n\in\mathbb{N}} \overline{V}_n = \bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$ and player II wins. □

Now we turn to the Banach–Mazur game, which is considered the starting point of the theory of infinite games.

Definition 4.20. [1, Ch 8.] Let X be a nonempty topological space and let $A \subseteq X$. The **Banach–Mazur game**, denoted by $G^{**}(A)$, is defined as follows: players I and II take turns in playing nonempty open sets

$$\begin{array}{lccccc} \text{I} & U_0 & & U_1 & & \ldots \\ \text{II} & & V_0 & & V_1 & \end{array}$$

so that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$. Player II wins this run of the game if and only if $\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} U_n \subseteq A$, otherwise I wins.

Theorem 4.21 (Banach–Mazur, Oxtoby). *[1, Thm 8.33] In the game $G^{**}(A)$ the following hold:*

(i) A is comeagre $\iff$ II has a winning strategy in $G^{**}(A)$.

(ii) If X is Choquet and there is a metric d on X whose open balls are open in X, then A is meagre in some nonempty open set $\iff$ I has a winning strategy in $G^{**}(A)$.

Proof. $(i)(\implies)$ Let (W_n) be a sequence of dense open sets such that $\bigcap_{n\in\mathbb{N}} W_n \subseteq A$. Let II play $V_n = U_n \cap W_n$. Then $\bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} W_n \subseteq A$, hence II wins.

$(\impliedby)$ As in the proof of Theorem 4.16.

$(ii)(\implies)$ Let us assume that A is meagre in a nonempty set U_0. Let (W_n) be a sequence of dense open sets in U_0 such that $\bigcap_{n\in\mathbb{N}} W_n \subseteq A^c$. Since U_0 is Choquet, player I has a winning strategy in the following game:

$$\begin{array}{llll} \text{I} & & U_1 & & \ldots \\ \text{II} & V_0 & & V_1 & \end{array}$$

so that $U_0 \supseteq V_0 \supseteq U_1 \supseteq \ldots,\ U_n, V_n$ nonempty open and I wins if $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$. Let σ be the winning strategy of I in this game. From here we can describe a winning strategy in the Banach–Mazur game for I by starting with U_0. Then II plays $V_0 \subseteq U_0$, to which I responds with the unique U_1 for which $(W_0 \cap V_0, U_1) \in \sigma$. Now II plays $V_1 \subseteq U_1$ and I responds with the unique U_2 for which $(W_0 \cap V_0, U_1, W_1 \cap V_1, U_2) \in \sigma$ and so on. This way, $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$ and $\bigcap_{n\in\mathbb{N}} U_n \subseteq \bigcap_{n\in\mathbb{N}} W_n \subseteq A^c$, so I wins.

$(\impliedby)$ Suppose that I has a winning strategy σ and the first move is U_0 according to it. We can construct another winning strategy σ' for I which also starts with U_0 and in the nth move it produces U_n with $\operatorname{diam}(U_n) < 2^{-n}$ for all $n \geq 1$. This is possible because in the nth move I may pretend that $\operatorname{diam}(V_{n-1}) < 2^{-n}$ by choosing a $V'_{n-1} \subseteq V_{n-1}$ with $\operatorname{diam}(V'_{n-1}) < 2^{-n}$ and responding by σ. This way $\bigcap_{n\in\mathbb{N}} U_n$ is a singleton contained in A^c. As in the first part and Theorem 4.16, by "starting the game" after U_0, we get that A is meagre in U_0. $\square$

Remark 4.22. Let X be a Polish space and $A \subseteq X$. If A has the BP, then $G^{**}(A)$ is determined by Proposition 2.51 and Theorem 4.21.

We will need another variant of the Banach–Mazur game.

Definition 4.23. Two games G and G' are **equivalent** if I (resp. II) has a winning strategy in G if and only if I (resp. II) has a winning strategy in G'.

Definition 4.24. [1, Ch 8.] Let X be a topological space. A collection of nonempty open sets in X is a **weak basis** if any nonempty open set contains one of them.

Remark 4.25. [1, Ch 21.] Let X be a Choquet space with a topology τ that admits a metric d whose open balls are open in (X, τ). Let $\mathcal{W}$ be a weak basis for (X, τ) and

I	U_0		U_1		$\ldots$
II		V_0		V_1	

let $A \subseteq X$. This variant of the Banach–Mazur game is defined as follows: players I and II take turns in playing nonempty open sets

so that $U_n, V_n \in \mathcal{W}$, $\operatorname{diam}(U_n), \operatorname{diam}(V_n) < 2^{-n}$, $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$. Player II wins this run of the game if and only if $\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} U_n \subseteq A$, otherwise I wins.

As in the proof of Theorem 4.21, if a player has a winning strategy, since X is Choquet, she can modify the winning strategy so that $\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} U_n$ becomes a singleton. Therefore, in the above definition of the game we can replace $\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} U_n \subseteq A$ with $\bigcap_{n\in\mathbb{N}} \overline{V}^d_n = \bigcap_{n\in\mathbb{N}} \overline{U}^d_n \subseteq A$.

It is easy to check that this variant is essentially equivalent to the original one from Definition 4.20, so Theorem 4.21 applies to this variant as well.

Definition 4.26. [1, Ch 21.] Let X be a Choquet space with a topology τ that admits a metric d whose open balls are open in (X, τ). Let $\mathcal{W}$ be a weak basis for (X, τ) and let $F \subseteq X \times \mathcal{N}$. The **unfolded Banach–Mazur game**, denoted by $G^{**}_u(F)$, is defined as follows: players I and II take turns in playing nonempty open sets and additionally II plays $y(n) \in \mathbb{N}$ in the nth round

I	U_0		U_1		$\ldots$
II		$y(0), V_0$		$y(1), V_1$	

so that $U_n, V_n \in \mathcal{W}$, $\operatorname{diam}(U_n), \operatorname{diam}(V_n) < 2^{-n}$, $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$. Player II wins this run of the game if and only if $\bigcap_{n\in\mathbb{N}} \overline{V}^d_n \times \{y\} = \bigcap_{n\in\mathbb{N}} \overline{U}^d_n \times \{y\} \subseteq F$, otherwise I wins.

Theorem 4.27. *[1, Thm 21.8] Let X be a Choquet space with a topology τ that admits a metric d whose open balls are open in (X, τ). Let $F \subseteq X \times \mathcal{N}$ and $A = \operatorname{pr}_X F$. Then*

(i) I has a winning strategy in $G^{**}_u(F) \implies A$ is meagre in a nonempty open set.

(ii) II has a winning strategy in $G^{**}_u(F) \implies A$ is comeagre.

Proof. (ii) If II has a winning strategy in $G^{**}_u(F)$, then she clearly has a winning strategy in $G^{**}(A)$ and we can apply Theorem 4.21.

(i) Let σ be a winning strategy for I and let U_0 be the first move according to it. Our goal is to show that A is meagre in U_0. Now fix a nonempty sequence $u \in \mathbb{N}^{<\mathbb{N}}$. The sequence $(U_0, V_0, U_1, V_1, \ldots, U_n)$ with $n \leq \operatorname{length}(u)$ or $(U_0, V_0, U_1, V_1, \ldots, U_n, V_n)$ with $n < \operatorname{length}(u)$ is compatible with σ, u if $(U_0, (u(0), V_0), U_1, (u(1), V_1), \ldots, U_n)$, respectively $(U_0, (u(0), V_0), U_1, (u(1), V_1), \ldots, U_n, (u(n), V_n))$ is in σ. As in the proof of Theorem 4.16, we can construct for every u a tree T_u of sequences compatible with σ, u such that

1. For any $(U_0, V_0, U_1, V_1, \ldots, U_n) \in T_u$, the family of sets
$$\mathcal{U} = \{U_{n+1} \colon (U_0, V_0, U_1, V_1, \ldots, U_n, V_n, U_{n+1}) \in T_u\}$$
is pairwise disjoint and $\bigcup \mathcal{U}$ is dense in U_n if $n < \operatorname{length}(u)$.

2. If $u \subseteq u'$, then T_u is the restriction of $T_{u'}$ to the sequences as above with $n \leq \operatorname{length}(u)$, respectively $n < \operatorname{length}(u)$.

Then the set $W_u = \bigcup\{U_{\operatorname{length}(u)} \colon (U_0, V_0, U_1, V_1, \ldots, U_{\operatorname{length}(u)}) \in T_u\}$ is open and dense in U_0 for every $u \in \mathbb{N}^{<\mathbb{N}}$. Let $G = \bigcap_{u \in \mathbb{N}^{<\mathbb{N}}} W_u$. Then G is comeagre in U_0 and it suffices to show that $G \subseteq A^c$, that is, if $x \in G$, then for all $y \in \mathcal{N}$ we have $(x, y) \notin F$. Fix $y \in \mathcal{N}$. Then $x \in \bigcap_{u \in \mathbb{N}^{<\mathbb{N}}} W_u \subseteq \bigcap_{n \in \mathbb{N}} W_{y|n}$, so there is a unique $(U_0, V_0, \ldots, U_n, V_n, \ldots)$ such that $x \in U_n$ for each n and
$$(U_0, (y(0), V_0), U_1, (y(1), V_1), \ldots, U_n, (y(n), V_n), \ldots) \in [\sigma].$$
Thus, $(x, y) \notin F$ and the proof is complete. $\square$

Proposition 4.28. *Let X be a Polish space. If $F \subseteq X \times \mathcal{N}$ is closed, then $G_u^{**}(F)$ is determined.*

Proof. Consider the map $f \colon [S] \to X \times \mathcal{N}$, $(U_0, (y(0), V_0), \ldots) \mapsto (x, y)$ where $\{x\} = \bigcap_{n \in \mathbb{N}} \overline{U}_n^d$ (see Remark 4.25). It suffices to show that this function is continuous because then N_{II} is closed and, by Theorem 4.14, it is determined.

Let us fix any $\varepsilon > 0$ and $n \in \mathbb{N}$. If $f((U_0, (y(0), V_0), \ldots)) = (x, y)$, we need a neigbourhood of $(U_0, (y(0), V_0), \ldots)$ which maps to $B(x, \varepsilon) \times [(y(0), y(1), \ldots, y(n-1))]$. Choose $k \in \mathbb{N}$ such that $k \geq n$ and $2^{-k} < \varepsilon$. Then $f([(U_0, (y(0), V_0), \ldots, U_k)]) \subseteq B(x, 2^{-k}) \times [(y(0), \ldots, y(k-1))] \subseteq B(x, \varepsilon) \times [(y(0), y(1), \ldots, y(n-1))]$. $\square$

Corollary 4.29. *Let X be a Polish space and A an analytic set. Then A is either meagre in a nonempty open set or comeagre in X.*

Proof. By Theorem 4.6, there exists $F \subseteq X \times \mathcal{N}$ closed such that $A = \operatorname{pr}_X F$. Then, by Proposition 4.28, we have that $G_u^{**}(F)$ is determined and consequently, Theorem 4.27 implies that A is either meagre in a nonempty open set or comeagre in X. $\square$

4.2 Analytic sets have the BP

Theorem 4.30 (Lusin–Sierpiński). *[1, Thm 21.6] Let X be a Polish space and $A \subseteq X$ an analytic set. Then A has the BP.*

Proof. Let $V_0, V_1, \dots$ be a countable basis of X. For a fixed $n \in \mathbb{N}$, by Lemma 2.12 and Theorem 4.6, we have that V_n is a Polish space and $A \cap V_n$ is analytic in V_n, so Corollary 4.29 implies that either A is comeagre in V_n or there exists a nonempty open set $V_n' \subseteq V_n$ such that A is meagre in V_n'. We may assume that this V_n' is basic open since if a set is meagre in an open set, then it is meagre in every open subset of it. Thus, for every $n \in \mathbb{N}$ the set A^c is either meagre in V_n or comeagre in some nonempty basic open $V_n' \subseteq V_n$. Hence, by Proposition 2.51, A^c has the BP, and this, together with Proposition 2.47, implies that A has the BP, which concludes the proof. □

4.3 Analytic sets are universally measurable

To prove our other main theorem, we need that the density-topology on $(0,1)$ is Choquet.

Lemma 4.31. *[1, Ex 17.47] Let $A \subseteq (0,1)$ be a Lebesgue measurable set and let $x \in \operatorname{d-int} A$. Then for every $\varepsilon > 0$ there is a nonempty closed set $P \subseteq A$ such that $x \in \operatorname{d-int} P$ and $\operatorname{diam}(P) < \varepsilon$.*

Proof. Let $r_0 > 0$ such that $r_0 < \min(x, 1-x, \frac{\varepsilon}{2})$. By definition, x is a density point of A, so there is a strictly decreasing sequence (r_n) in $(0, r_0)$ such that $\lim\limits_{n\to\infty} r_n = 0$ and for all $0 < r \leq r_n$ we have $\lambda(A \cap B(x,r)) > (1 - \frac{1}{2^{n+1}})\lambda(B(x,r))$. Now let P_n be a closed set such that $P_n \subseteq (B(x,r_n) \setminus B(x,r_{n+1})) \cap A$ and $\lambda(P_n) \geq (1 - \frac{1}{2^{n+1}})\lambda((B(x,r_n) \setminus B(x,r_{n+1})) \cap A)$. Set $P = \{x\} \cup \bigcup\limits_{n\in\mathbb{N}} P_n$, then x is a density point of P. It remains to prove that P is closed. Let (y_n) be a sequence in P that converges in $(0,1)$. If the set $I = \{k \in \mathbb{N} \colon \exists n \in \mathbb{N}\ y_n \in P_k\}$ has infinitely many members, then $y_n \to x$. Otherwise, $\{y_n \colon n \in \mathbb{N}\} \subseteq \bigcup\limits_{k\in I} P_k$, which is a compact set and therefore contains the limit. □

Lemma 4.32. *[1, Ex 17.47] The space $(0,1)$ with the density topology is a Choquet space.*

Proof. We will describe a winning strategy for player II in the Choquet game on $((0,1), \tau_d)$. In the nth step player I plays U_n d-open. Let $x_n \in U_n$. By Lemma 4.31, there is a nonempty closed set $P_n \subseteq U_n$ (in the Euclidean topology) with $x_n \in \operatorname{d-int} P_n$ and $\operatorname{diam}(P_n) < \frac{1}{n+1}$. Now let II play $\operatorname{d-int} P_n$. Then we have $\bigcap\limits_{n\in\mathbb{N}} U_n = \bigcap\limits_{n\in\mathbb{N}} P_n \neq \emptyset$. Therefore, II wins the Choquet game and $(0,1)$ is a Choquet space. □

Theorem 4.33 (Lusin)**.** *[1, Thm 21.10] In a Polish space X every analytic set is universally measurable.*

Proof. Let $S \subseteq X$ be an analytic set. We will prove some propositions, which will reduce this general setting to a much simpler one.

Proposition 4.34. *Let μ be a σ-finite measure on a measurable space $(X, \mathcal{M})$. Then there exists a probability measure ν on $(X, \mathcal{M})$ such that for all $A \subseteq X$ $\mu(A) = 0 \iff \nu(A) = 0$.*

Proof. Let (A_n) be a sequence of sets in $\mathcal{M}$ such that $\bigcup_{n\in\mathbb{N}} A_n = X$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. Then let us define $T = \sum_{n=0}^{\infty} 2^{-n-1}\dfrac{\mu(A_n)}{1+\mu(A_n)}$, where the sum is clearly convergent. It is straightforward to verify that the probability measure $\nu(A) = \frac{1}{T}\sum_{n=0}^{\infty} 2^{-n-1}\dfrac{\mu(A \cap A_n)}{1+\mu(A_n)}$ (for all $A \in \mathcal{M}$) is indeed equivalent to μ. $\square$

Proposition 4.35. *Let $\mathcal{M}$ be a σ-algebra on X that contains all singletons in X. Let μ be a finite measure on the measurable space $(X, \mathcal{M})$. Then there exist a discrete measure μ_d and a continuous measure μ_c on $(X, \mathcal{M})$ such that $\mu = \mu_d + \mu_c$.*

Proof. Let $D_n = \{x \in X\colon \mu(\{x\}) > \frac{1}{n}\}$ for all $n \in \mathbb{N}^+$ and let $D = \{x \in X\colon \mu(\{x\}) > 0\}$. Thus, $D = \bigcup_{n\in\mathbb{N}^+} D_n$. Every D_n is finite since μ is finite, hence D is countable. For a set $A \in \mathcal{M}$ let $\mu_d(A) = \mu(A \cap D)$ and $\mu_c(A) = \mu(A \setminus D)$. Cleary, μ_d is discrete and μ_c is continuous. $\square$

Let μ be a σ-finite Borel measure on X. By definition, the σ-algebra of measurable sets is generated by the σ-ideal of nullsets and the Borel sets. By propositions 4.34 and 4.35, we can assume, without loss of generality, that μ is a continuous probability measure.

Proposition 4.36. *[1, Thm 17.41] Let X be a Polish space and let μ be a continuous Borel probability measure on X. Then there exists a measure-preserving Borel isomorphism $(X, \mu) \to ((0,1), \lambda)$.*

By Proposition 4.36 and Theorem 4.6, we can assume that $X = (0,1)$ and μ is Lebesgue measure.

Let $P = S^c$ and $\mu_*(H) = \sup\{\mu(A)\colon A \subseteq P, A \text{ Borel}\}$ for all $H \subseteq (0,1)$. Then there are Borel sets $A_n \subseteq P$ such that $\mu_*(P) - \frac{1}{n+1} < \mu(A_n)$ for all $n \in \mathbb{N}$. Let $A = \bigcup_{n\in\mathbb{N}} A_n$. Clearly, $A \subseteq P$ is Borel and $\mu(A) = \mu_*(P)$. Now for the set $P' = P \setminus A$ we have $\mu_*(P') = 0$ and P' is co-analytic. If $\mu(P') = 0$, then since $P = P' \cup A$, we have that P is μ-measurable and so is S. Therefore, it remains to prove that $\mu(P') = 0$.

By Lemma 4.32, the space $(0,1)$ with the density topology is Choquet and the density topology is finer than the Euclidean topology. Then we can apply the double topology variant of the general unfolded Banach–Mazur game (presented in Definition 4.26) to $X = (0,1), \tau =$ the Euclidean topology and $\tau' =$ the density topology, since there is a closed set $F \subseteq (0,1) \times \mathcal{N}$ (where $(0,1)$ carries the Euclidean topology) such that $\mathrm{pr}_{(0,1)} F = P'^c$. By Theorem 4.27, we have that either

P'^c is comeagre or else P'^c is meagre in a nonempty open set in the density topology. In the first case, by Theorem 3.37, P' is of measure zero, and we are done. In the second case, let U be a nonempty open set in the density topology so that $U \setminus P'$ is meagre. By Theorem 3.37, $\mu(U \setminus P') = 0$. Then there exists a Borel set B such that $U \setminus P' \subseteq B$ and $\mu(B) = 0$. Since $U \setminus B \subseteq P'$ is measurable and $\mu(U \setminus B) > 0$, we get that $\mu_*(P') > 0$, which is a contradiction. The proof is complete. $\square$

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Szakdolgozat címe:
Measure and Category

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2023. 06. 07.

Miklós Csenge

a hallgató aláírása