# NYILATKOZAT

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A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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# Contents

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# Introduction

Singularity theory studies the local behaviour of an algebraic set $X$. In smooth points, $X$ locally looks like a linear subspace. However, there can be singular points where the local picture is much more complicated. For example, $X$ can intersect itself. The main idea is then to find a parametrization from a smooth algebraic set $\tilde{X}$, called a resolution.

The first chapter serves as an introduction to singularity theory. For curve singularities, the resolution is essentially unique and can be given by blowing up points. The resolution of surface singularities is much more complicated as several different resolutions can exist. Nevertheless, resolution graphs encode a lot of information about the singularity. We conclude this chapter by the fundamental theorem of Hironaka, stating that resolutions exist for every algebraic set.

The second chapter is devoted to simple surface singularities. These are an important class of singularities arising from group actions. We give a realization as singularities of algebraic sets and calculate the corresponding resolution graphs.

In the third chapter, we give a quick overview of the representation theory of finite groups with special emphasis on irreducible representations. From irreducible representations we construct the so-called McKay graph. The resolution graphs of simple surface singularities are then reconstructed as the McKay graph of the corresponding group.

The McKay correspondence was first proved by case-by-case calculations. In the final chapter, we give a more geometric approach and mention several reformulations of the original correspondence.

As algebraic geometry gets highly abstract very quickly, several proofs are omitted. Throughout the thesis, the emphasis is on understanding key examples and working out some crucial calculations in detail. In the same vein, I structured the thesis to emphasise the geometric insights opposed to the formal treatment of the literature.

# Chapter 1

# Singularities

## 1.1 Curve singularities

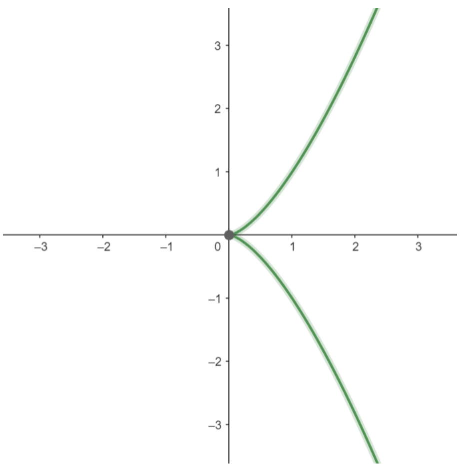
### 1.1.1 Plane algebraic curves

**Definition 1.1.1.** *An affine algebraic curve* $C = \{(x, y) \mid f(x, y) = 0\}$ *is the zero set of a polynomial* $f$ *with complex coefficients.*

**Definition 1.1.2.** *The degree of a curve is simply the degree of the polynomial* $f$*.*

One can find several examples in the high-school curriculum such as lines and conics having degrees 1 and 2. In some sense, these are the simplest curves.

If we increase the degree, the behaviour of the curves gets much more complicated. Exceptional points can appear already for degree 3 and it is hard to interpret the local picture.




The main tool in analysis for local investigations is the Implicit function theorem, which states that if the differential of $f$, $df = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ does not vanish at a point $p$

of the curve, then the curve is smooth at $p$ with tangent vector orthogonal to the differential at $p$.

However, at the special points where the differential vanishes the curve is not smooth and the tangent space is more difficult to define.

**Definition 1.1.3.** *The curve $C$ is singular at a point $p \in C$ if $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$. The singular locus of a curve is the set of singular points.*

**Example 1.1.4.** *We calculate the singular points of the cusp given by the equation $y^2 - x^3 = 0$.*

The differential is $(\frac{\partial(y^2-x^3)}{\partial x}, \frac{\partial(y^2-x^3)}{\partial y}) = (-3x^2, 2y)$. As expected from the picture, the cusp has only one singularity, at the origin.

By the previous example, it is natural to think that singular points form a finite set, but we have to be careful with curves such as $C = \{(x, y) \mid x^2 = 0\}$. In this case the whole curve will be singular. To avoid this problem we have to consider irreducible curves.

**Definition 1.1.5.** *A curve $C$ is irreducible if the defining polynomial $f$ is irreducible.*

**Theorem 1.1.6.** *The singular locus of an irreducible curve $C$ is finite.*

*Proof.* By definition, the singular locus is the intersection of the curves corresponding to $f$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. We show that already the intersection of $f$ and one of its partial derivatives is finite. This will follow from the more general lemma below: The intersection of two curves satisfying some mild conditions is always finite.

**Lemma 1.1.7** (Shafarevich)**.** *Let $f, g \in \mathbb{C}[x, y]$ be polynomials, and $f$ irreducible. If $f$ does not divide $g$, then the system $f(x, y) = g(x, y) = 0$ has a finite number of solutions.*

*Proof of the Lemma.* We follow the proof in Shafarevich's book [15]. The main idea is to work in the Euclidean ring $\mathbb{C}(x)[y]$ instead of $\mathbb{C}[x, y]$. By Gauss's lemma, since $f$ and $g$ were relatively prime in $\mathbb{C}[x, y]$, the same holds in our new ring. Now we can use that the new ring is Euclidean, thus there exist $r, q$ such that $rf + qg = 1$. Multiplying by the denominators of $r$ and $q$, we arrive at the equation $\tilde{r}f + \tilde{q}g = h$, where $h$ is a polynomial in $x$.

If $(x_1, y_1)$ is a solution of $f(x, y) = g(x, y) = 0$, then the LHS of the equation is zero. Therefore $x_1$ must be a root of $h$. As $h$ has a finite number of roots, the possible $x$ coordinates for solutions is finite. The proof is finished by a similar argument for the $y$ coordinate. $\square$

The theorem immediately follows from the Lemma as $f$ is irreducible and the partial derivatives have lower degrees. So the only possibility for $f$ dividing the partial derivatives is if both partial derivatives are zero. But then $f$ is constant contradicting the assumption that $f$ is irreducible. □

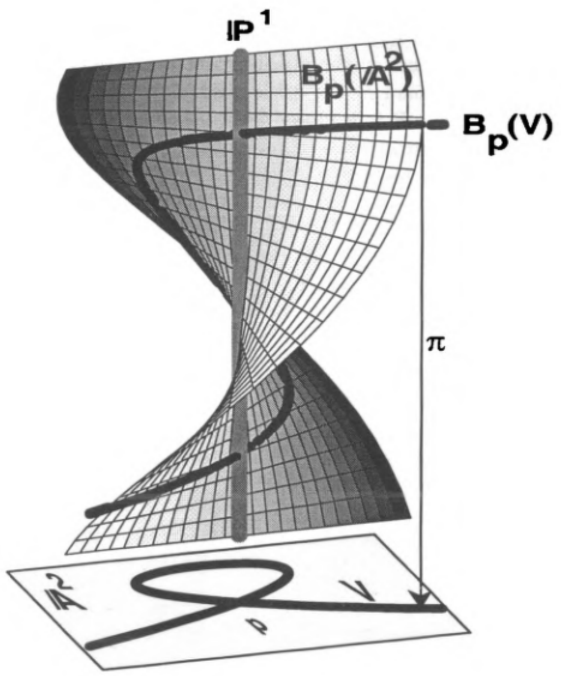
**Corollary 1.1.8.** *A curve has finitely many singularities if and only if every irreducible factor in the decomposition of $f$ has exponent one.*

### 1.1.2 Blowup

In this chapter, we introduce the basic device for constructing resolutions. It is a local surgery technique called blowup. The general idea is that each blowup makes the singularity "more simple" and after a sequence of blowups we get a smooth curve.

We motivate the definition by the following example:

Several singularities (nodes) locally look like two intersecting lines. The main problem is that both lines are perfectly reasonable candidates for the tangent. Therefore we replace the point with the parameter space of all lines through the point. This will split the two lines as they correspond to different points in the parameter space. The following picture is from [17].



**Definition 1.1.9.** *Let $B$ be the set of all pairs $(x, L)$, where $L$ is a line through the origin and $x \in L$. That is, $B = \{(x, L) \in \mathbb{C}^2 \times \mathbb{CP}^1 |\ x \in L\}$. This is called the blowup of the complex plane at the origin.*

There is a natural projection from the blowup to the complex plane.

$$\pi : B \to \mathbb{C}^2$$
$$(x, L) \mapsto x.$$

This space $B$ has the desired properties. The fiber of any point other than the origin is a single point in $B$, namely $(x, L)$ where $L$ is the unique line through the origin and $x$. On the other hand, the fiber of the origin is a projective line as the origin lies on all the lines going through the origin.

We can also define the blowup algebraically. Let $x_1, x_2$ be the coordinates of $x$, and $a, b$ homogeneous coordinates of $\mathbb{CP}^1$. Then $x \in L$ is equivalent to $ax_2 = bx_1$. Thus $B = \{(x_1, x_2) \times (a : b) \in \mathbb{C}^2 \times \mathbb{CP}^1 |\ ax_2 = bx_1\}$. The projection $\pi$ restricted to $\pi^{-1}(\mathbb{C}^2 \backslash \{0\})$ is an algebraic ismorphism.

Remark: We can define the blowup at an arbitrary point by using a translation.

Now that we understand the blowup of the plane we can turn to the blowup of plane curves.

**Definition 1.1.10.** *Let $C$ be a plane curve. The blowup of $C$ at a point $P$ is the Zariski-closure of the preimage $\pi^{-1}(C \backslash \{P\})$ in $B$, where $B$ is the prevously defined blowup of the plane at $P$.*

Intuitively, the preimage $\pi^{-1}(C \backslash \{P\})$ lacks finitely many points in the fiber of the origin which will be in the Zariski-closure.

**Example 1.1.11.** *We calculate the blowup of the curve given by the equation $xy = 0$. This curve is the union of the two coordinate axes in the plane.*

The blowup of this curve at the origin is given by the following system of equations:

$$xy = 0$$
$$ay = bx.$$

We get a better picture by looking at the charts of the projective line. On the first chart $b = 1$ and we get the equations:

$$xy = 0$$
$$ay = x.$$

We are interested in the preimage of the complement of the origin. Therefore $a$ must be 0. Consequently, $x = 0$ and $y$ can be arbitrary nonzero complex number. The Zariski-closure will be the whole line. We get a line at the other chart similarly. The previously intersecting lines are now disjoint hence smooth.

Perhaps somewhat surprisingly blowing up works for more complicated singularities as well.

**Example 1.1.12.** *The blowup of the cusp $x^3 = y^2$ at the origin is smooth.*

The equations for the blowup are

$$x^3 = y^2$$
$$ay = bx.$$

On the chart $b = 1$, we obtain $(ay)^3 = y^2$. Equivalently, $y^2(a^3y - 1) = 0$. Thus the blowup on this chart is the smooth curve given by the equation $(a^3y - 1) = 0$.

On the second chart $a = 1$, we have $x^3 = (bx)^2$. By rearranging, we get $x^2(x - b^2) = 0$. Therefore the exceptional divisor will be the point $x = y = b = 0, a = 1$ and the blowup on this chart is smooth as well, given by $(x - b^2)$.

**Theorem 1.1.13.** *For every curve $C$, there exists a finite sequence of blowups such that the resulting curve $\tilde{C}$ is smooth.*

This smooth $\tilde{C}$ is a resolution of $C$ with the map given by the blowups. The main difficulty of the proof is to measure the "complexity" of the singularity. For details see, Kollár's book on resolution of singularities [5]. Also Fulton's book [3] is a wonderful introduction to algebraic curves.

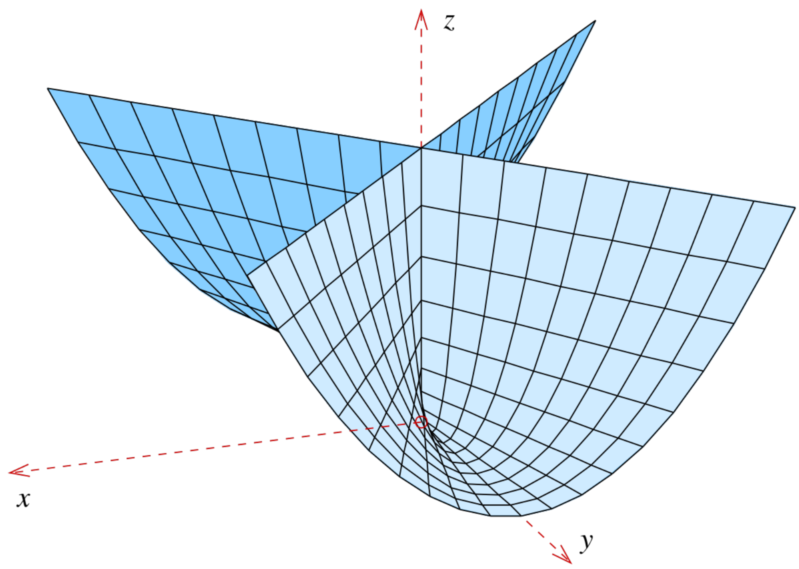
## 1.2 Surface singularities

With a thorough understanding of plane curves and their singularities, the general focus is shifted to the study of higher dimensional varieties, most notably to surfaces. Even though the classification of smooth surfaces was achieved by Enriques and the Italian school, and later a modern classification was provided by Kodaira, singular surfaces still remain a flourishing part of algebraic geometry with many open questions.

**Definition 1.2.1.** *An affine algebraic (hyper)surface $S$ in $\mathbb{C}^3$ is the zero set of a complex polynomial on $\mathbb{C}^3$.*

**Definition 1.2.2.** *A point $p \in S$ is singular if the differential of the defining polynomial of $S$ vanishes at $p$.*

A new phenomenon appearing for surfaces is that the singular locus of an irreducible surface can be one dimensional. For example, the Whitney umbrella given by $x^2 = zy^2$ is singular among the $z$ axis.




The figure is from [4].
To avoid this complication it is natural to work with normal surfaces, where the singular locus has codimension 2, therefore the singularities are isolated. In this case, we can examine the singularity with a (slightly changed) blow up at a point.

**Definition 1.2.3.** *The blowup of $\mathbb{C}^3$ at the origin is the closed subset $B$ of $\mathbb{C}^3 \times \mathbb{CP}^2$ defined by the equations $\{x_i y_j = x_j y_i |\ 1 \leq i, j \leq 3\}$ where $x_i$ are the affine coordinates of $\mathbb{C}^3$ and the $y_i$ are homogeneous coordinates of $\mathbb{CP}^2$.*

Similarly to the curve case, there exists a natural projection $\pi$ to $\mathbb{C}^3$.

**Definition 1.2.4.** *The blowup of a surface $S$ at a point $p$ is the Zariski closure of the preimage $\pi^{-1}(S\backslash\{P\})$ in $B$. The exceptional divisor is the preimage of $P$ under $\pi$.*

**Example 1.2.5** (The Double Cone)**.** *The Double Cone is given by the equation $x^2 + y^2 - z^2 = 0$.*

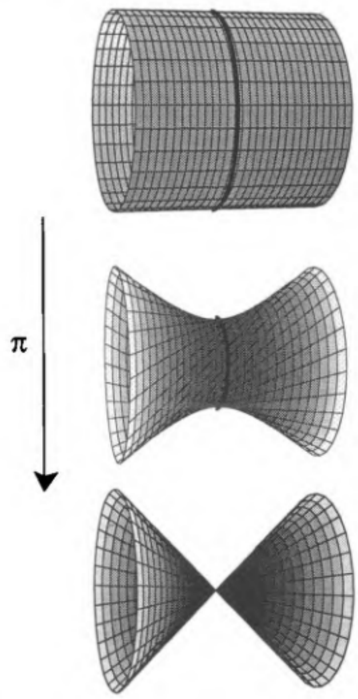
If we blow up at the origin, (calling the homogeneous coordinates $a$, $b$, $c$), we arrive at the system of equations:

$$\begin{aligned} x^2 + y^2 - z^2 &= 0 \\ xb &= ya \\ yc &= zb \\ za &= xc. \end{aligned}$$

To understand the exceptional divisor we consider the affine chart $c = 1$ of $\mathbb{CP}^2$. Thus we get the new equations,

$$\begin{aligned} x^2 + y^2 - z^2 &= 0 \\ xb &= ya \\ y &= zb \\ za &= x. \end{aligned}$$

We then obtain $z^2(a^2+b^2-1) = 0$. The exceptional divisor will be the intersection of the algebraic sets given by $z = 0$ and $a^2+b^2-1 = 0$. As the picture below suggests, the real solutions form a circle and the complex solutions form a complex projective line. The following picture is from [17].




### 1.2.1 The resolution graph

Generally, it is not enough to blow up points to get a resolution even for normal surfaces. Zariski proved that for every surface singularity a resolution can be obtained by a sequence of blowing up points and normalizations.

Let $S$ be a normal surface and $\pi : \tilde{S} \to S$ be a resolution. Then the exceptional divisor $E$ will be 1-dimensional, therefore it is a union of curves $E_1, E_2, ..., E_n$. (We can choose $\pi$ so that the curves intersect transversally.)

**Definition 1.2.6.** *The resolution graph is constructed as follows. Let the vertices be the irreducible components $E_i$ of the exceptional divisor. If two irreducible components $E_i$ and $E_j$ intersect in $k$ points, then we draw $k$ edges between $E_i$ and $E_j$.*

Mumford in [8] showed that the link $L$ can be obtained from the resolution graph by the plumbing construction. Thus the resolution graph encodes the abstract topological type of the singularity as, by Milnor [7], $S$ is locally homeomorphic a cone on $L$. Therefore the resolution graph is of primary interest as it can be used to calculate the fundamental group of the link and also construct invariants of the singularity. For more on surfaces, see [2] and [9].

## 1.3 Resolution in higher dimension

In this chapter we mainly follow the book An Invitation to Algebraic Geometry by K.E. Smith, L. Kahanpaeae, P.Kekaelaeinen, and W.N. Traves [17].

### 1.3.1 Affine algebraic sets

**Definition 1.3.1.** *An affine algebraic set $X$ in $\mathbb{C}^n$ is the common zero set of a collection of polynomials on $\mathbb{C}^n$. That is, if $\mathcal{F} = \{F_i\}_{i \in I}$, then $X = \{x \in \mathbb{C}^n | F_i(x) = 0, \forall i \in I\}$. We will use the notation $V(\mathcal{F})$ for $X$.*

In the abstract theory it is useful to allow several polynomials because affine algebraic sets will be closed under intersection.

**Lemma 1.3.2.** *The intersection of affine algebraic sets is an affine algebraic set.*

*Proof.* Let $X_\alpha = V(\mathcal{F}_\alpha), \alpha \in \mathcal{A}$ be affine algebraic sets. Then

$$\bigcap_{\alpha \in \mathcal{A}} V(\mathcal{F}_\alpha) = V(\cup_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha).$$

□

**Lemma 1.3.3.** *The union of finitely many affine algebraic sets is an affine algebraic set as well.*

*Proof.* It is enough to prove the statement for the union of two affine algebraic sets. We claim that

$$V(\mathcal{F}) \cup V(\mathcal{G}) = V(\mathcal{F}\mathcal{G}).$$

It's easy to check that every point in the union is contained in the RHS. The main task is to prove that the RHS is contained in the LHS. For a proof by contradiction let $x \in V(\mathcal{F}\mathcal{G})$ but $x \notin V(\mathcal{F})$ and $x \notin V(\mathcal{G})$. Since $x \notin V(\mathcal{F})$ there exists an $f \in \mathcal{F}$ such that $f(x) \neq 0$. Similarly, there is a $g \in \mathcal{G}$ with $g(x) \neq 0$. But then $f \cdot g(x) \neq 0$ and $f \cdot g \in V(\mathcal{F}\mathcal{G})$ which in a contradiction as $x \in V(\mathcal{F}\mathcal{G})$. □

The whole space $\mathbb{C}^n = V(0)$ and $\emptyset = V(1)$ are affine algebraic sets. Thus we can define a topology with the affine algebraic sets as closed sets.

**Definition 1.3.4.** *The topology on $\mathbb{C}^n$ where the closed sets are exactly the affine algebraic sets in $\mathbb{C}^n$ is called the Zariski topology on $\mathbb{C}^n$.*

This is indeed a topology by the previous two lemmas. Affine algebraic sets are closed in the usual Euclidean topology as well, but the converse is false. The Zariski topology is much coarser than the Euclidean topology.

Now we define the maps between affine algebraic sets, called morphisms.

**Definition 1.3.5.** *A morphism between affine algebraic sets $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ is the restriction of a polynomial map between the ambient spaces $\mathbb{C}^n$ and $\mathbb{C}^m$, where by polynomial map we mean that every component is a polynomial.*

**Definition 1.3.6.** *A morphism $\varphi$ of affine algebraic sets $X \xrightarrow{\varphi} Y$ is an isomorphism if there exist an inverse morhism, that is, $\varphi$ is bijective and $\varphi^{-1}$ is a morphism as well. Two affine algebraic sets are said to be isomorphic if there exists an isomorphism between them.*

### 1.3.2 Projective algebraic sets

Algebraic geometers prefer to work in projective space rather than affine space for several reasons. As the projective space is a compactification of affine space, algebraic sets will be compact and also the intersection theory is much nicer.

**Definition 1.3.7.** *The $n$-dimensional complex projective space $\mathbb{CP}^n$ is the parameter space of lines in $\mathbb{C}^{n+1}$ going through the origin.*

Alternatively, $\mathbb{CP}^n$ can be defined by the so-called homogeneous coordinates. Take $\mathbb{C}^{n+1} \backslash \{0\}$ and factor out with the $\mathbb{C}^*$ action of multiplication by a nonzero constant. We denote the equivalence class of $(x_0, x_1, ..., x_n)$ by $(x_0 : x_1 : ... : x_n)$.

What are the projective algebraic sets? Polynomials in $n+1$ variables are not functions on $\mathbb{CP}^n$ as they are (usually) not constant on an equivalence class.

**Definition 1.3.8.** *A polynomial is called homogeneous if all the nonzero terms have the same degree.*

Equivalently, $P(\lambda x_0, \lambda x_1, ..., \lambda x_n) = \lambda^{\deg P} \cdot P(x_0, x_1, ..., x_n)$ for every $\lambda \in \mathbb{C}$. In particular, $P(x_0, x_1, ..., x_n) = 0 \Rightarrow P(\lambda x_0, \lambda x_1, ..., \lambda x_n) = 0$ therefore the zero set of homogeneous polynomials is well-defined.

**Definition 1.3.9.** *A projective algebraic set in $\mathbb{CP}^n$ is the common zero set of a collection of homogeneous polynomials in $n+1$ variables.*

We can define the Zariski topology for the projectice space similarly to the affine case. The definition of morphisms becomes a bit more technical.

**Definition 1.3.10.** *Let $X \subseteq \mathbb{CP}^n$ and $Y \subseteq \mathbb{CP}^m$ be projective algebraic sets. We call a map $\varphi : X \to Y$ a morphism of projective algebraic sets if the following holds: For every $p \in X$ there exists a nonempty Zariski-open neighborhood $U$ of $p$, and homogeneous polynomials $F_0, F_1, ..., F_m \in \mathbb{C}[x_0, x_1, ..., x_n]$ such that $\varphi|_U : U \to W$ agrees with the polynomial map*

$$U \longrightarrow W$$

$$u \longmapsto (F_0(u) : F_1(u) : ... : F_m(u)).$$

Notice that every component of the polynomial map must have the same degree and they can't simultaneously vanish in $U$. Moreover, for different points in $V$ we might have to choose different neighborhoods and polynomials.

### 1.3.3 Quasi-projective algebraic sets

We have developed affine and projective algebraic sets seperately. These are special cases of the more general notion of a quasi-projective algebraic set.

**Definition 1.3.11.** *A quasi-projective algebraic set $X$ is a locally Zariski-closed subset of $\mathbb{CP}^n$ i.e. $X$ is the intersection of a Zariski-open and a Zariski-closed set.*

Every affine algebraic set $X = V(\mathcal{F})$ is quasi-projective as the projective algebraic set corresponding to the homogenization of $\mathcal{F}$ intersected with the correct chart is $X$ itself. Projective algebraic sets are quasi-projective as well since the ambient projective space is Zariski-open.

The morphisms of quasi-projective algebraic sets are defined the same way as morphisms of projective algebraic sets.

We will also need the product of two quasi-projective algebraic sets. The product of affine algebraic sets can be easily understood but for the quasi-projective case we need a new tool, the Segre embedding.

**Definition 1.3.12.** *The Segre embedding is the following map from the product of projective spaces into a higher dimensional projective space:*

$$\mathbb{CP}^n \times \mathbb{CP}^m \xrightarrow{\Sigma_{n,m}} \mathbb{CP}^{(n+1)(m+1)-1}$$

$$((x_0 : x_1 : ... : x_n), (y_0 : y_1 : ... : y_m)) \longmapsto (x_0 y_0 : x_0 y_1 : ... : x_n y_m).$$

**Theorem 1.3.13.** *The image of the Segre embedding is a projective variety. Moreover, if $X \subseteq \mathbb{CP}^n$ and $Y \subseteq \mathbb{CP}^m$ are quasi-projective algebraic sets, then the image of $\Sigma_{n,m}|_{X \times Y}$ is a quasi-projective algebraic set.*

### 1.3.4 Hironaka's theorem

**Definition 1.3.14.** *A morphism of quasi-projective algebraic sets $X \xrightarrow{\pi} Y$ is a projective morphism if $X$ is a closed subset of the product variety*

$$X \subseteq Y \times \mathbb{CP}^n$$

*and $X \xrightarrow{\pi} Y$ is the restriction of the projection onto $Y$.*

**Definition 1.3.15.** *Let $\pi$ be a morphism of quasi-projective algebraic sets $X \xrightarrow{\pi} Y$. If there exists a dense open subset $U \subseteq X$ such that $\pi|_U$ is an isomorphism onto some dense open subset in $Y$, then $\pi$ is called birational.*

**Theorem 1.3.16** (Hironaka)**.** *For every quasi-projective algebraic set $Y$, there exists a smooth quasi-projective algebraic set $\tilde{Y}$ with a projective birational morphism $\tilde{Y} \xrightarrow{\pi} Y$.*

On the proof:

We have seen that blowing up points is not enough even for surfaces. The idea is to blow up higher dimensional algebraic subsets of the singular locus as well. Hironaka managed to show that a sequence of carefully chosen blowups will eventually terminate thus giving a resolution. Even though Hironaka's proof can be considered algorithmic, it is hard to use it for practical calculations.

# Chapter 2

# Simple surface singularities

In this chapter, we take a closer look at an important family of singular surfaces.

**Definition 2.0.1.** *Let $G$ be a finite subgroup of $SU(2)$. A simple surface singularity is a singularity of the orbit space $\mathbb{C}^2/G$.*

To make sense of this definition we have to construct these orbit spaces as algebraic surfaces, but as a first step we classify the finite subgroups of $SU(2)$.

## 2.1 The finite subgroups of $SU(2)$

In this section we closely follow Lindh [6]. Using the quaternions, it can be shown that there exists a $\phi$ homomorphism from $SU(2)$ to $SO(3)$ with kernel $\{\pm I\}$. Thus it is crucial to understand the finite subgroups of $SO(3)$.

It is a well-known fact that every element of $SO(3)$ is a rotation by some angle around an axis. Therefore we can represent every element of a finite subgroup $G$ by a pair of antipodal points where the corresponding axis intersects the unit sphere. We can get a lot of information by the action of $G$ on these antipodal pairs.

**Lemma 2.1.1** (Burnside)**.** *Let $G$ be a finite group acting on a finite set $X$. We denote by $X^g$ the elements fixed by $g$. Then the number $N$ of orbits is*

$$N = \frac{1}{|G|} \sum_{g \in G} |X^g| \tag{2.1.1}$$

Using this formula we get the number of orbits instantly. Every element of $G$ except the identity fixes exactly two points. Therefore,

$$N = \frac{1}{|G|}(2(|G| - 1) + |X|) \tag{2.1.2}$$

By rearranging the terms, we get

$$2(1 - \frac{1}{|G|}) = N - \frac{|X|}{|G|} \tag{2.1.3}$$

The orbits partition $X$ thus $|X| = \sum_{i=1}^{N} G(x_i)$, where each $x_i$ is in a different orbit. We can rewrite the previous equation as

$$2(1 - \frac{1}{|G|}) = N - \frac{|X|}{|G|} = N - \frac{\sum_{i=1}^{N} G(x_i)}{|G|} = \sum_{i=1}^{N}(1 - \frac{G(x_i)}{|G|}). \tag{2.1.4}$$

Moreover, by the Orbit-Stabilizer theorem

$$2(1 - \frac{1}{|G|}) = \sum_{i=1}^{N}(1 - \frac{G(x_i)}{|G|}) = \sum_{i=1}^{N}(1 - \frac{1}{|G_{x_i}|}). \tag{2.1.5}$$

If $|G| \geq 2$, then $1 \leq 2(1 - \frac{1}{|G|}) < 2$. On the other hand, the cardinality of the stabilizer of $x_i$ is at least 2 as the identity and the rotation through $x_i$ fixes $x_i$. Thus every term on the RHS is at least $\frac{1}{2}$. Consequently, $N \leq 3$, the number or orbits is at most 3. Also as the LHS is at least than 1 and every term of the RHS is smaller than 1, $N$ must be at least 2.

If $N = 2$, then

$$2(1 - \frac{1}{|G|}) = (1 - \frac{G(x_1)}{|G|}) + (1 - \frac{G(x_2)}{|G|}). \tag{2.1.6}$$

By rearranging the terms, we obtain

$$\frac{2}{|G|} = \frac{G(x_1)}{|G|} + \frac{G(x_2)}{|G|}. \tag{2.1.7}$$

Therefore,

$$2 = G(x_1) + G(x_2) = |X|. \tag{2.1.8}$$

Thus there are only two antipodal points. Every element of $G$ acts as a rotation around the same axis. Then $G$ is a cyclic group.

For the case $N = 3$, we have the equation:

$$2(1 - \frac{1}{|G|}) = (1 - \frac{1}{|G_{x_1}|}) + (1 - \frac{1}{|G_{x_2}|}) + (1 - \frac{1}{|G_{x_3}|}). \tag{2.1.9}$$

Equivalently,

$$\frac{1}{|G_{x_1}|} + \frac{1}{|G_{x_2}|} + \frac{1}{|G_{x_3}|} = 1 + \frac{2}{|G|}. \tag{2.1.10}$$

Since the LHS is greater than 1, the equation has only finitely many solutions, see the table below. These groups arise as the symmetry groups of polyhedrons.

| Groups | As symmetries | \|Stabilizers\| | $\|G\|$ |
|---|---|---|---|
| $D_n$ | regular $n$-agon | 2, 2, $n$ | $2n$ |
| $\mathbb{T}$ | Tetrahedron | 2, 3, 3 | 12 |
| $\mathbb{O}$ | Octahedron | 2, 3, 4 | 24 |
| $\mathbb{D}$ | Dodecahedron | 2, 3, 5 | 60 |

Now we can classify the finite subgroups of $SU(2)$.

**Theorem 2.1.2.** *Every finite subgroup of $SU(2)$ is a preimage of a finite subgroup of $SO(3)$ by $\phi$ or a cyclic group of odd order.*

*Proof.* If $G$ is an even subgroup of $SU(2)$ then by Cauchy's theorem it has an element of order 2. The only order 2 element of $SU(2)$ is $-I$. Since the kernel of $\phi$ is $\{\pm I\}$, $G$ is a preimage indeed.

Secondly, if $G$ is and odd subgroup, then $\phi$ is an isomorphism restricted to $G$ and $\phi(G)$ is an odd subgroup of $SO(3)$. Using the classification, $\phi(G)$ can only be a cyclic group. $\square$

The preimages of $D_n$, $\mathbb{T}$, $\mathbb{O}$, and $\mathbb{I}$ are denoted as $\mathbb{B}D_n$, $\mathbb{BT}$, $\mathbb{BO}$, and $\mathbb{BD}$.

## 2.2 Orbit spaces as affine algebraic sets

In this section we realize the orbit spaces $\mathbb{C}^2/G$ as algebraic sets. The fundamental idea is to understand algebraic sets by the (polynomial) functions on them.

### 2.2.1 Radical ideals

Recall that an affine algebraic set $X$ is the common zero set of a collection of polynomials $\mathcal{F}$ i.e. $X = V(\mathcal{F})$. We can assume that $\mathcal{F}$ is an ideal as the common zero set of a collection of polynomials is the same as the common zero set of the ideal generated by them. However, it is important to realize there is still ambiguity in the definition as an algebraic set can correspond to several ideals.

An ideal gives rise to an algebraic set by taking the common zero set. Conversely, if $X$ is an algebraic set, then the set of polynomial functions vanishing on $X$, denoted by $I(X)$, is an ideal. Notice that ideals arising as $I(X)$ have the following property.

**Definition 2.2.1.** *An ideal $I$ is radical if $f^n \in I$ implies $f \in I$ for every positive integer $n$.*

For a polynomial $f$, if $f^n$ vanishes at a point, then also $f$ vanishes there. Therefore $I(X)$ is a radical ideal, indeed. If we restrict ourselves to radical ideals, we get a bijective correspondence.

**Theorem 2.2.2.** *The previously defined functions $V$ and $I$ are bijections between affine algebraic sets and radical ideals. Moreover, $I$ is the inverse of $V$.*

This is a direct consequence of Hilbert's Nullstellensatz, which can be found in every standard book on algebraic geometry, for example, in Perrin [10].

### 2.2.2 The coordinate ring of an algebraic set

The ring of (polynomial) functions on an algebraic set $X$ is called the coordinate ring of $X$.

**Definition 2.2.3.** *Let $X$ be an algebraic set. Then the coordinate ring of $X$ is*

$$A(X) = \mathbb{C}[x_1, x_2, ..., x_n]/I(X)$$

*where $I(X)$ is the ideal of polynomials vanishing on $X$.*

The name comes from the fact that $A(X)$ is isomorphic to the ring of polynomial functions on the ambient space $\mathbb{C}^n$ restricted to $X$.

**Lemma 2.2.4.** *The coordinate ring*

$$A(X) = \mathbb{C}[x_1, x_2, ..., x_n]/I(X)$$

*is a finitely generated algebra over* $\mathbb{C}$ *and has no nilpotent elements.*

*Proof.* $A(X)$ is generated by the restrictions of the coordinate functions $x_1, ..., x_n$ thus it is finitely generated. Let $f$ be a nilpotent element of $A(X)$, that is, $f^n = 0$ for some positive integer $n$. Equivalently, $f^n \in I(X)$. Since $I(X)$ is radical, $f \in I(X)$. So $f = 0$ in $A(X)$. $\square$

The converse is also true.

**Theorem 2.2.5.** *If $A$ is a finitely generated* $\mathbb{C}$*-algebra that has no nilpotent elements (i.e. it is reduced), then $A$ is a coordinate ring of an affine algebraic set.*

*Proof.* Since $A$ is finitely generated, there is a surjective homomorphism

$$\varphi : \mathbb{C}[y_1, y_2, ..., y_k] \twoheadrightarrow A$$

taking $y_i$ to the $i$th generator of $A$. By the first isomorphism theorem,

$$A \cong \mathbb{C}[y_1, y_2, ..., y_k]/\ker\varphi.$$

We have to check that $\ker\varphi$ is a radical ideal. If $f^n \in \ker\varphi$ i.e $\varphi(f^n) = 0$, then since $\varphi$ is a homomorphism, $\varphi(f)^n = 0$. But the only nilpotent element in $A$ is 0, so $\varphi(f) = 0$. Therefore $\ker\varphi$ is nilpotent indeed. Then by Theorem 2.2.2., $\ker\varphi = I(X)$ for some algebraic set $X$. $\square$

Thus a realization of an object as an algebraic set can be done by showing that the ring of functions on that object is a reduced finitely generated $\mathbb{C}$-algebra.

One could argue that there is still ambiguity about the constructed algebraic set $X$ as we get different algebraic sets from different surjective homomorphisms onto $A$. However, the constructed algebraic sets have isomorphic coordinate rings. It can be shown that the coordinate rings of two algebraic sets are isomorphic if and only if they are isomorphic as algebraic sets. The surjective homomorphisms onto $A$ correspond to different embeddings of an “abstract” algebraic set into affine space. A more modern approach would be to take $X = \text{Spec } A$.

### 2.2.3 Invariant polynomials

**Definition 2.2.6.** *A group action on a vector space $V$ is a map*

$$\eta : G \times V \to V$$

*such that $\eta(e, v) = v$ and $\eta(g, \eta(h, v)) = \eta(gh, v)$ holds for $\forall g \in G, v \in V$. We also make the assumption that the action is linear, that is, $\eta(g, \cdot)$ is a linear map from $V$ to $V$.*

We will use the notation $g \cdot v$ for $\eta(g, v)$.

The main question is what the polynomial functions on the orbit space $V/G$ are.

The action of $G$ on $V$ induces an action on the polynomials on $V$ by

$$g \cdot f(x_1, x_2, ..., x_n) = f(g^{-1} \cdot (x_1, x_2, ..., x_n)).$$

**Definition 2.2.7.** *The invariant polynomials of a group action by $G$ on an $n$-dimensional complex vector space $V$, denoted by $\mathbb{C}[x_1, x_2, ..., x_n]^G$, are the polynomials on $V$ that are invariant under the action of the entire group $G$, that is,*

$$g \cdot f = f \tag{2.2.1}$$

*for every $g \in G$.*

These are indeed functions on $V/G$ as an invariant polynomial is constant on an orbit.

**Example 2.2.8** (Symmetric polynomials)**.** *If $G$ is the symmetric group acting on $V$ by permuting the basis vectors, then the invariant polynomials are exactly the symmetric polinomials.*

The invariant polynomials form a subring of the polynomials on $V$ hence the ring of invariant polynomials is reduced. We now have to show that it is finitely generated.

Before proving this we define the Reynolds operator which will play a crucial role.

**Definition 2.2.9.** *The Reynolds operator $\rho : \mathbb{C}[x_1, x_2, ..., x_k] \to \mathbb{C}[x_1, x_2, ..., x_k]^G$ is defined as*

$$\rho(f) := \frac{1}{|G|} \sum_{g \in G} g \cdot f. \tag{2.2.2}$$

The essential property of the Reynolds operator that it fixes the invariant polynomials. Moreover, it preserves the degree.

**Theorem 2.2.10** (Hilbert's finiteness theorem)**.** *Let $G$ be a finite group acting on a finite dimensional vector space. Then the ring of invariant polynomials is finitely generated.*

*Proof.* We follow the proof from Lindh [6]. As $G$ acts linearly, a $p$ polynomial is invariant if and only if the homogeneous parts of $p$ are invariant. Therefore it is natural to work primarily with homogeneous polynomials.

Let $I$ be the ideal genarated by the nonconstant homogeneous invariant polynomials. By Hilbert's basis theorem the polynomial ring over the complex numbers is Noetherian, thus every ideal is finitely generated. Let $f_1, f_2, ..., f_k$ be the generators of $I$.

We show that the invariant polynomials are exactly the elements of the subalgebra $\mathbb{C}[f_1, f_2, ..., f_k]$. The elements of $\mathbb{C}[f_1, f_2, ..., f_k]$ are invariant polynomials indeed since $\forall i, f_i$ is invariant and the sums and products of invariant polynomials are invariant as well.

To prove that every invariant polynomial is in $\mathbb{C}[f_1, f_2, ..., f_k]$, we proceed by induction on the degree. If $f$ is a constant polynomial, then it is in $\mathbb{C}[f_1, f_2, ..., f_k]$ trivially. Now let $f$ be a homogeneous invariant polynomial of degree $d$. By definition $f \in I$, thus $f$ is expressible as

$$f = \sum_{i=1}^{k} h_i f_i. \tag{2.2.3}$$

Applying the Reynolds operator, we obtain

$$f = \rho(f) = \rho(\sum_{i=1}^{k} h_i f_i) = \sum_{i=1}^{k} \rho(h_i f_i) = \sum_{i=1}^{k} \rho(h_i) f_i. \tag{2.2.4}$$

Hence $\rho(h_i)$ is invariant and $\deg(h_i) < \deg(f)$. By induction, $\rho(h_i) \in \mathbb{C}[f_1, f_2, ..., f_k]$. Consequently, $f \in \mathbb{C}[f_1, f_2, ..., f_k]$. As the homogeneous invariant polynomials generate the invariant ring, the proof is finished. $\square$

## 2.3 Generators and relations

For finite subgroups of $SU(2)$, the ring of invariant polynomials and the corresponding algebraic sets were computed explicitly by Klein, and more recently by Dolgachev.

We shall cover the case of cyclic groups without going too much into the details.

The cyclic group of order $n$ as a subgroup of $SU(2)$ is given by

$$k \mapsto \begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}$$

where $\varepsilon$ is the first $n$th root of unity. It acts on $\mathbb{C}^2$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon^k x \\ \varepsilon^{-k} y \end{pmatrix}.$$

Notice that the action on a monomial is just multiplication by a scalar. In particular, if $f$ is invariant, then every monomial of $f$ is invariant. It is easy to check that $x^n, y^n$, and $xy$ are invariants and we show that every invariant is generated by these. Let $x^i y^j$ be an invariant monomial. Then

$$\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix}$$

acts on $x^i y^j$ as $(\varepsilon x)^i (\varepsilon^{-1} y)^j$. Since $x^i y^j$ is invariant $\varepsilon^i \varepsilon^{-j}$ must be 1 i.e. $i \equiv j$ modulo $n$. Thus $x^i y^j$ is in the subring generated by $x^n, y^n$ and $xy$.

We have a surjective homomorphism

$$\varphi : \mathbb{C}[r, s, t] \twoheadrightarrow \mathbb{C}[x^n, y^n, xy]$$

$$r \mapsto x^n$$

$$s \mapsto y^n$$

$$t \mapsto xy.$$

To find the corresponding algebraic set, we have to understand $\ker \varphi$ i.e. we need relations between $x^n, y^n$ and $xy$. Certainly $x^n y^n - (xy)^n$ is a relation. Therefore $rs - t^n$ must be in $\ker \varphi$. We accept it without proof that $rs - t^n$ generates $\ker \varphi$ and the corresponding algebraic set is $V(rs - t^n)$.

The generators of the invariant rings and relations between them can be computed by using so-called Grundformen, for details see Dolgachev [1]. The following table is from Lindh [6].

| Subgroup of $SL_2(\mathbb{C})$ | Generating set | Relation between generators |
|---|---|---|
| $C_n$ | $x = \Psi_1^n$<br>$y = \Psi_2^n$<br>$z = \Psi_1\Psi_2$ | $xy + z^n = 0$ |
| $\mathbb{BD}_n$ | $x = \sqrt[n]{4}\Psi_3^2$<br>$y = \frac{1}{2\sqrt[n]{4}}\Psi_1\Psi_2$<br>$z = \Psi_3\Psi_2^2$ | $z^2 + x(y^2 + x^n) = 0$ |
| $\mathbb{BT}$ | $x = \Psi_1$<br>$y = \sqrt[3]{4}\Psi_2\Psi_3$<br>$z = i(\Psi_2^3 + \Psi_3^3)$ | $z^2 + x^4 + y^3 = 0$ |
| $\mathbb{BO}$ | $x = \sqrt[3]{108}\Psi_1^2$<br>$y = -\frac{1}{\sqrt[9]{108}}\Psi_2$<br>$z = \Psi_1\Psi_3$ | $z^2 + x(y^3 + x^2) = 0$ |
| $\mathbb{BD}$ | $x = \Psi_1$<br>$y = \Psi_2$<br>$z = \sqrt[5]{-1728}\Psi_3$ | $x^2 + y^3 + z^5 = 0$ |

## 2.4 Resolution of simple surface singularities

The resolution of simple surfaces now can be computated by blowups.

First, we consider the surface corresponding to the cyclic group of order 2 given by the equation $xy - z^2 = 0$. Blowing up at the origin, we obtain

$$xy - z^2 = 0$$
$$xb = ya$$
$$yc = zb$$
$$za = xc.$$

On the chart $a = 1$, we get

$$xy - z^2 = 0$$
$$xb = y$$
$$yc = zb$$
$$z = xc.$$

Thus $x(xb) - (xc)^2 = 0$. Equivalently, $x^2(b - c^2) = 0$. Thus the blowup is given by the equation $(b - c^2) = 0$ hence it is smooth on this chart. The exceptional divisor is the intersection of $x = 0$ and $(b - c^2) = 0$, so it is a complex projective line after compactification. As the equation is symmetric in the variables $x$ and $y$, the blowup is also smooth on the chart $b = 1$.

On the remaining chart $c = 1$, we have the equations

$$xy - z^2 = 0$$
$$xb = ya$$
$$y = zb$$
$$za = x.$$

Thus $z^2(ab - 1) = 0$. So the blowup is smooth on this chart as well and we get the same complex projective line after compactification. Therefore the resolution graph is just one vertex without any edges. Note that this is surface is isomorphic to the Double Cone.

Now we calculate the resolution graph of the simple surface singularity corresponding to the cyclic group of order 3. After blowing up at the origin, we get

$$xy - z^3 = 0$$
$$xb = ya$$
$$yc = zb$$
$$za = xc.$$

Similarly to the case of cyclic group of order 2, the blowup will be smooth on the charts $a = 1$ and $b = 1$ since there will be a linear term in their equations.

On the chart $c = 1$, we have the equations

$$xy - z^3 = 0$$
$$xb = ya$$
$$y = zb$$
$$za = x.$$

Thus we obtain $z^2(ab - z) = 0$. The equation of the blowup is $ab - z = 0$ which is a smooth surface. Furthermore, the exceptional divisor is the intersection of $ab - z = 0$ and $z = 0$. So it has the equation $ab = 0$ i.e. it is the union of the two coordinate axes $a = 0$ and $b = 0$. Therefore the exceptional divisor has two components intersecting at the origin, hence the resolution graph is two vertices connected by an edge.

The resolution graph of the simple surface singularity corresponding to the cyclic group of order $n$ can be computed inductively. Similarly to the previous cases, the blowup will be smooth on the first two charts. While on the chart $c = 1$, we get $z^2(ab - z^{n-2}) = 0$. Thus the equation of the blowup will be $(ab - z^{n-2}) = 0$, the surface corresponding to the cyclic group of order $n - 2$. Moreover, the exceptional divisor will be the intersection of $(ab - z^{n-2}) = 0$ and $z = 0$, hence it is the union of the two coordinate axes $a = 0$ and $b = 0$. Since the surface is still singular on this chart we need to perform more blowups. It can be shown that the resolution graph will be a path graph on $n - 1$ vertices. The two coordinate axes $a = 0$ and $b = 0$ after the first blowup correspond to the two endpoints of the path graph. The two components produced by the second blowup will be the vertices adjacent to the endpoints and so on.

The resolution graphs of the remaining simple surface singularities can be found in the following table, which is from Miles Reid's notes [12].

| Name | Equation | Group | Resolution graph |
|---|---|---|---|
| $A_n$ | $x^2+y^2+z^{n+1}$ | cyclic $\mathbb{Z}/(n+1)$ | $\circ - \circ \cdots \circ$ |
| $D_n$ | $x^2+y^2z+z^{n-1}$ | binary dihedral $\mathrm{BD}_{4(n-2)}$ | $\circ - \circ - \circ \cdots \circ$<br>$\vert$<br>$\circ$ |
| $E_6$ | $x^2+y^3+z^4$ | binary tetrahedral | $\circ - \circ - \circ - \circ - \circ$<br>$\vert$<br>$\circ$ |
| $E_7$ | $x^2+y^3+yz^3$ | binary octahedral | $\circ - \circ - \circ - \circ - \circ - \circ$<br>$\vert$<br>$\circ$ |
| $E_8$ | $x^2+y^3+z^5$ | binary icosahedral | $\circ - \circ - \circ - \circ - \circ - \circ - \circ$<br>$\vert$<br>$\circ$ |

# Chapter 3

# The McKay graph

In this chapter, we reconstruct the resolution trees of simple surface singularities using ideas from representation theory. We mainly use the notes of Ed Segal [13] and the book of Serre [14] for the first three sections.

**Definition 3.0.1.** *A representation of a finite group $G$ on a finite dimensional complex vector space $V$ is a homomorphism $\rho : G \to GL(V)$ of $G$ to the group of automorphisms of $G$. Sometimes we call $V$ itself the representation of $G$.*

Note that a representation of a group is the same as a group action on $V$ with $\rho(g) = \eta(g, \cdot)$.

- The representation that maps every element of $G$ to the identity is called the trivial representation.

- One of the most important representations is the regular representation. Let $V$ be a vector space of dimension $|G|$. Pick a basis and label it with the elements of $G$. Then every $g \in G$ will act on the basis by left multiplication, that is,

  $$\rho_{reg}(h)b_g = b_{hg}.$$

  We defined $\rho_{reg}(h)$ on every basis element, thus we can linearly extend it to get a linear map from $V$ to $V$. It is easy to check that this linear map is invertible. For more details on regular representations, see Theorem 3.3.2.

## 3.1 Maschke's theorem

**Definition 3.1.1.** *A subrepresentation of a representation* $(V, \rho)$ *is a vector subspace* $W$ *of* $V$ *which is invariant under* $G$*. That is, for every* $g \in G$*,* $x \in W$ *implies* $\rho(g)x \in W$*. Equivalently,* $\rho(g)$ *restricted to* $W$ *is an isomorphism of* $W$*. Thus we get a* $\rho^W : G \to GL(W)$ *representation by restriction.*

**Definition 3.1.2.** *If* $(V, \rho)$ *and* $(W, \psi)$ *are representations, then we define the direct sum of these representations as*

$$\rho \oplus \psi : G \to GL(V \oplus W) \tag{3.1.1}$$

$$(\rho \oplus \psi)(g)(v, w) := (\rho(g)v, \psi(g)w). \tag{3.1.2}$$

**Theorem 3.1.3** (Maschke)**.** *Let* $\rho : G \to GL(V)$ *be a representation and let* $W$ *be a subrepresentation. Then there exists a (complementary) subrepresentation* $U$ *with* $U \oplus W = V$*.*

*Proof.* By definition, a complementary subrepresentation $U$ is a complementary subspace of $W$. As complementary subspaces are exactly the kernels of projections onto $W$, our task is to find the correct projection.

Let $W'$ an arbitrary complement of $W$, and let $p$ be the corresponding projection onto $W$. We claim that the following projection will be sufficient:

$$\pi := \frac{1}{|G|} \sum_{g \in G} \rho(g) \cdot p \cdot \rho(g)^{-1}. \tag{3.1.3}$$

We have to check that $\pi$ is a projection onto $W$. Firstly, $\pi$ maps $V$ onto $W$ since $p$ maps $V$ onto $W$ and $\rho(g)$ maps $W$ to $W$. Secondly, $\pi|_W$ is the identity as

$$(\rho(g) \cdot p \cdot \rho(g)^{-1})|_W = \rho(g)|_W \cdot p|_W \cdot \rho(g)^{-1}|_W = \rho(g)|_W \cdot I \cdot \rho(g)^{-1}|_W = I. \tag{3.1.4}$$

To finish the proof we show that the kernel of $\pi$ is invariant under $G$. The main observation is that $\pi$ commutes with $\rho(h)$ for every $h \in G$. Indeed

$$\rho(h) \cdot \pi \cdot \rho(h)^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho(h) \cdot \rho(g) \cdot p \cdot \rho(g)^{-1} \rho(h)^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho(hg) \cdot p \cdot \rho(hg)^{-1} = \pi. \tag{3.1.5}$$

Therefore if $x \in \ker \pi$ then $\rho(g)x \in \ker \pi$ as

$$\pi \rho(g)x = \rho(g)\pi x = \rho(g)0 = 0. \tag{3.1.6}$$

Thus $U = \ker \pi$ is a subrepresentation. $\square$

**Definition 3.1.4.** *A representation $V$ of $G$ is irreducible if is has no subrepresentations other than $V$ and $0$.*

**Example 3.1.5.** *Every 1 dimensional representation is irreducible.*

Analogously to the Fundamental Theorem of Number theory, every representation can be decomposed into irreducible representations.

**Theorem 3.1.6.** *Every representation is a direct sum of irreducible representations.*

*Proof.* Let $V$ be a representation of $G$. If $V$ is irreducible, then we are done. Otherwise, $V$ has a proper subrepresentation $W$ with complementary subrepresentation $U$ by Maschke's theorem. We continue this process for $W$ and $U$. As $V$ is finite dimensional, the process will terminate in finitely many steps, giving the desired decomposition of $V$ into irreducible representations. $\square$

## 3.2 Schur's lemma and abelian groups

**Definition 3.2.1.** *Let $\rho : G \to GL(V)$ and $\psi : G \to GL(W)$ be representations. A morphism from $(V, \rho)$ to $(W, \psi)$, also called a $G$-linear map, is a linear map $T : V \to W$ with the additional property*

$$T \circ \rho(g) = \psi(g) \circ T \qquad \forall g \in G. \tag{3.2.1}$$

*Equivalently, the following diagram commutes for every $g \in G$*

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow{\psi(g)} & W. \end{array}$$

**Lemma 3.2.2.** *If $T$ is a morphism of representations from $(V, \rho)$ to $(W, \psi)$, then* $\ker T$ *is a subrepresentation of $V$ and* $\operatorname{Im} T$ *is a subrepresentation of $W$.*

*Proof.* We show that $\ker T$ is a subrepresentation. That is, if $v \in \ker T$ then $\rho(g)v \in \ker T$ for all $g \in G$. This is straightforward:

$$T\rho(g)v = \psi(g)Tv = \psi(g)0 = 0. \tag{3.2.2}$$

$\operatorname{Im} T$ is a subrepresentation as well. Since if $w \in \operatorname{Im} T$ i.e. $w = Tv$ for some $v \in V$, so

$$\psi(g)w = \psi(g)Tv = T\rho(g)v \in \operatorname{Im} T \qquad \forall g \in G. \tag{3.2.3}$$

□

**Theorem 3.2.3** (Schur's Lemma)**.** *Let $\rho : G \to GL(V)$ and $\psi : G \to GL(W)$ be irreducible representations of $G$.*

1. *Every morphism $T : V \to W$ is either an isomorphism or the zero map.*

2. *Let $T : V \to V$ be a morphism. Then $T = \lambda Id$ for some $\lambda \in \mathbb{C}$.*

*Proof.* 1. By the previous lemma $\ker T$ is a subrepresentation of $V$ and $\operatorname{Im} T$ is a subrepresentation of $W$. On the other hand, $V$ and $W$ are irreducible representations. Therefore $\ker T = V$ or $0$ and similarly $\operatorname{Im} T = W$ or $0$. If $\ker T = V$ or $\operatorname{Im} T = 0$ then $T$ is the zero map. Otherwise, $\ker T = 0$ i.e T is injective and $\operatorname{Im} T = W$ so $T$ is surjective. Thus $T$ is an isomorphism.

2. Since we are working over $\mathbb{C}$, every linear map has an eigenvalue. Let $\lambda$ be an eigenvalue of $T$. We claim that $T - \lambda Id$ is also a morphism from $V$ to $V$. Indeed

$$(T - \lambda Id)\rho(g)v = T\rho(g)v - \lambda Id\rho(g)v = \rho(g)Tv - \rho(g)(\lambda v) = \rho(g)(T - \lambda Id)v. \tag{3.2.4}$$

But $T - \lambda Id$ has a nontrivial kernel, therefore $T - \lambda Id$ is the zero map by part 1. Consequently, $T = \lambda Id$.

□

Schur's lemma almost completely solves the representation theory of abelian groups.

**Theorem 3.2.4.** *Every irreducible representation of an abelian group is* 1-*dimensional.*

*Proof.* Let $(V, \rho)$ be an irreducible representation of an abelian group $G$. Then $\rho(h)$ is a morphism from $V$ to $V$ for every $h \in G$ as the following diagram clearly commutes.

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ {\scriptstyle \rho(h)}\downarrow & & \downarrow{\scriptstyle \rho(h)} \\ W & \xrightarrow{\rho(g)} & W \end{array}$$

Thus, by the second part of Schur's lemma, every element of $G$ is mapped to a scalar multiple of the identity. Consequently, every subspace of $V$ is $G$-invariant i.e. a subrepresentation. Therefore $V$ is irreducible only if it has dimension 1. □

**Corollary 3.2.5.** *Let $\rho : G \to GL(V)$ be a representation of an abelian group $G$. Then the elements of $\rho(G)$ are simultaneously diagonalizable.*

*Proof.* We can decompose $V$ as the direct sum

$$V = U_1 \oplus U_2 \oplus ... \oplus U_n \tag{3.2.5}$$

of irreducible representations. By the previous theorem each $U_i$ has dimension 1. Picking a vector $u_i$ from each $U_i$, we get a desired basis where $\rho(g)$ is diagonal for all $g \in G$. □

**Example 3.2.6** (The representations of cyclic groups)**.** *Let $Z_n$ be the cyclic group of order $n$ and $g$ a generator.*

Since $Z_n$ is abelian, we first classify the $\rho : Z_n \to GL(\mathbb{C})$ 1-dimensional representations. As $g$ is a generator of $Z_n$, we have $g^n = 1$ and $\rho$ is a homomorphism, therefore $\rho(g)^n = 1$ also holds. On the other hand, if $\varepsilon$ is an arbitrary $n$th root of unity, then we obtain a 1-dimensional representation of $Z_n$ by setting $\rho(g^k) = \varepsilon^k$. Thus these are all the 1-dimensional representations of $Z_n$.

Let $\rho : Z_n \to GL(v)$ be an arbitrary representation of $Z_n$. Then it is a direct sum of irreducible representations thus $\rho(g)$ is a diagonal matrix with eigenvalues that are $n$th roots of unity and $\rho(g^k) = \rho(g)^k$.

## 3.3 More on decompositions into irreducible representations

In 3.1 we proved that every representation can be decomposed as a direct sum of irreducible representations. A good analogue for irreducible representations are the prime numbers. Therefore, in this chapter, we establish the equivalent of the fundamental theorem of arithmetic, the decomposition is unique up to permutations of the irreducible elements. However, at the end of the chapter we will see that the representation theory of finite groups is in some sense simpler than arithmetic as there are only finitely many irreducible representations.

**Theorem 3.3.1.** *Let $\rho : G \to GL(V)$ be a representation. The decomposition of $V$ into irreducible representations*

$$V = U_1 \oplus U_2 \oplus ... \oplus U_k \tag{3.3.1}$$

*is unique up to a permutation of the irreducible representations $U_1, U_2, ..., U_k$.*

*Proof.* Our main task is to determine the order of an irreducible representation $W$ in a decomposition. By Schur's lemma, it is natural to consider the morphisms from $V$ to $W$. Let us denote the morphisms from $V$ to $W$ by $\mathrm{Hom}_G(V, W)$. Then

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(\bigoplus_{i=1}^{k} U_i, W) = \bigoplus_{i=1}^{k} \mathrm{Hom}_G(U_i, W). \tag{3.3.2}$$

Consequently,

$$\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(\bigoplus_{i=1}^{k} U_i, W) = \sum_{i=1}^{k} \dim \mathrm{Hom}_G(U_i, W). \tag{3.3.3}$$

By Schur's lemma, $\dim \mathrm{Hom}_G(U_i, W) = 1$ if $U_i$ and $W$ are ismorphic. Otherwise, $\dim \mathrm{Hom}_G(U_i, W) = 0$. Therefore the order of $W$ in a decomposition is exactly $\dim \mathrm{Hom}_G(V, W)$ which depends only on $V$, not on the decomposition. □

The next theorem sheds light on why regular representations are so important.

**Theorem 3.3.2.** *Every irreducible representation $W$ of $G$ appears in the decomposition of the regular represetation of $G$. Moreover, the number of factors isomorphic to $W$ in the decomposition is exactly* $\dim W$.

*Proof.* Since the number of factors isomorphic to $W$ is $\dim \mathrm{Hom}_G(V_{reg}, W)$ we have to construct an isomorphism between $W$ and $\mathrm{Hom}_G(V_{reg}, W)$. Recall that $b_e \in V_{reg}$ is the basis element corresponding to the identity in $G$. Define a function

$$T : \mathrm{Hom}_G(V_{reg}, W) \to W \tag{3.3.4}$$

by the evaluation at $b_e$, that is

$$T(f) := f(b_e). \tag{3.3.5}$$

We claim that $T$ is an isomorphism. First we check that $T$ is linear. Indeed

$$T(f+g) = (f+g)(b_e) = f(b_e) + g(b_e) = T(f) + T(g) \tag{3.3.6}$$

and

$$T(\lambda f) = (\lambda f)(b_e) = \lambda f(b_e) = \lambda T(f). \tag{3.3.7}$$

$T$ is injective since if $T(f) = f(b_e) = 0$ then

$$f(b_g) = f(\rho_{reg}(g) b_e) = \rho_W(g) f(b_e) = \rho_W(g) 0 = 0. \tag{3.3.8}$$

Thus such an $f$ vanishes on every basis element of $V_{reg}$ therefore $f = 0$.

To finish the proof we have to show that $T$ is surjective i.e. for every $w \in W$ there exists a morphism $f$ such that $f(b_e) = w$. If $f \in \mathrm{Hom}_G(V_{reg}, W)$, then $f$ is completely determined by $f(b_e)$ since

$$f(b_g) = f(\rho_{reg}(g) b_e) = \rho_W(g) f(b_e). \tag{3.3.9}$$

Therefore the only possible morphism $f$ with $f(b_e) = w$ is the linear extension of $f(b_g) := \rho_W(g) w$. We have to check that this is a morphism. By definition it is linear. Moreover, for every $b_g$ basis vector in $V_{reg}$

$$f(\rho_{reg}(h) b_g) = f(b_h g) = \rho_W(hg) f(b_e) = \rho_W(h) \rho_W(g) f(b_e) = \rho_W(h) f(b_g). \tag{3.3.10}$$

Thus $f$ is a morphism between $V_{reg}$ and $W$ indeed. $\square$

**Corollary 3.3.3.** *Every finite group $G$ has only finitely many irreducible representations.*

*Proof.* Let

$$V_{reg} = {U_1}^{d_1} \oplus {U_2}^{d_2} \oplus \ldots \oplus {U_k}^{d_k}$$

be the decomposition of the regular representation into irreducible factors. Then by the previous theorem, $d_i = \dim W_i$ and every irreducible representation of $G$ appers in the decomposition. Consequently,

$$\dim V_{reg} = \sum_{i=1}^{k} d_i^2. \tag{3.3.11}$$

But $\dim V_{reg}$ is just $|G|$. Therefore, we obtain

$$|G| = \sum_{i=1}^{k} d_i^2 \tag{3.3.12}$$

where $k$ is the number of all irreducible representations of $G$.

Since $d_i^2 \geq 1$ for every $i$, we have

$$|G| \geq \#\{\text{irreducible representations of } G\}. \tag{3.3.13}$$

$\square$

**Corollary 3.3.4.** *The number of irreducible representations = $|G|$ if and only if $G$ is abelian.*

*Proof.* If $G$ is abelian, this is a direct consequence of Equation (3.3.12) since every irreducible representation of an abelian group is 1-dimensional.

On the other hand, if the number of irreducible representations is exactly the order of $G$, then by Equation 3.3.12, every irreducible representation is 1-dimensional. Thus the regular representation is the direct sum of 1-dimensional representations. In other words, it is simultaneously diagonalizable. Since the regular representation is injective i.e. it is an isomorphism onto its image, $G$ must be abelian. $\square$

## 3.4 McKay's construction

First we need to define a new operation of representations, the tensor product.

**Definition 3.4.1.** *Let $\rho : G \to GL(V)$ and $\psi : G \to GL(W)$ be representations of $G$. Then their tensor product is defined by*

$$\rho \otimes \psi : G \to GL(V \otimes W)$$

$$\rho \otimes \psi(g) := \rho(g) \otimes \psi(g)$$

.

**Definition 3.4.2.** *Let $G$ be a finite subgroup of $SU(2)$ and $\rho_{Nat} : G \to GL(\mathbb{C}^2)$ be the so-called natural representation defined by the inclusion of $G$ into $GL(\mathbb{C}^2)$. The McKay graph is a directed multi-graph with vertices corresponding to the irreducible representations of $G$, denoted by $\{V_i\}$. If*

$$V_i \otimes \mathbb{C}^2 = \bigoplus_{j=1}^{k} V_j^{m_{ij}}$$

*is the irreducible decomposition of $V_i \otimes \mathbb{C}^2$, then we draw $m_{ij}$ (directed) edges from $V_i$ to $V_j$.*

**Example 3.4.3** ($G = Z_n$)**.** *The natural representation of $Z_n$ is generated by*

$$\rho_{Nat}(1) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{n-1} \end{pmatrix}$$

*where $\varepsilon$ is the first $n$th root of unity.*

The irreducible representations are 1-dimensional and given by

$$\rho_i(1) = \begin{pmatrix} \varepsilon^i \end{pmatrix}.$$

Thus

$$\rho_i \otimes \rho_{Nat}(1) = \begin{pmatrix} \varepsilon^i \end{pmatrix} \otimes \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} \varepsilon^i \cdot \varepsilon & 0 \\ 0 & \varepsilon^i \cdot \varepsilon^{-1} \end{pmatrix} =$$
$$\begin{pmatrix} \varepsilon^{i+1} & 0 \\ 0 & \varepsilon^{i-1} \end{pmatrix} = \rho_{i+1}(1) \oplus \rho_{i-1}(1).$$

We conclude that

$$V_i \otimes \mathbb{C}^2 = V_{i+1} \oplus V_{i-1}$$

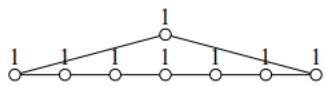
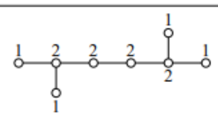
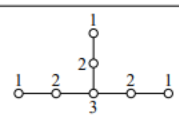
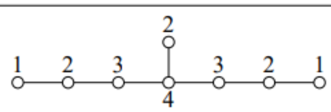
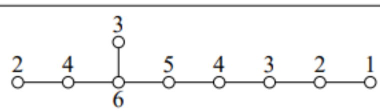
where the indices are taken modulo $n$. Therefore the McKay graph is a cycle of $n$ vertices.

The McKay graph can be computed explicitly for every finite subgroup of $SU(2)$ using character theory. For a method using Dixon's restricted character algorithm see [6].

For a more uniform approach one can prove the following properties of McKay graphs:

- The McKay graph is connected.
- The McKay graph is undirected, that is, $m_{ij} = m_{ji}$.
- The McKay graph has no self-loops, that is, $m_{ii} = 0$.
- The McKay graph is a simple graph i.e. $m_{ij} \in \{0, 1\}$.

Using these properties and some additional constraints on the dimensions of irreducible representations, a full classification of McKay graphs can be found, for example, in Sun [18]. The results are summarized in the following table from the same article.

| Finite subgroup of $SU(2)$ | | Affine simply laced Dynkin diagram | |
|---|---|---|---|
| $\mathbb{Z}/n\mathbb{Z}$ | $\langle x \mid x^n = 1 \rangle$ | $\widetilde{A}_{n-1}$ |   |
| $\mathbb{B}D_{2n}$ | $\langle x, y, z \mid x^2 = y^2 = y^n = xyz \rangle$ | $\widetilde{D}_{n-2}$ |   |
| $\mathbb{BT}$ | $\langle x, y, z \mid x^2 = y^3 = z^3 = xyz \rangle$ | $\widetilde{E}_6$ |   |
| $\mathbb{BO}$ | $\langle x, y, z \mid x^2 = y^3 = z^4 = xyz \rangle$ | $\widetilde{E}_7$ |   |
| $\mathbb{BD}$ | $\langle x, y, z \mid x^2 = y^3 = z^5 = xyz \rangle$ | $\widetilde{E}_8$ |   |

# Chapter 4

# Geometric McKay correspondence

**Theorem 4.0.1** (McKay correspondence)**.** *The McKay graph of $G$ with the trivial representation deleted is the resolution graph of the simple surface singularity $\mathbb{C}^2/G$.*

The first proof of McKay's theorem was given by case-by-case computation. A direct relation between irreducible representations and the irreducible components of the exceptional divisor was found by Gonzalez-Sprinberg and Verdier. We present the basic idea without proofs.

Let $G$ be a finite subgroup of $SU(2)$ and $S = \mathbb{C}^2/G$ the associated simple surface singularity. Since $G$ acts freely, the quotient map $\dot{\mathbb{C}}^2 \to \dot{\mathbb{C}}^2/G = S \backslash 0$ is a principal $G$-bundle. From an irreducible representation $V$ we can construct a vector bundle $\tilde{V} = \dot{\mathbb{C}}^2 \times_G V$ over $S\backslash\{0\}$. If $\pi : \tilde{S} \to S$ is the minimal resolution which is an isomorphism between $\tilde{S}\backslash\pi^{-1}(0)$ and $S\backslash 0$, then we can pull back the bundle $\tilde{V}$ by $\pi|_{\tilde{S}\backslash\pi^{-1}(0)}$. The pullback bundle $\pi^* V$ can be extended to a bundle $B_V$ on the whole of $\tilde{S}$. Then the first Chern class $c_1(B_V) \in H^2(\tilde{S}, \mathbb{Z})$ is dual to the class of the corresponding irreducible component of the exceptional divisor $E_V \in H_2(\tilde{S}, \mathbb{Z})$. Furthermore, the correspondence

$$V \mapsto B_V$$

can be extended to an isomorphism between the representation ring and the Grothendieck ring $K(\tilde{S})$ of vector bundles from $K$-theory. For more details, see [16].

A correspondence in the opposite direction would be also very interesting. The starting point could be that the fundamental group of the link $L$ is isomorphic to $G$ as the quotient map restricted to the unit sphere $S^3$ in $\mathbb{C}^2$ gives a covering $S^3 \to S^3/G = L$. By the plumbing construction of Mumford, we get the fundamental group as generators and relations. However, I don't have a good way to construct a representation from a generator corresponding to an irreducible component of the exceptional divisor.

The classical McKay correspondence had been reformulated several times. Artin and Verdier gave an interpretation in terms of reflexive $\mathcal{O}_X$ modules. More recently, Ito and Nakamura suggested an approach by Hilbert schemes. Furthermore, higher dimensional analogues of the McKay correspondence have been studied extensively. The first hint came from string theorists around 1985. They proved that if $G$ is a finite subgroup of $SL(3, \mathbb{C})$ and $Y \to X = \mathbb{C}^3/G$ is a crepant resolution of $X$, then the Euler number of $Y$ equals the number of irreducible representations of $G$. Since then a homological McKay correspondence had been provided for finite subgroups of $SL(n, \mathbb{C})$. For further details, see Reid [11].

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