The Allen-Cahn equations on metric graphs

A semigroup theoretical approach

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Declaration

STATEMENT

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Introduction

The theory of semigroups, originating in the mid-20th century, has been instrumental in developing various approaches for studying evolutionary systems. Our focus lies primarily in the application of semigroup theory to the analysis and modeling of evolutionary systems on metric graphs. Semigroups provide a robust mathematical framework for capturing the dynamic behavior of networks over time, encompassing changes in topology, node behavior, and edge dynamics.
The study of evolutionary equations on metric graphs has been a subject of research since the early 20th century, gaining increased interest in recent years due to its practical applications in theoretical physics, biology, and engineering. For instance, when studying the separation process in multi-component alloy systems or solidification and fracture dynamics, metric graphs naturally arise as a means to represent the underlying network structure. This field of study is known by various names, such as dynamics on networks, one-dimensional ramified spaces, or quantum graphs within certain theoretical physics communities. Exploring this area requires interdisciplinary tools from graph theory, partial differential equations (PDEs), mathematical physics, and other relevant disciplines, depending on the specific problem at hand.
An important example in this context is the Allen-Cahn equation, a reaction-diffusion equation, which serves as a useful model for processes such as the separation process. Considering the stochastic version of the Allen-Cahn equation, which incorporates thermal fluctuations, becomes crucial for capturing the inherent randomness in these phenomena. Thus, it is both justifiable and of great interest to contribute to mathematical research in this specific area by developing results for more general cases. In a previous work by Sikolya Eszter and Mihály Kovács [9], the stochastic Allen-Cahn equation on a metric graph with non-local type Kirchhoff boundary conditions was studied. The present thesis builds upon their work by investigating whether their results also hold in the case of non-compact graphs.

The structure of this thesis unfolds in a coherent manner, beginning with Chapter 1, which serves as a gateway to the theory of semigroups. By delving into the fundamental definitions and concepts of semigroup theory, such as generators, strongly continuous and analytic semigroups, and key generation theorems like the Hille-Yosida and Lumer-Philips theorems, we lay the groundwork for establishing well-posedness results and exploring the dynamics of PDEs, this part is mainly based on the first and second chapter of [6]. Furthermore, this chapter introduces the methods of bilinear forms and their associated operators, highlighting their relevance as tools for proving well-posedness within the framework of semigroup theory, and we refer to the first chapter of [15] for these tools.

Chapter 2 introduces the fascinating realm of metric graphs, which provide a geometric framework for studying PDEs on graph structures with parameterized edges. This chapter embarks on a journey through the captivating world of metric graphs, elucidating the concept of parametrization and distinguishing between compact and non-compact graphs. Non-compact graphs, with their edges extending infinitely in one direction, open up exciting possibilities for modeling real-world phenomena. We investigate the differential operators defined on metric graphs, particularly the Laplacian, and the accompanying vertex conditions, such as continuity conditions and the Kirchhoff law. By incorporating the theory of metric graphs into our study, we establish a framework that allows for the application of semigroup theory. For this chapter, we refer to the first chapter of [3] and the second chapter of [20].

Chapter 3 delves into the intriguing field of stochastic PDEs on Banach spaces. This chapter offers a comprehensive survey of this technical and intricate theory, aiming to provide a solid foundation without delving into intricate details. Central to this theory is the stochastic integral with respect to Cylindrical Brownian motion, specifically for operator-valued functions defined on Hilbert spaces with their image spaces in Banach spaces. The construction of the stochastic integral for step operator-valued functions, employing gamma-radonifying operators and gamma-summing norms, forms a key component of this chapter. Furthermore, we explore the extension of the stochastic integral to operator-valued processes, despite the generalized Itô isometry only holding up to an isomorphism. Well-posedness results for linear and semilinear stochastic PDEs, including those with additive and multiplicative noise, are presented. These results lay the foundation for our analysis of the stochastic Allen-Cahn equations on non-compact metric graphs in the fourth chapter. The third chapter is mainly based on the lecture notes [19, Chapter 6 and 13], while the results on the semilinear stochastic pde, is based on the papers [18] and [11] by J.M.A.M. van Neerven.

The fourth chapter which is the core of the thesis, focuses on investigating the well-posedness of the Stochastic Allen-Cahn equations on a non-compact metric graph. The chapter begins by establishing the well-posedness of the deterministic part of this evolutionary problem in the $\mathbf{L}^2$-space framework. The framework and settings introduced in the second chapter, which involves metric graphs, are

employed to convert the problem into the framework of semigroup theory. The methods and theories established in the first chapter are then utilized to prove the well-posedness of the deterministic case. The reason for proving the deterministic case first is to extend the well-posedness to all $\mathbf{L}^p$ spaces, not just p=2. This extension is accomplished through the well-posedness for p=2 and the application of extension techniques that will be further elaborated in the fourth chapter.
The aim of establishing the well-posedness for the deterministic problem in all $\mathbf{L}^p$ spaces, rather than just $\mathbf{L}^2$, is directly related to the objective of proving the well-posedness of the stochastic Allen-Cahn equations on Banach spaces, specifically $\mathbf{L}^p$ spaces. This expansion is crucial, as the results would otherwise fall within the realm of Hilbert spaces, where they are already well-established. However, our investigation ends after extending the well-posedness to all $\mathbf{L}^p$ spaces, that is because, our tools won't be enough to prove a necessary result in the case of $\mathbf{L}^p$, $p \neq 2$ for non-compact graphs.
The second part of the fourth chapter addresses the well-posedness of the stochastic evolutionary problem on compact graphs, which was already successfully investigated in [9, Section 3], where they leveraged the well-posedness of the deterministic case. However, before they focused on the specific stochastic Allen-Cahn equations, a more general problem was examined: semilinear stochastic PDEs on metric graphs. The well-posedness of this general problem was proved by utilizing the theory presented in the third chapter, which incorporates the well-posedness of semilinear stochastic PDEs. The well-posedness of the deterministic case also played a role in establishing certain assumptions necessary for the application of the last theorem 3.20, presented in the third chapter.
After addressing the general semilinear stochastic PDE with multiplicative noise on the compact graph, the chapter transitions to the original problem of the stochastic Allen-Cahn equations. It is demonstrated that the well-posedness of the original problem can be deduced as a special case of the well-posedness of the general problem throughout some manipulations. In this chapter, we will refer to results when needed, but notable papers from which the investigation for the non-compact case is inspired are [9] which handles the same problem but on a compact graph, [13] and [14] which treat a deterministic evolutionary equation on a compact graph and also utilizes the methods of forms, and one paper that should also be mentioned is [7], which proves well-posedness results for general first and second order PDE's on compact and non-compact graphs with general vertex boundary conditions, the results of this paper proceeds the results established in the first part of the fourth chapter, the difference is that in that paper they use other methods than forms to prove those results.

By harmonizing the theories of semigroups, stochastic PDEs on Banach spaces, and metric graphs, we embark on a remarkable journey that combines their unique strengths and insights. This integration not only enhances our understanding of PDEs but also allows us to explore new avenues of research and applications. The amalgamation of these theories enables us to analyze the dynamics of PDEs in diverse settings, bridging the gap between abstract mathematical concepts and their tangible manifestations in the real world. Through the interplay of semigroup theory, stochastic PDEs, and the framework of metric graphs, we unveil the hidden beauty and versatility of PDEs, paving the way for further advancements and applications in various scientific and engineering domains.

Contents

Chapter 1

Semigroups and bilinear forms

In this chapter, we will collect the basic results of the theory of semigroup theory, along with a brief introduction to bilinear form concepts and notations. We will also explore semigroup-form characterizing results, which will be utilized as tools to prove our results in the fourth chapter. The first section is based on [6], and for the second section, we refer to [15]. Throughout this chapter, we set X to be a Banach space and H a real Hilbert space, and we denote the non-negative real half-line $[0, \infty)$ by $\mathbb{R}^+$.

1.1 Some semigroup theory

Consider the following Abstract Cauchy Problem (ACP),

$$ACP \begin{cases} \dot{u}(t) = Au(t), & \text{for } t \geq 0; \\ u(0) = x. \end{cases}$$

Where $A : D(A) \subset X \to X$ is a linear operator, $x \in X$ is the initial value, and $u : \mathbb{R}_+ \to X$.

Solving differential equations analytically can often be challenging or even impossible. In order to understand the behavior of a system, an alternative approach is to use the theory of dynamical systems or solve the problem numerically. However, before applying these approaches, it is important to determine whether the system has a solution and if that solution is unique. Additionally, we may want to study the regularity of the solutions. The theory of semigroups provides a useful framework for addressing these questions.

Definition 1.1 (One-parameter semigroup)
A family $(T(t))_{t\geq 0}$, $T : \mathbb{R}_+ \to \mathcal{L}(X)$ *of bounded linear operators on* X *is called a*

semigroup on X if it satisfies the following functional equation :

$$(FE)\begin{cases} T(t+s)=T(t)T(s) & \text{for all } t,s\geq 0; \\ T(0)=I & \text{where } I \text{ is the identity operator of } X. \end{cases} \tag{1.1}$$

The exponential function is a valuable tool in solving certain types of finite-dimensional dynamical systems. The notion of a semigroup is its generalization for infinite-dimensional systems on Banach spaces. For this generalization to make sense, the semigroups must be able to describe the time evolution of the dynamical system throughout the functional equation (FE) as in the finite-dimensional case. This functional equation is, in fact, identical to the algebraic semigroup property, which gives rise to the name "semigroup." By understanding the relationship between the exponential function and the semigroup, we can better appreciate the power and flexibility of these mathematical concepts in solving abstract systems.

In some linear finite cases, e.g $u(t)' = Mu(t)$ where for simplicity say that M is some diagonal constant matrix, then the following exponential function $u(t) = e^{tM}u_0$, where $u_0 \in \mathbb{R}$ is the initial condition, fully captures the dynamics of the evolutionary systems, this is thanks to the fact that $(e^{tM}u_0)$ is a solution to the differential equation describing the system, and this is because $t \mapsto e^{tM}$ is differentiable. To generalize this, it is desirable for the semigroups to be differentiable as well. While the (uniform) continuity of the mappings: $t \in \mathbb{R}^+ \mapsto T(t) \in \mathcal{L}(X)$, is a sufficient condition for differentiability [6, Theorem 1.3.7], there exist many natural semigroups that do not satisfy uniform continuity, but instead, satisfy a weaker form of continuity known as strong continuity. This type of semigroup is known as strongly continuous semigroup (C_0-semigroup) and constitute a large class of semigroups that can provide solutions to abstract systems under certain additional conditions. By studying the properties of C_0-semigroup, we can gain a deeper understanding of the behavior of dynamical systems and develop more effective tools for analyzing and solving these systems, such as stability and control analysis.

Definition 1.2 (Strongly continuous semigroup)
$(T(t))_{t\geq 0}$ is a C_0-semigroup, if $(T(t))_{t\geq 0}$ is a semigroup and it is strongly continuous, i.e.

$$\forall x \in X \text{ the orbit mappings } \quad t \mapsto T(t)x \in X \quad \text{are continuous on } \mathbb{R}^+.$$

Proposition 1.3
Let $(T(t))_{t\geq 0}$ be a semigroup on X. Then $(T(t))_{t\geq 0}$ is a C_0-semigroup iff there exist

$K, \delta > 0$ and a dense subset D of X such that :

$$i) \quad \|T(t)\| \leq K, \quad \forall t \in [0, \delta],$$

$$ii) \quad \lim_{t \to 0} T(t)x = x, \quad \forall x \in D.$$

Usually, many semigroups satisfy property i) of Proposition 1.3, therefore by this proposition it is enough to prove strong continuity on a dense subset rather than on the whole Banach space X.

Proposition 1.4
If $(T(t))_{t\geq 0}$ is a C_0-semigroup, then $(T(t))_{t\geq 0}$ is exponentially bounded, i.e there exist, $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0. \tag{1.2}$$

Remark 1.5
If $M = 1$ and $\omega = 0$, i.e $\|T(t)\| \leq 1$, we say $(T(t))_{t\geq 0}$ is a contraction C_0-semigroup.

If X is finite-dimensional, the behavior of a dynamical system $\{e^{At}\}_{t\geq 0}$ can be fully characterized by its corresponding matrix A, where matrices with different spectral properties generate different dynamical systems. Additionally, the matrix A can be obtained by taking the derivative of e^{At} at $t = 0$. This analogy motivates the introduction of the notion of a generator for infinite-dimensional systems, which is a key tool for characterizing and analyzing the behavior of semigroups.

Definition 1.6 (Generator)
Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. We call the operator $(A, D(A))$ on X the generator of $(T(t))_{t\geq 0}$ if,

$$\begin{cases} Ax & := \lim_{h \to 0^+} \frac{1}{h}(T(h)x - x); \\ D(A) & := \{x \in X : \quad t \mapsto T(t)x \in X \quad \textit{is differentiable at t=0}\} \end{cases}$$

Being familiar with some notions of semigroup theory, we will now formulate how semigroups provide solutions to abstract Cauchy problems.

Definition 1.7
We say that the ACP is well-posed if,

1. *$\forall x \in D(A)$, there exists a unique solution $u(., x)$ of the ACP,*

2. *$D(A)$ is dense in X,*

3. *$\forall (x_n)_{n\in\mathbb{N}} \subset D(A)$, if $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} u(t, x_n) = 0$ uniformly in compact intervals $[0, t_0]$.*

Theorem 1.8 (C_0-semigroup and well-posedness)
If the operator A from the ACP is closed, then the ACP is well-posed iff A is the generator of a C_0-semigroup.

Now we will present two famous theorems that provide sufficient conditions for the operator to generate a C_0-semigroup, but first, we need to introduce the notion of the resolvent of an operator.

Definition 1.9 (Resolvent)
Let $(A, D(A))$ be a linear closed operator on X, where $D(A) \subset X$ and X is Banach space. We call the following set :

$$\rho(A) =: \{\lambda \in \mathbb{C} : (\lambda - A) : D(A) \to X \text{ is bijective } \}$$

the resolvent set of A, and for $\lambda \in \rho(A)$ we introduce the so-called resolvent operator of A defined as :

$$R(\lambda, A) := (\lambda - A)^{-1},$$

which is bounded on X by the closed graph theorem.

Theorem 1.10 (Contraction case,Hille-Yosida,1948)
For a linear operator $(A, D(A))$ on a Banach space X, the following properties are all equivalent :

1. *$(A, D(A))$ generates a strongly continuous contraction semigroup,*
2. *$(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $Re\lambda > 0$ one has $\lambda \in \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{1}{Re\lambda}$.*

It turns out that if the operator A is dissipative then there is a simpler generation theorem in the sense that we do not have to deal with the resolvent estimates.

Definition 1.11 (Dissipative operators)
A linear operator $(A, D(A))$ on X is called dissipative if

$$\|(\lambda - A)x\| \geq \lambda\|x\|$$

for all $\lambda > 0$ and $x \in D(A)$.

Theorem 1.12 (Lumer-Philips,1961)
Let $(A, D(A))$ be a closed, densely defined, dissipative operator on X, then the followings are equivalent :

1. *A generates a contraction C_0-semigroup;*

2. *range*$(\lambda - A)$ *is dense in* X *for all* $\lambda > 0$.

Now, we will present another type of semigroup characterized by a property of its generator. We will use the terminology given by [6].

Definition 1.13 (Sectorial operators)
Let $(A, D(A))$ *be a closed linear operator with a dense domain in* X*. We say that* A *is sectorial if there exist* $0 \leq \delta \leq \frac{\pi}{2}$ *such that :*

1. $\Sigma_{\frac{\pi}{2}+\delta} =: \{\lambda \in \mathbb{C} : |arg\lambda| < \frac{\pi}{2} + \delta\}\backslash\{0\} \subset \rho(A)$

2. $\forall \epsilon \in (0, \delta)$*, there exists* $M_\epsilon \geq 1$ *such that*

$$\|R(\lambda, A)\| \leq \frac{M_\epsilon}{|\lambda|}, \text{ for all } 0 \neq \lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\delta-\epsilon}$$

Definition 1.14 (Analytic semigroups)
A family $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ *is called an analytic semigroup on* X *if*

1. $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ *is a semigroup on* Σ_δ*, satisfying the functional equation (1.1),*

2. *The map* $z \to T(z)$ *is analytic on* Σ_δ*,*

3. $\lim_{\Sigma_{\delta'}\ni z\to 0} T(z)x = x$ *for all* $x \in X$ *and* $0 < \delta' < \delta$.

Theorem 1.15 (Characterization of analytic semigroups)
$(A, D(A))$ *generates an analytic semigroup* $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ *on* $X \iff A$ *is sectorial.*

1.2 Brief introduction to bilinear forms and associated operators

Our future investigations will focus on studying the properties of the Laplace operator subjected to some boundary conditions. One of the key objectives of our analysis is to establish that this operator is a generator of a semigroup. To achieve this goal, while referring to [15], we introduce the tools of bilinear forms and their associated operators in this section, as they offer useful techniques for conducting such analyses.

Definition 1.16
We call a mapping $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{R}$ *a bilinear form on the Hilbert space* H *if it satisfies*

$$\mathfrak{a}(\alpha u + v, h) = \alpha\mathfrak{a}(u, h) + \mathfrak{a}(v, h) \text{ and } \mathfrak{a}(u, \alpha v + h) = \alpha\mathfrak{a}(u, v) + \mathfrak{a}(u, h)$$

for $u, v, h \in D(\mathfrak{a})$*, where* $D(\mathfrak{a})$ *is a linear subspace of* H*.*

A bilinear form can have many properties, but for our purposes, we will focus on the ones that are most useful. Specifically, we will see that certain properties of a bilinear form can give us sufficient conditions for the associated operator (see Definition 1.19 below) to be the generator of a C_0-semigroup. This is why it is important to study the properties of bilinear forms and how they relate to the operators they are associated to.

Definition 1.17

We say that $\mathfrak{a}$ *is,*

1. *densely defined if* $D(\mathfrak{a})$ *is dense in* H*,*
2. *accretive if* $\mathfrak{a}(u,u) \geq 0$ *for all* $u \in D(\mathfrak{a})$*,*
3. *continuous if there exists* $M \geq 0$ *such that :* $|\mathfrak{a}(u,u)| \leq M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}}$*, for all* $u, v \in D(\mathfrak{a})$*, where* $\|u\|_{\mathfrak{a}} = \sqrt{\mathfrak{a}(u,u) + \|u\|^2}$*, and* $\|.\|$ *is the induced norm from the inner product of* H*,*
4. *closed if* $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$ *is a complete space,*
5. *symmetric if* $\mathfrak{a}(u,v) = \mathfrak{a}(v,u)$ *for all* $u, v \in D(\mathfrak{a})$*.*

Proposition 1.18

Let $\mathfrak{a}$ *be a densely defined, accretive, continuous,closed (DACC) form. Then* $\|.\|_{\mathfrak{a}}$ *is a norm on* $D(\mathfrak{a})$*, we call it the norm associated with* $\mathfrak{a}$*. Moreover* $\|.\|$ *and* $\|.\|_{\mathfrak{a}}$ *are equivalent on* H*.*

Proof. Let $I : (H, \|.\|_{\mathfrak{a}}) \to (H, \|.\|)$, be the identity operator. We have:

$$\|u\|_{\mathfrak{a}} = \sqrt{\mathfrak{a}(u,u) + \|u\|^2} \implies \|u\| \leq \|u\|_{\mathfrak{a}} \implies I \text{ is continuous}$$

I is bijective, hence by the closed graph theorem, $I^{-1} = I$ is also continuous,therefore there exist $C \geq 0$ such that : $\|u\|_{\mathfrak{a}} \leq C\|.\|$ for all $u \in H$

□

Using the Riesz representation theorem, these forms above induce a unique operator, which we call the operator associated with the form. The usefulness of the method of forms, as will be seen later, stems from the fact that if a form enjoys nice properties then the negative of its associated operator will be the generator of a C_0-semigroup. However, because we would like later to work directly with the associated operator rather than its negative, we therefore define the associated operator in an equivalent but different way than what can be usually found in the litterature.

Definition 1.19 (Associated operator)
Let $\mathfrak{a}$ be a DACC form on H. The operator B defined as

$$\begin{cases} D(B) := \{u \in D(\mathfrak{a}) : \exists w \in H \text{ such that }, \mathfrak{a}(u,\phi) = \langle w, \phi \rangle_H \quad \forall \phi \in D(\mathfrak{a})\} \\ \quad Bu := -w \end{cases}$$

is called the operator associated with the form $\mathfrak{a}$.

In the sequel, we will apply the Lumer-Philips generation theorem 1.12, to prove a C_0-semigroup characterization result using forms. The next proposition serves the purpose of preparing the assumption stated in Theorem 1.12.

Proposition 1.20
Let B be the operator associated with a DACC form. Then

1. *B is densely defined,*
2. *$\forall \lambda > 0$, $(\lambda - B) : D(B) \to H$ is invertible,*
3. *$(\lambda - B)^{-1}$ is bounded,*
4. *$\|\lambda(\lambda - B)^{-1} f\| \leq \|f\|$, $\forall \lambda > 0, f \in H$.*

Proof. We refer to [15, Proposition 1.22]. □

Proposition1.20, says that the operator associated with a DACC form, is densely defined and dissipative.

Proposition 1.21
If $\mathfrak{a}$ is symmetric, then the associated operator B is self-adjoint.

Theorem 1.22
Let $\mathfrak{a}$ be a DACC on H, and B its associated operator. Then B is the generator of a contraction C_0-semigroup on H $\iff$ B is closed, moreover, this semigroup is also analytic.

Proof. Using Proposition 1.20, we get that B is densely defined and dissipative. $range(\lambda - B)$ is dense in H follows from [15, Theorem 1.49]. Then we apply Theorem 1.12. Concerning the analyticity of the semigroup, for this part of the proof we refer to [15, Theorem 1.50]. □

Remark 1.23
In the previous theorem, B had to be closed so that it generates a C_0-semigroup. If the form $\mathfrak{a}$ happens to be symmetric, then we can use Proposition 1.21 and the fact that B is densely defined to satisfy the closedness property.

Chapter 2

Metric graphs

This chapter provides an overview of metric graphs, which are directed graphs having parametrized edges on intervals. In the first section, we introduce the concept and framework of metric graphs. The second section discusses differential operators on metric graphs, boundary conditions, and their interpretations. We rely on [3] and [20] as our main references for this chapter.

2.1 Introduction to the framework and settings of metric graphs

Definition 2.1 (Undirected graph)
An undirected graph is a triple $G = (V, E, \Phi)$, where V is the set of vertices, E the set of edges and $\Phi : E \to V \times V$ maps every edge $e \in E$ to an unordered set of vertices, that is $\Psi(e) = \{u, v\}$.

Definition 2.2 (Directed graph)
$G = (V, E, \Phi)$ is said to be a directed graph if Φ maps each edge to an ordered pair of vertices, i.e for $e \in E$ we have $\Phi(e) = (u, v)$. We say that e is a directed edge, and u and v are the origin and terminal of e.

Definition 2.3 (Edge parametrization)
Let $G = (V, E, \Phi)$ be a directed graph, we say that $e \in E$ with $\Phi(e) = (u, v)$ is a parameterized edge, if we assign to e a positive length $l_e \in (0, \infty)$ and parameterize e on $(0, l_e)$.

Parametrizing the edge e on $[0, l_e]$, means that for $s \in [0, l_e]$ we have a parameter $x_e(s) \in e$, then we can set the endpoints by setting, $x_e(0) = u$ and $x_e(l_e) = v$. To abbreviate these endpoints settings we use directly the notation $e(0) = u$ and $e(l_e) = v$.

To avoid the notation technicalities of edge parametrization, we will in the sequel only state the following : Let $e \in E$ where $\Phi(e) = (u, v)$, be a parametrized edge on $[0, l_e]$.
Here is an example of parametrizing and edge e on [0,1]

The concept of edge parametrizing is basically a way of moving from the discrete settings of graphs to a continuous one. The reason for moving to the continuous setting is that later, we would like to define the Laplacian on a graph G and to be able to do that, we will also need to consider the topological functional space $\mathbf{L}^2(G)$ which consists of square-integrable functions on G. Which is defined as the direct sum of all the $\mathbf{L}^2$ spaces on the intervals assigned to the edges through the parametrization. Meanwhile, the common vertices shared by the adjacent parametrized edges will be identified through some vertex boundary conditions.

Definition 2.4 (Metric graph)
Let $G = (V, E, \Phi)$ be a directed graph without any isolated vertex. By parametrizing all the edges in E, we call G a metric graph and we say that G is

1. *finite, if there are finite numbers of edges and vertices,*
2. *non-compact if G has leads which are infinitely long edges, i.e edges parameterized on $[0, \infty)$.*

The mapping Φ describes the incidence relation between the vertices and the edges, one way to present this relation at once is to use the incoming and outgoing incidence matrices.

Definition 2.5 (Incidence matrix for a non-compact metric graph)
Let G=(V,E,Φ) be a non-compact and finite metric graph such that V=$\{v_1, v_2, ..., v_n\}$ and E=$\{\utilde{e}_1, ..., \utilde{e}_k, \tilde{e}_{k+1}, ..., \tilde{e}_m\}$ where $\utilde{e}_j$ denotes a directed edge for $j \in \{1, ..., k\}$, $\tilde{e}_j$ denotes a lead for $j \in \{k+1, ..., m\}$ and $n, m \in \mathbb{N}$, set $s := m - k$,. Φ induces the outgoing and incoming incidence matrices for the directed edges.

$$\utilde{\Phi}^+ := \left(\utilde{\Phi}^+_{ij}\right)_{n\times m} \left[\left(\utilde{\phi}^+_{ij}\right)_{n\times k} \middle| \, [0]_{n\times s} \right]_{n\times m} \quad \utilde{\Phi}^- := \left(\utilde{\Phi}^-_{ij}\right)_{n\times m} = \left[\left(\utilde{\phi}^-_{ij}\right)_{n\times k} \middle| \, [0]_{n\times s} \right]_{n\times m}$$

and an additional outgoing incidence matrix corresponding to the incidence between the vertices and the leads, denoted by

$$\tilde{\Phi}^+ := \left[[0]_{n\times k} \middle| \left(\tilde{\phi}^+_{ij}\right)_{n\times s} \right]_{n\times m}$$

These matrices are defined by,

$$\underset{\sim}{\phi}_{ij}^{+} := \begin{cases} 1, & \textit{if } \underset{\sim}{e}_j(0) = v_i, \\ 0, & \textit{otherwise,} \end{cases} \quad ; \quad \underset{\sim}{\phi}_{ij}^{-} := \begin{cases} 1, & \textit{if } \underset{\sim}{e}_j(1) = v_i, \\ 0, & \textit{otherwise,} \end{cases}$$

$$\tilde{\phi}_{ij}^{+} := \begin{cases} 1, & \textit{if } \tilde{e}_j(0) = v_i, \\ 0, & \textit{otherwise.} \end{cases}$$

The reason for the block notation of these previous matrices is to be able to define the incidence matrix $\Phi := \underset{\sim}{\Phi}^{-} + \underset{\sim}{\Phi}^{+} + \tilde{\Phi}^{+}$.

2.2 Differential operators on metric graphs

Let $G(V, E, \Phi)$ be a non-compact, finite metric graph, with $|V| = n$ and $|E| = m$ such that E contains k directed edges parametrized on $[0, 1]$ and $s := m - k$ leads. We mean by a differential operator on the graph G an operator acting on functions defined on the parametrized edges, for more details we refer to [20]. We will define a function u on G as the $|E|$-tuple of functions $\underset{\sim}{u}_j$ and $\tilde{u}_j$ on the intervals $[0.1]$ and $[0, \infty)$ respectively, i.e :

$$u : (\underset{\sim}{u}_1, ..., \underset{\sim}{u}_k, \tilde{u}_{k+1}, ..., \tilde{u}_m) \text{ where } \underset{\sim}{u}_j : [0, 1] \to \mathbb{R} \text{ and } \tilde{u}_j : [0, \infty) \to \mathbb{R}$$

However, it is important to note that G includes a set of vertices that represent the boundary relationships between edges. For example, to ensure that the functions on G are continuous, the coordinate functions must not only be continuous on their respective intervals but also maintain continuity at the boundaries between adjacent edges.

We now introduce some sets and notations.

1. For $v_i \in V$ let $\Gamma(v_i)^{+} := \{j \in \{1, ..., m\} : \underset{\sim}{e}_j(1) = v_i\}$, denote the set of all incoming edges to v_i.

2. $\Gamma(v_i)^{-} := \{j \in \{1, ..., m\} : \underset{\sim}{e}_j(0) = v_i \text{ or } \tilde{e}_j(0) = v_i\}$, denote the set of all outgoing edges to v_i.

3. $\Gamma(v_i) := \Gamma(v_i)^{+} \cup \Gamma(v_i)^{-} = \{j \in \{1, ..., m\} : \underset{\sim}{e}_j(1) = v_i \text{ or } \underset{\sim}{e}_j(0) = v_i \text{ or } \tilde{e}_j(0) = v_i\}$

4. Let u_j denote either $\underset{\sim}{u}_j$ or $\tilde{u}_j$.

Definition 2.6 (Continuity condition)
We denote by $C(G)$ the set of continuous functions on G. We say that u is continuous on G or $u \in C(G)$ if :

1. $u \in C([0,1])^k \times C([0,\infty))^s$
2. $\forall j \in \Gamma(v_i)$, $u_j(v_i) := q_i \ \forall i \in \{1,...,n\}$, *where* $q_i \in \mathbb{R}$

Remark 2.7
If $u \in C(G)$,we denote the values of $\underset{\sim}{u}_j$ and $\tilde{u}_j$ for all $j \in \{1,...,m\}$ at the boundary 0 *and* 1 *by :*

$$\underset{\sim}{u}_j(v_i) := \begin{cases} \underset{\sim}{u}_j(0), & \text{if } \underset{\sim}{e}_j(0) = v_i, \\ 0, & \text{otherwise} \end{cases} \quad ; \quad \underset{\sim}{u}_j(v_i) := \begin{cases} \underset{\sim}{u}_j(1), & \text{if } \underset{\sim}{e}_j(1) = v_i, \\ 0, & \text{otherwise} \end{cases}$$

$$; \quad \tilde{u}_j(v_i) := \begin{cases} \tilde{u}_j(0), & \text{if } \tilde{e}_j(0) = v_i, \\ 0, & \text{otherwise} \end{cases}$$

The second condition ensures that the values of u at the vertices $v_i \in V$ are uniquely defined and that all adjacent edges share the same value q_i at v_i.
The continuity condition happens to be imposed on many problems, for instance, the metric graph we will work within our investigation in Chapter 4 will be equipped with a Laplacian, therefore we will consider functions defined on Sobolev spaces, $u \in (H^2(0,1))^k \times (H^2(0,\infty))^s$, consequently u must be continuous. This example serves as a justification for exploring this continuity condition, and now we will mention another type of vertex boundary condition, the Kirchhoff law.

Definition 2.8 (Kirchhoff law)
Let $u \in C(G)$, then we set $q =: (q_1,...,q_n)^T$ to be the set of all uniquely determined values of the vertices by u, i.e $q_i := u_j(v_i)$, for some $j \in \Gamma(v_i)$. We say that :

1. q_i *is the incoming flow at* v_i *if* $q_i := \sum_{j \in \Gamma^+(v_i)} u_j(1)$.
2. q_i *is the outgoing flow at* v_i *if* $q_i := \sum_{j \in \Gamma^-(v_i)} u_j(0)$.
3. u *satisfy the Kirchhoff law on* G *if* $\forall\ i = 1,...,n$, q_i *is both the incoming and outgoing flow at* v_i, *i.e*

$$\sum_{j=1}^{m} \phi_{ij} u_j(v_i) = 0, \quad \forall \quad v_i \in V \tag{2.1}$$

Remark 2.9 1. *If the right-hand side of 2.1, is non-zero and instead depends on the common values of some non-adjacent vertices to v_i, then we call such vertex* *condition a non-local Kirchhoff law. e.g we can have have at v_i :*

$$\sum_{j=1}^{m} \phi_{ij} u_j(v_i) = \sum_{l=1}^{n} q_l b_l \tag{2.2}$$

Where the b_l are some real coefficients.

2. *To ensure the existence of the vector of common values in the left-hand side of the flow equations, it is necessary for $u \in C(G)$ to be continuous. On the other hand, the functions u_j's in the right-hand side can be replaced by their derivatives, provided that these derivatives exist. The choice of using the functions u_j's or their derivatives depends on the interpretation of the given flow relative to the problem being considered.*

Chapter 3

Stochastic PDE's

The fourth chapter focuses on the semilinear stochastic equation with multiplicative noise. To present an existence and uniqueness result at the end of the chapter, we first provide a survey on the existence and uniqueness of a stochastic linear equation with additive and multiplicative noise, without delving too deep into the technical details to avoid detracting from the main results of the thesis. In the first section, we discuss the construction of stochastic integrals for operator-valued functions and processes. In the second section, we address the existence and uniqueness of the results for the three types of stochastic equations. Definitions and brief explanations will be provided when necessary and the proofs will be omitted. Our discussion is primarily based on Chapters 6 and 13 of J.M.A.M. van Neerven's lecture notes ([19]), and we also refer to the survey and paper by J.M.A.M. van Neerven [18] and [11] for the theoretical background leading to the well-posedness of the semilinear stochastic equation. Throughout this chapter, unless specified otherwise, X and H refer to a Banach space and a Hilbert space respectively, and $[0, T]$ is an interval of $\mathbb{R}^+$ where $0 < T$, while for any space Banach Y, we denote its dual by Y^*, and if Y is Hilbert then we denote the inner product associated to Y, by $\langle ., . \rangle_Y$.

3.1 Stochastic intgeration

In this section, we will look at how to define stochastic integrals with respect to a cylindrical Brownian motion on a Banach space. First, we will construct the stochastic integral for $\mathcal{L}(H, X)$ operator-valued functions and then we will consider the more delicate problem of constructing stochastic integrals for $\mathcal{L}(H, X)$ operator valued stochastic processes.

$\mathcal{L}(H, X)$ operator-valued function

The construction follows similarly to how it's done in Hilbert spaces, by constructing the integral first for step functions and extending the integral to a larger class using isometry. The analogy in Banach spaces is that the γ-radonifying operators will play the role of by Hilbert-Schmidt operators in Hilbert spaces. First, we present some preliminary definitions and notations.

For $h \in H$ and $x \in X$,we denote by $h \otimes x$ the operator in $\mathcal{L}(H, X)$ defined as:

$$(h \otimes x)h' := \langle h, h' \rangle_H x, \quad h' \in H.$$

The H-cylindrical Brownian motion is considered instead of a simple Brownian motion because the stochastic PDE considered in the fourth chapter has a random term that involves space-time noise. This means that the equation takes into account spatial and temporal fluctuations, and depending on H, the cylindrical Brownian motion is the appropriate mathematical model for this stochastic part.

Definition 3.1 (Cylindrical Brownian motion)
Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\mathcal{H}$ a Hilbert space, consider the following mapping :

$$\mathcal{W} : \mathcal{H} \to \mathbf{L}^2(\Omega)$$

If the following properties hold:

1. *$\forall h \in \mathcal{H}$, the random variables $\mathcal{W}h$ are Gaussian,*

2. *$\forall h_1, h_2 \in \mathcal{H}$, $\mathbb{E}(\mathcal{W}h_1 \cdot \mathcal{W}h_2) = \langle h_1, h_2 \rangle_{\mathcal{H}}$,*

3. *$\mathcal{H} := \mathbf{L}^2(0, T; H)$, where H is a Hilbert space. Then we call the mapping $\mathcal{W}$ an H-Cylindrical Brownian motion. We denote it by W_H and it's output is given by*

$$W_H(t)h := \mathcal{W}(1_{(0,t)} \otimes h), \quad t \in [0, T], \text{ and } h \in H.$$

When dealing with Hilbert spaces, the $\mathbf{L}^2$-norm is a crucial tool for estimating stochastic integrals through the Ito isometry. Additionally, Hilbert-Schmidt operators aid in constructing integrals with respect to cylindrical Brownian motion. However, these concepts no longer apply in the more general setting of Banach spaces. To extend stochastic integrability beyond the Hilbert space framework, it is necessary to introduce the γ-summing norm and the γ-radonifying operators. These tools provide a way to carry out stochastic integrability in Banach spaces, where the traditional Hilbert space techniques cannot be employed.

Definition 3.2 (γ-summing operators)
Let X be a Banach space, a linear operator $S \in \mathcal{L}(H, X)$ is said to be γ-summing if

the following norm is finite,

$$\|S\|_{\gamma_p^\infty(H,X)} := sup \left(\mathbb{E}\| \sum_{j=1}^{N} \gamma_j S h_j \|^p \right)^{\frac{1}{p}} < \infty,$$

for some $1 \leq p < \infty$ *and equivalently for all* p*, where :*

1. γ_j *are real-valued Gaussian variables for* $j \in \{1, ..., N\}$*,*

2. $\{h_1, ..., h_N\}$ *is a finite orthonormal system of* H*,*

3. *the supremum is taken over all finite orthonormal systems of* H*.*

The space of all γ*-summing operators in* $\mathcal{L}(H, X)$ *is denoted by* $\gamma_p^\infty(H, X)$*.*

Definition 3.3
An operator $S \in \mathcal{L}(H, X)$ *is said to be a finite rank operator if it can be written as :*

$$S = \sum_{n=1}^{N} h_n \otimes x_n,$$

where :

1. $\{h_1, ..., h_N\}$ *is a finite orthonormal system of* H*,*

2. $x_1, ..., x_N$*, are arbitrary in* X*.*

Proposition 3.4
If $S \in \mathcal{L}(H, X)$ *is a finite rank operator, then* $S \in \gamma_p^\infty(H, X)$*.*

Definition 3.5 (γ-radonifying operators)
The space of γ*-radonifying operators in* $\mathcal{L}(H, X)$*, denoted by* $\gamma(H, E)$ *is defined as the closure of all finite rank operators in* $\mathcal{L}(H, X)$ *with respect to* $\gamma_p^\infty(H, E)$*-norm. While for* $R \in \gamma(H, E)$*, the* γ*-radonifying-norm is defined as :*

$$\|R\|_{\gamma(H,E)} := \|R\|_{\gamma_p^\infty(H,E)}.$$

Having introduced these necessary notions, we can start the process of constructing the stochastic integral in a Banach space X.

Definition 3.6
A function $f : [0, T] \to \mathcal{L}(H, X)$ *is said to be a step function if :*

$$f = 1_{(a,b)} \otimes (h \otimes x), \quad \text{where } 0 \leq a < b \leq T, \quad h \in H, \text{ and } x \in X$$

The stochastic integral for a step function f with respect to W_H is defined as :

$$\int_0^T f dW_H := W_H(1_{(a,b)} \otimes h) \otimes x = (W_H(b)h - W_H(a)h) \otimes x \in \mathbf{L}^2(\Omega; X).$$

Remark 3.7 *1. By linearity we can extend the integral to finite rank step functions.*

2. Any step function $f : [0,T] \mapsto \mathcal{L}(H,X)$ uniquely defines a bounded operator $R_f \in \mathcal{L}(\mathbf{L}^2(0,T;H),E)$, [19, 6.2] by :

$$R_f g := \int_0^T f(t)g(t)dt, \quad g \in \mathbf{L}^2(0,T;H).$$

The purpose of the first remark is to simplify the proofs when extending integrability to a larger class of $\mathcal{L}(H,X)$ operator-valued functions by working with finite rank step functions. The second remark is important because, as we will see later, defining the stochastic integral for a representative R_f of a function f is a useful approach for extending the stochastic integral. The next theorem generalizes the Ito-isometry for $\mathcal{L}(H,X)$ operator-valued finite rank functions and enables us to extend the stochastic integral to a broader class of $\mathcal{L}(H,X)$ operator-valued functions.

Theorem 3.8 (Ito-isometry)
For all finite rank step functions $f : [0,T] \to \mathcal{L}(H,X)$ we have :

1. $R_f \in \gamma(\mathbf{L}^2(0,T;H),X)$,

2. $\int_0^T f dW_H$ *is Gaussian,*

3. $\mathbb{E}\| \int_0^T f dW_H\|^2 = \|R_f\|^2_{\gamma(\mathbf{L}^2(0,T;H),X)}$.

Using Remark 3.7.2. and Theorem 3.8.3. we can define a linear mapping

$$J_T^{W_H} : R_f \mapsto \int_0^T f dW_H$$

which uniquely extends to an isometric embedding

$$J_T^{W_H} : \gamma(\mathbf{L}^2(0,T;H),X) \to L^2(\Omega;X).$$

Hence the stochastic integral for operators $R \in \gamma(\mathbf{L}^2(0,T;H),X)$ is well defined by $J_T^{W_H}(R)$.However, we are aiming to define the stochastic integral for a class of functions taking values in $\mathcal{L}(H,X)$.
Drawing from Remark 3.7.1., we can construct a broader collection of functions

by considering those $\mathcal{L}(H, X)$-valued functions f that correspond to an operator $R_f \in \gamma(\mathbf{L}^2(0, T; H), X)$. Yet, the means of selecting such a set of functions remains unclear. The subsequent definition characterizes stochastic integrability through the limit of finite rank step functions, while the ensuing theorem provides a resolution to the aforementioned inquiry.

Definition 3.9
A function $f : (0, T) \to \mathcal{L}(H, X)$ is said to be stochastically integrable with respect to W_H if there exists a sequence of finite rank step functions $f_n : (0, T) \to \mathcal{L}(H, X)$ such that :

1. *$\forall\ h \in H$, $\lim_{n\to\infty} f_n h = fh$ in measure;*
2. *$\exists$ an X-valued random variable I such that : $I = \lim_{n\to\infty} \int_0^T f_n dW_H$ in probability.*

Let X^* denote the dual of X, for $x^* \in X^*$,let $(f^* x^*)(t) := f^*(t)x^*$.

Theorem 3.10
A strongly measurable function $f : (0, T) \to \mathcal{L}(H, X)$ is stochastically integrable with respect to W_H if,

1. *$f^* x^* \in \mathbf{L}^2(0, T; H)$ for all $x^* \in X^*$,*
2. *There exists an operator $R \in \gamma(\mathbf{L}^2(0, T; H), X)$ such that for all $g \in \mathbf{L}^2(0, T; H)$ and $x^* \in E^*$ we have*

$$\langle Rg, x^* \rangle = \int_0^T \langle f(t)g(t), x^* \rangle \, dt.$$

As a consequence of the generalized Ito-Isometry Theorem 3.8, the "if" statement in the previous theorem can be replaced with and "iff" statement.

$\mathcal{L}(H, X)$ operator-valued processes

We now address the challenging task of defining the stochastic integral for operator-valued processes in $\mathcal{L}(H, X)$. However, this generalization cannot be applied to any Banach space and requires a specific geometric property known as the UMD property. While we won't delve into the technicalities of this property in this chapter, we refer the interested reader to Chapter 12 of [19] for more information. $\mathbf{L}^p$ spaces on any domain, satisfy this property [19, Theorem 12.4.,Definition 12.14]. Therefore, the results in this thesis will be stated for such spaces, but they can be generalized to Banach spaces that satisfy the UMD property. Similar to the previous construction, we begin with step processes, but instead of using an isometric mapping, we use an isomorphic one that can handle the integration of these processes. We then provide a generalized version of Theorem 3.10, which gives a sufficient condition for the integrability of such processes.

Definition 3.11
A function $f : (0,T) \times \Omega \to \mathcal{L}(H,X)$ is said to be a finite rank adapted step process with respect to the filtration $F = (\mathcal{F})_{t\in[0,T]}$ if,

$$f(t,\omega) = \sum_{m=1}^{M}\sum_{n=1}^{N} 1_{(t_{n-1},t_n)}(t) 1_{A_{mn}}(\omega) \sum_{j=1}^{k} h_j \otimes x_{jmn},$$

where :

1. $0 \le t_0 < ... < t_N \le T$,
2. *For $n = 1, ..., N$, the sets $A_{1n}, ..., A_{Mn} \subset \mathcal{F}_{t_{n-1}}$ are disjoint,*
3. *$h_1, ..., h_k \in H$ and $\forall j, m, n$ the vectors x_{jmn} belong to X.*

The stochastic integral for a finite rank adapted step process function f with respect to W_H is defined as

$$\int_0^T f(t)dW_H := \sum_{m=1}^{M}\sum_{n=1}^{N} 1_{A_{mn}} \sum_{j=1}^{k}(W_H(t_n)h_j - W_H(t_{n-1})h_j)x_{jmn}.$$

As in the previous case for $\mathcal{L}(H,X)$ operator-valued functions, we would like to construct a larger class of stochastically integrable processes.

Theorem 3.12 (Ito Isomorphism)
Let X be an $\mathbf{L}^q$ space and fix $1 < p < \infty$. For all finite rank adapted processes $f : (0,T) \times \Omega \to \mathcal{L}(H,X)$ we have :

$$\mathbb{E}\| \int_0^T f(t)dW_H\|^p \approx_{p,X} \|R_f\|^p_{\gamma(\mathbf{L}^2(0,T;H),X)}$$

Where " $\approx_{p,E}$ " means that the estimation is up to some constant depending only on p and X.

Definition 3.13
A process $f : (0,T) \times \Omega \to \mathcal{L}(H,X)$ is said to be $\mathbf{L}^p$-stochastically integrable with respect to W_H, if there exists a sequence of finite rank adapted processes $f_n : (0,T) \times \Omega \to \mathcal{L}(H,X)$ such that :

1. *$\forall\ h \in H$: $\lim_{n\to\infty} f_n h = fh$ in measure;*

2. $\exists$ *a random variable* $I \in \mathbf{L}^p(\Omega; X)$ *such that :*

$$I = \lim_{n\to\infty} \int_0^T f_n dW_H \text{ in } \mathbf{L}^p(\Omega; X).$$

Definition 3.14 (H-strongly measurable process)
A process $f : (0,T) \times \Omega \to \mathcal{L}(H, X)$ *is said to be* H*-strongly measurable if* $\forall h \in H$*, the process* $fh : (0,T) \times \Omega \to X$ *is strongly measurable.*

Theorem 3.15
Let X *be an* $\mathbf{L}^q$ *space and fix* $1 < p < \infty$*. An* H*-strongly measurable adapted process* $f : (0,T) \times \Omega \to \mathcal{L}(H, X)$*, is* L^p *stochastically integrable with respect to* W_H *if*

1. $f^* x^* \in \mathbf{L}^p(\Omega; \mathbf{L}^2(0,T; H))$ *for all* $x^* \in X^*$*,*
2. *There exists an operator* $R \in \mathbf{L}^p(\Omega; \gamma(\mathbf{L}^2(0,T;H), X))$ *such that :* $\forall g \in \mathbf{L}^2(0,T;H)$ *and* $x^* \in X^*$*,*

$$\langle Rg, x^* \rangle = \int_0^T \langle f(t)g(t), x^* \rangle \, dt, \text{ in } L^p(\Omega).$$

3.2 Linear and semilinear stochastic PDE

Linear stochastic equation with additive noise

Consider the following stochastic equation on the Banach state space X:

$$\begin{cases} dU(t) = AU(t)dt + BdW_H(t), \quad t \in [0,T], \\ U(0) = x. \end{cases} \tag{3.1}$$

Where :

1. W_H is a cylindrical Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$,
2. $B \in \gamma(H, X)$ is bounded,
3. $(A, D(A))$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X.

Definition 3.16
A weak solution of (3.1) is an X*-valued process* $\{U(t)\}_{t\in[0,T]}$ *which has a strongly measurable version with the following properties:*

1. *Almost surely, the paths* $t \mapsto U(t)$ *are integrable,*

2. $\forall t \in [0,T]$ *and* $x^* \in D(A^*)$ *we have almost surely,*

$$\langle U(t), x^* \rangle = \langle x, x^* \rangle + \int_0^t \langle U(s), A^* x^* \rangle \, ds + W_H(t) B^* x^*.$$

Theorem 3.17 (Existence and uniqueness,[19])
Let X be an L^p space with $p \in [2,\infty)$, $B \in \mathcal{L}(H,X)$, and $t \mapsto T(t)B$ is stochastically integrable.Then the stochastic equation 3.1 has a unique weak solution given by the stochastic convolution formula :

$$U(t) = T(t)x + \int_0^T T(t-s) B dW_H(s).$$

Proof. The proof follows from [19, Corollary 8.11], Theorem 8.6) and the fact that L^p satisfies the UMD-property. □

The uniqueness of the solution to the stochastic differential equation is a consequence of the geometric properties of the L^p space with $p \in [2,\infty)$. While the stochastic integrability of $t \mapsto T(t)B$ ensures the well-definedness of the stochastic convolution and hence the existence of a solution. Moreover, if $B \in \gamma(H,E)$, a strong solution can be obtained.

Linear stochastic equation with multiplicative noise

Consider the following stochastic equation on the state space $X := \mathbf{L}^p(D)$, where $(D, \mathcal{D}, \mu)$ is a σ-finite measure space and $p \in (1,\infty)$:

$$\begin{cases} dU(t) = AU(t)dt + B(U(t))dW_H(t), & t \in [0,T], \\ U(0) = u_0. \end{cases} \tag{3.2}$$

Where :

1. W_H is a cylindrical Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$,
2. $(A, D(A))$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X,
3. $B : \Omega \to \mathcal{H}, \mathcal{X})$.

For $\theta \geq 0$, consider the following space :

$V_\theta^p(\Omega; \gamma(\mathbf{L}^2(0,t), X)) := closure\{$ all finite rank adapted step processes
$f : (0,T) \times \Omega \to \mathcal{L}(H,X)$ such that : $s \mapsto (t-s)^{-\theta} f(s) \in \mathbf{L}^p(\Omega; \gamma(\mathbf{L}^2(0,t), X))\}$

The existence and uniqueness of 3.2 are established in $V_\theta^p(\Omega; \gamma(\mathbf{L}^2(0,t), E))$, because in this space, under the assumption that A generates an analytic C_0-semigroup, and $\theta < \frac{1}{2}$, then thanks to the geometry (Pisier's [19, 14.2], and UMD properties) of X, the stochastic integral would be well defined [19, Lemma 14.8].

Definition 3.18
A strongly measurable process $U : [0, T] \times \Omega :\to X$ is called a mild V_θ^p-solution of 3.2, if $U \in V_\theta^p(\Omega; \gamma(\mathbf{L}^2(0, t), X))$ and for all $t \in [0.T]$

$$U(t) = T(t)u_o + \int_0^t T(t-s)B(U(s))dW_H \textit{ almost surely} .$$

Theorem 3.19
For $\theta < \frac{1}{2}$. If

1. *A is the generator of an analytic C_0-semigroup,*
2. *$B : X \to \gamma(H, X)$ is Lipschitz continuous,*
3. *$u_0 \in \mathbf{L}^p(\Omega, \mathcal{F}_0; X)$.*

Then there exists a unique mild V_θ^p-solution U of 3.2.

Semilinear stochastic equation with multiplicative noise

Consider the following stochastic equation on the state space $X := \mathbf{L}^p(D)$, where $(D, \mathcal{D}, \mu)$ is a σ-finite measure space.

$$\begin{cases} dU(t) = (AU(t) + F(t, U(t)))dt + G(t, U(t))dW_H(t), & t \in [0, T_0] , \\ U(0) = u_0, \end{cases} \tag{3.3}$$

Before stating the existence and uniqueness result of this equation, we first introduce some terminology:

1. fix $X^0 := X$, and let B be a Banach space with norm $\|.\|$,
2. from [6, II,5], we know that if $(A, D(A))$ generates an analytic C_0-semigroup of contractions $(T(t))_{t\geq 0}$ on the Banach space X. Then $\forall\ \alpha \in (0, 1)$ the following fractional domains spaces are Banach spaces :
$$X^\alpha := D((-A)^\alpha), \quad \|v\|_\alpha := \|(-A)^\alpha v\|, \quad v \in D((-A)^\alpha),$$
3. for $u \in B$ we define the subdifferential of the norm at u as the set $\partial\|u\| := \{u^* \in B^* : \|u^*\| = 1 \text{ and } \langle u, u^*\rangle = 1\}$.

Theorem 3.20 (Existance and uniqueness theorem,[18])
Suppose that the following assumptions hold:

1. *(A,D(A)) is densely defined, closed and sectorial on X*

2. *For $\theta \in \left[0, \frac{1}{2}\right)$ we have continuous dense embedding : $X^\theta \hookrightarrow B \hookrightarrow X$*

3. *Let $(T(t))_{t\geq 0}$ be an analytic C_0-semigroup generated by $(A, D(A))$. Assume that $(T(t))_{t\geq 0}$ restricted to B i.e $T_{|B}$ is a contraction semigroup with dissipative generator $A_{|B}$*

4. *The map $F : [0,T] \times \Omega \times B \to B$ is locally Lipschitz continuous in the sense that for all $r > 0$, there exists a constant $L_F^{(r)}$ such that*

$$\|F(t,\omega,u) - F(t,\omega,v)\| \leq L_F^{(r)} \|u - v\|$$

   for all $\|u\|, \|v\| \leq r$ and $(t,\omega) \in [0,T] \times \Omega$ and there exists a constant $C_{F,0} \geq 0$ such that

$$\|F(t,\omega,0)\| \leq C_{F,0}, \quad t \in [0,T], \omega \in \Omega.$$

   Moreover, for all $u \in B$ the map $(t,\omega) \mapsto F(t,\omega,u)$ is strongly measurable and adapted. Finally, for suitable constants $a, b \geq 0$ and $N \geq 1$ we have

$$\langle Au + F(t, u + v), u^* \rangle \leq a(1 + \|v\|)^N + b\|u\|$$

   for all $u \in D\left(A|_B\right), v \in B$ and $u^* \in \partial\|u\|$.

5. *There exist constants $a'', b'', m' > 0$ such that the function $F : [0,T] \times \Omega \times B \to B$ satisfies*

$$\langle F(t,\omega,u+v) - F(t,\omega,v), u^* \rangle \leq a''(1 + \|v\|)^{m'} - b''\|u\|^{m'}$$

   for all $t \in [0,T], \omega \in \Omega, u, v \in B$ and $u^* \in \partial\|u\|$, and

$$\|F(t,v)\| \leq a''(1 + \|v\|)^{m'}$$

   for all $v \in B$

6. *Let $\gamma\left(H, X^{-\kappa_G}\right)$ denote the space of γ-radonifying operators from H to $X^{-\kappa_G}$ for some $0 \leq \kappa_G < \frac{1}{2}$. Then the map $G : [0,T] \times \Omega \times B \to \gamma\left(H, X^{-\kappa_G}\right)$ is locally Lipschitz continuous in the sense that for all $r > 0$, there exists a constant $L_G^{(r)}$ such that*

$$\|G(t,\omega,u) - G(t,\omega,v)\|_{\gamma(H,X^{-\kappa_G}).} \leq L_G^{(r)} \|u - v\|$$

   for all $\|u\|, \|v\| \leq r$ and $(t,\omega) \in [0,T] \times \Omega$. Moreover, for all $u \in B$ and $h \in H$ the map $(t,\omega) \mapsto G(t,\omega,u)h$ is strongly measurable and adapted. Finally, G

is of linear growth, that is, for suitable constant c',

$$\|G(t,\omega,u)\|_{\gamma(H,X^{-\kappa_G}).} \leq c'(1+\|u\|)$$

for all $(t,\omega,u) \in [0,T] \times \Omega \times B$.

7. $2 < q < \infty$, $0 \leq \theta < \frac{1}{2}$, *and* $0 \leq k_G < \frac{1}{2}$ *with* $\theta + k_G < \frac{1}{2} - \frac{1}{q}$

Then for all $u_0 \in \mathbf{L}^q(\Omega, \mathcal{F}_0, \mathcal{P}; B)$, *there exists a unique global mild solution* $U \in \mathbf{L}^q(\Omega, C([0,T]; B))$ *of 3.3.*

Chapter 4

Application

Let $G = (V, E, \Phi)$ be a connected, non-compact, and finite metric graph, with $|V| = n$ and $|E| = m$ such that E contains k directed edges parameterized on $[0, 1]$ and $s := m - k$ leads, we consider below (4.1), stochastic Allen-Cahn equations with multiplicative noise on each edge of the graph G, subject to continuity (b) and non-local Kirchhoff (c) boundary vertex conditions. As discussed in the introduction, the objective of this thesis is to investigate the existence and uniqueness of solutions to this evolutionary problem on the non-compact graph G. While the regularity of solutions could be analyzed, our primary focus in this thesis is on whether this stochastic problem is also well-posed as in the case of a compact graph [9]. It is important to note that the vertex conditions chosen are not arbitrary but have a direct impact on the well-posedness of the system. Specifically, the self-adjointness of the underlying differential operator, which in our case is the Laplacian, is closely related to the choice of vertex conditions. [3, Chapter 1, Theorem 1.4.4], provides three equivalent necessary and sufficient conditions that the vertex conditions must satisfy for the operator to be self-adjoint.

$$
\begin{cases}
\dot{u}_j(t,x) = & u_j''(t,x) - p_j(x)u_j(t,x) \\
& +\beta_j^2 u_j(t,x) - u_j(t,x) \\
& +g_j\left(t,x,u_j(t,x)\right)\frac{\partial w_j}{\partial t}(t,x), \quad t\in(0,\infty), \begin{cases} x\in(0,1) \text{ if } j\in\{1,\dots,k\} \\ x\in(0,\infty) \text{ if } j\in\{k+1,\dots,m\} \end{cases} & (a) \\
u_j\left(t,v_i\right) = & u_\ell\left(t,v_i\right) =: q_i(t), \qquad t\in(0,T], \forall j,\ell\in\Gamma\left(\mathrm{v}_i\right), i=1,\dots,n, & (b) \\
[Mq(t)]_i = & -\sum_{j=1}^m \phi_{ij}u_j'\left(t,v_i\right), \qquad t\in(0,T], i=1,\dots,n, & (c) \\
u_j(0,x) = & \mathrm{u}_j(x), \qquad \begin{cases} x\in[0,1] \text{ if } j\in\{1,\dots,k\} \\ x\in[0,\infty) \text{ if } j\in\{k+1,\dots,m\} \end{cases} & (d)
\end{cases}
\tag{4.1}
$$

The main objective of this chapter is to place the problem (4.1) within the framework of Theorem 3.20. However, we will see in the fourth section that with the methods we used this won't be possible, therefore starting from the end of the fourth section, we will only present the results for the compact part of G, by forsaking the leads, these results are already established in [9, Section 3]. The approach they used to the problem was by considering the well-posedness of a more general version of (4.1). Specifically, they focused on studying semilinear stochastic equations with multiplicative noise on a compact graph, while maintaining the same vertex conditions as in (4.1). By analyzing the well-posedness of this general version, they directly obtained the corresponding well-posedness results for (4.1) as a special case. Their methodology offers the advantage of providing greater control in satisfying the assumptions imposed by Theorem 3.20. Some of these assumptions pertain to the well-posedness of the deterministic counterpart of the problem, which is essentially a heat equation with a reaction term. Therefore, the first three sections of this chapter are dedicated to addressing the existence and uniqueness of solutions for the deterministic system on the non-compact graph G. By establishing the well-posedness of the deterministic problem, we lay the groundwork for subsequently applying Theorem 3.20 to the more general semilinear stochastic equations on the compact part of G.

4.1 Heat equation on a metric graph

Consider the following deterministic version of (4.1) which are heat equations on each edge of G with boundary of the same vertex conditions :

$$
\begin{cases}
\dot{u}_j(t,x) = & u_j''(t,x) - p_j(x)u_j(t,x) \qquad t \in (0,\infty), \begin{cases} x \in (0,1) \text{ if } j \in \{1,\dots,k\}, \\ x \in (0,\infty) \text{ if } j \in \{k+1,\dots,m\}, \end{cases} \quad (a) \\
u_j(t,v_i) = & u_\ell(t,v_i) =: q_i(t), \qquad t \in (0,\infty), \forall j,\ell \in \Gamma(v_i), i = 1,\dots,n, \quad (b) \\
[Mq(t)]_i = & -\sum_{j=1}^m \phi_{ij} u_j'(t,v_i), \qquad t \in (0,\infty), i = 1,\dots,n, \quad (c) \\
u_j(0,x) = & \mathrm{u}_j(x), \qquad \begin{cases} x \in [0,1] \text{ if } j \in \{1,\dots,k\}, \\ x \in [0,\infty) \text{ if } j \in \{k+1,\dots,m\}. \end{cases} \quad (d)
\end{cases}
\tag{4.2}
$$

Explanation of the equations in (HE)

1. (a) The heat equation with a reaction term posed on edges and leads, where

$$
\dot{u}_j(t,x) = \frac{\partial u_j}{\partial t}(.,x)\ ,\ \left(u_j''\right)(t,x) = \frac{\partial^2 u_j}{\partial^2 x}(t,.)
$$

   and $p = [p_1, ..., p_m]^T$ is a non-negative continuous function on the edges of G.

2. (b) Is the vertex continuity condition mentioned in Definition 2.6, and we use the same notation $q = [q_1, ..., q_n]^T$ to denote the common values of u at all the vertices.

3. (c) Is the non-local Kirchhoff condition at each vertex v_i similar to 2.2, the difference here is that we consider the derivatives of the flow, and the coefficients are given by $M = (b_{ij})_{n\times n}$ which is a real, symmetric, and negative semidefinite matrix.

4. (d) The initial conditions of each equation on the edges.

4.2 Boundary spaces, operators, and the ACP

In this section, our goal is to reframe the deterministic problem (4.2) as an abstract Cauchy problem (ACP) with an operator $(A, D(A))$. We will approach the solution of this problem in the setting of Hilbert spaces. Due to the presence of the differential operator in (4.2), it is appropriate to work with weak derivatives and consider $\mathbf{L}^2$ functions on the graph G. Consequently, we define the state space for the evolution system as follows:

$$
\mathbf{E}_2 = \underset{\sim}{\mathbf{E}}_2 \times \tilde{\mathbf{E}}_2 := (\mathbf{L}^2(0,1))^k \times (\mathbf{L}^2(0,\infty))^s.
$$

Proposition 4.1
$\mathbf{E}_2$ *is a Hilbert space with the natural inner product:*

$$\langle u, w\rangle_{\mathbf{E}_2} := \sum_{j=1}^{k}\int_0^1 u_j(x)w_j(x)dx + \sum_{j=k+1}^{m}\int_0^{\infty} u_j(x)w_j(x)dx, \quad u, w \in \mathbf{E}_2.$$

Proof. $\mathbf{L}^2(0,1)$ and $\mathbf{L}^2(0,\infty)$ are Hilbert spaces with their natural inner product, using ([4], Chapter 1, Definition 6.1) the proof follows. $\square$

The way we will convert the (4.2) into an ACP, is to first consider a maximal operator A_{max}, then we will write the vertex condition into a feedback operator, and incorporate this feedback into A_{max} by domain perturbation of $D(A_{max})$.

Maximal operator

Definition 4.2
Consider the space of continuous functions vanishing at infinity denoted by $C_0([0,\infty))$, *defined as the restriction of the set of continuous functions with compact support* $C_c(\mathbb{R})$ *to* $[0,\infty)$ *i.e :*

$$C_0([0,\infty)) := C_c(\mathbb{R})_{|[0,\infty)}$$

Let's first introduce the continuity boundary operator L defined by:

$$\begin{cases} D(L) := \left\{\; u \in (C[0,1])^k \times (C_0([0,\infty)))^s : u_j(v_i) = u_l(v_i) \quad \forall j, l \in \Gamma(v_i), i = 1, ..., n, \right\} \\ \quad Lu := (q_1, \cdots, q_n)^\top = q \in \mathbb{R}^n; q_i = u_j(v_i) \text{ for some } j \in \Gamma(v_i), i = 1, ..., n. \end{cases}$$

Let's define now the maximal operator A_{max} on $\mathbf{E}_2$ as :

$$\begin{cases} D(A_{\max}) := (H^2(0,1))^k \times (H^2(0,\infty))^s \cap D(L), \\ A_{\max} := \begin{pmatrix} \frac{\partial^2}{\partial x^2} - p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2}{\partial x^2} - p_m \end{pmatrix}_{(m\times m)} . \end{cases}$$

Abstract Cauchy problem(ACP)

Consider the **feedback operator**, defined in the following way :

$$\begin{cases} D(C) := D(A_{\max}), \\ Cu := -\tilde{\Phi}^{+}u'(0) - \underset{\sim}{\Phi}^{+}u'(0) + \underset{\sim}{\Phi}^{-}u'(1), \text{ see Definition 2.5.} \end{cases}$$

By perturbing $D(A_{max})$ with the feedback operator we can reformulate the (HE) into an ACP in the following way :

$$(ACP)\begin{cases} \dot{u}(t) = Au(t) \quad t > 0; \\ u(0) = \mathrm{u}^0 = (\mathrm{u}_1^0, \dots, \mathrm{u}_m^0)^\top, \end{cases}$$

where :

$$\begin{cases} A := A_{\max}, \\ D(A) := \{u \in D(A_{max}) : MLu = Cu\} \quad . \end{cases} \tag{4.3}$$

4.3 Well-posedness of the abstract Cauchy problem

The conversion of (4.2) to the (ACP), is in fact a conversion from the metric graphs setting to the framework of semigroup theory. For the (ACP) , we are now able to use the theory from Chapter 1 to treat the well-posedness of the (4.2). There are many approaches to studying the properties of the operator $(A, D(A))$ from (ACP). In our case, we will use forms methods. We will first define a bilinear form $\mathfrak{a}$ and its associated operator B, and prove that $\mathfrak{a}$ is symmetric and satisfies the DACC properties (see Definition 1.17), Showing that in fact B is nothing but the operator A from the (ACP), and using Theorem 1.22, we can prove the well-posedness. The results proven in the sequel are generalizations to results found in [9, Section 2] while the technical and detailed parts can be found in [13]and [14].

Consider the bilinear form $\mathfrak{a}$ defined on $\mathbf{E}_2$ by:

$$\begin{cases} \mathfrak{a}(u,w) = \sum_{j=1}^{k} \int_0^1 (u_j' w_j' + p_j u_j w_j) dx + \sum_{j=k+1}^{m} \int_0^\infty (u_j' w_j' + p_j u_j w_j) dx \\ \qquad\qquad - \sum_{i,h=1}^{n} b_{ih} q_h r_i, \\ D(\mathfrak{a}) := V := \left(H^1(0,1)\right)^k \times \left(H^1(0,\infty)\right)^s \cap D(L), \\ \text{ where } Lu = q \text{ and } Lw = r. \end{cases} \tag{4.4}$$

From $\mathfrak{a}$ we define its associated operator $(B, D(B))$ by :

$$\begin{cases} D(B) := \{u \in V : \exists w \in \mathbf{E}_2 \text{ such that} : \mathfrak{a}(u,\phi) = \langle w, \phi \rangle_{\mathcal{E}_2} \quad \forall \phi \in V\} \\ \quad Bu := -w, \text{ see Definition 1.19.} \end{cases} \tag{4.5}$$

Proposition 4.3
The associated operator $(B, D(B))$ of $\mathfrak{a}$ is $(A, D(A))$ in the (ACP).

Proof. Let $u \in D(A)$ then $\forall\ w \in V$ we have :

$$\mathfrak{a}(u,w) = \sum_{j=1}^{k} \int_0^1 u_j' w_j' dx + \sum_{j=k+1}^{m} \int_0^\infty u_j' w_j' dx + \sum_{j=1}^{k} \int_0^1 p_j u_j w_j dx$$

$$+\sum_{j=k+1}^{m}\int_0^\infty p_j u_j w_j dx - \sum_{i,h=1}^{n} b_{ih} q_h r_i.$$

Consider these individual parts:

$$(a) := \sum_{j=1}^{k}\int_0^1 u_j' w_j' dx, \quad (b) = \sum_{j=k+1}^{m}\int_0^\infty u_j' w_j' dx, \quad (c) = \sum_{i,h=1}^{n} b_{ih} q_h r_i,$$

$$(d) = \sum_{j=1}^{k}\int_0^1 p_j u_j w_j dx + \sum_{j=k+1}^{m}\int_0^\infty p_j u_j w_j dx,$$

then integrating by parts we have :

$$(a) = \sum_{j=1}^{k}\left[u_j' w_j\right]_0^1 - \sum_{j=1}^{k}\int_0^1 u_j'' w_j dx, \quad (b) = \sum_{j=k+1}^{m}\left[u_j' w_j\right]_0^\infty - \sum_{j=k+1}^{m}\int_0^\infty u_j'' w_j dx,$$

$$(c) = \langle Mq, r\rangle_{\mathbb{R}^n}.$$

From [20, Definition 2.2.2] we have :

$$w_j(0) = \sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{+} r_i, \quad \text{and} \quad w_j(1) = \sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{-} r_i,$$

then

$$\left[u_j' w_j\right]_0^1 = u_j'(1) w_j(1) - u_j'(0) w_j(0) = u_j'(1)\sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{-} r_i - u_j'(0)\sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{+} r_i$$

$$= \sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{-} u_j'(1) r_i - \sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{+} u_j'(0) r_i,$$

the edge e_j can only have one terminal vertex, this means that the coefficients $\underset{\sim}{\phi}_{ij}^{-}$ for fixed j are all zeros expect at one vertex, that is why inside the sum, $\sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{-} u_j'(1) r_i$ we can replace $u_j'(1)$ by $u_j'(v_1)$, and this is the same for $\sum_{i=1}^{n} \underset{\sim}{\phi}_{ij}^{+} u_j'(0) r_i$, hence :

$$\left[u_j' w_j\right]_0^1 = \sum_{i=1}^{n} (\underset{\sim}{\phi}_{ij}^{-} - \underset{\sim}{\phi}_{ij}^{+}) r_i u_j'(v_i).$$

Suppose tha $w \in (C_0([0,\infty]))^s$ this implies that

$$\left[u_j' w_j\right]_0^\infty = -u_j'(0) w_j(0) = -\sum_{i=1}^{n} \tilde{\phi}_{ij}^{+} r_i u_j'(v_i).$$

Hence we have :

$$\begin{cases} \sum_{j=1}^{k}\left[u_j'w_j\right]_0^1 = \sum_{i=1}^{n} r_i \sum_{j=1}^{k}(\underset{\sim}{\phi}_{ij}^{-} - \underset{\sim}{\phi}_{ij}^{+})u_j'(v_i), \\ \sum_{j=k+1}^{s}\left[u_j'w_j\right]_0^\infty = -\sum_{i=1}^{n} r_i \sum_{j=k+1}^{m} \tilde{\phi}_{ij}^{+}u_j'(v_i), \end{cases}$$

$$\implies \sum_{j=1}^{k}\left[u_j'w_j\right]_0^1 + \sum_{j=k+1}^{m}\left[u_j'w_j\right]_0^\infty = \sum_{i=1}^{n} r_i \sum_{j=1}^{m}(\underset{\sim}{\phi}_{ij}^{-} - \underset{\sim}{\phi}_{ij}^{+} - \tilde{\phi}_{ij}^{+})u_j'(v_i),$$

from the Kirchhoff condition in matrix form we have:

$$\sum_{j=1}^{m}(\underset{\sim}{\phi}_{ij}^{-} - \underset{\sim}{\phi}_{ij}^{+} - \tilde{\phi}_{ij}^{+})u_j'(v_i) = \sum_{h=1}^{n} b_{ih}q_h,$$

i.e it is equal to i-th coordinate of the vector $[Mq]$,

$$\implies \sum_{j=1}^{k}\left[u_j'w_j\right]_0^1 + \sum_{j=k+1}^{m}\left[u_j'w_j\right]_0^\infty = \sum_{i=1}^{n}\sum_{h=1}^{n} b_{ih}q_h r_i \quad \text{which is equal to (c)}$$

$$\implies (a)+(b)+(c)+(d) = -\Big[\sum_{j=1}^{k}\int_0^1 u_j''w_j dx + \sum_{j=k+1}^{m}\int_0^\infty u_j''w_j dx\Big]$$

$$+\Big[\sum_{j=1}^{k}\int_0^1 p_ju_jw_j dx + \sum_{j=k+1}^{m}\int_0^\infty p_ju_jw_j dx\Big],$$

$$= -\Big[\sum_{j=1}^{k}\int_0^1 (u_j'' - p_ju_j)w_j dx + \sum_{j=k+1}^{m}\int_0^\infty (u_j'' - p_ju_j)w_j dx\Big],$$

$$\implies \mathfrak{a}(u,w) = -\langle Au, w\rangle_{\mathbf{E_2}}$$

It is now sufficient to have that $(C_0([0,\infty)))^s$ is dense in $(H^1(0,\infty))^s$ in the H^1-norm, then the previous equality holds for every $w \in V$. This density is true [1, Theorem 3.22].

For the converse statement, we refer to [13, Lemma 3.4], because the arguments used are the same for non-compact graphs. □

Now, that we proved that $(A, D(A))$ from (ACP) is actually the associated operator $(B, D(B))$ to the form $\mathfrak{a}$. This means that instead of studying the properties of $(A, D(A))$ we can instead show that $\mathfrak{a}$ is DACC and then possibly use the theory developed in the first chapter.

Proposition 4.4

$\mathfrak{a}$ is densely defined, continuous, accretive, closed, and symmetric.

Proof. Densely defined: We have $V := (H^1(0,1))^k \times (H^1(0,\infty))^s \cap D(L)$.

The density for $V_1 := (H^1(0,1))^k \cap D(L)$ in $L^2((0,1))^k$ was established in [13, Lemma 3.1], but it also holds for $V_2 := (H^1(0,\infty))^s \cap D(L)$ in $L^2((0,\infty))^s$, since $(C_c^\infty(0,\infty))$ contains functions having compact support in $(0,\infty)$ which is a segment domain [1, Theorem 3.22], then we have the density inclusion $(C_c^\infty(0,\infty))^s \subset V_2 \subset (\mathbf{L}^2(0,\infty))^s$. Therefore the cartesian product of $V = V_1 \times V_2$ will be dense in $\mathbf{E}_2$.

Accretive: By assumption of M we have : $\sum_{i,h=1}^n b_{ih} q_i q_h \leq 0$

$$\implies -\sum_{i,h=1}^{n} b_{ih} q_i q_h \geq 0$$

$$\implies \mathfrak{a}(u,u) \geq 0 \implies \mathfrak{a} \quad \text{ is accretive.}$$

Symmetric: $\mathfrak{a}$ is real-valued $\implies$ $\mathfrak{a}$ is symmetric.

Closed: we have $V := (H^1(0,1))^k \times (H^1(0,\infty))^s \cap D(L)$, denote by

$$H := (H^1(0,1))^k \times (H^1(0,\infty))^s,$$

notice that V is a Hilbert space with the natural inner product :

$$\langle u, w\rangle_H := \langle u, w\rangle_{(H^1(0,1))^k} + \langle u, w\rangle_{(H^1(0,\infty))^s},$$

where

$$\begin{cases} \langle u, w\rangle_{(H^1(0,1))^k} := \sum_{j=1}^k \int_0^1 (u_j' w_j' + u_j w_j) dx, \\ \langle u, w\rangle_{(H^1(0,\infty))^s} := \sum_{j=k+1}^m \int_0^\infty (u_j' w_j' + u_j w_j) dx, \end{cases}$$

from [13, Lemma 3.1] we have that $\langle u, w\rangle_{(H^1(0,1))^k}$ is equivalent to the following inner product:

$$\langle u, w\rangle_{V_1} := \sum_{j=1}^{k} \int_0^1 u_j' w_j' dx \quad \text{for} \quad u, w \in V_1.$$

Proof for this can be found in [13, Lemma 3.1] where they used the Poincare inequality to prove it, but this inequality also holds for domains bounded in one direction, i.e it holds for $(0,\infty)$, [12, Theorem 12.17]. Therefore we also have that $\langle u, w\rangle_{(H^1(0,\infty))^s}$ is equivalent to the inner product:

$$\langle u, w\rangle_{V_2} := \sum_{j=k+1}^{m} \int_0^\infty u_j' w_j' dx \quad \text{for} \quad u, w \in V_2.$$

Therefore $\langle u, w\rangle_H$ is equivalent to $\langle u, w\rangle_V := \langle u, w\rangle_{V_1} + \langle u, w\rangle_{V_2}$.

Now recall that the form $\mathfrak{a}$ is defined on $\mathbf{E}_2$ by:

$$\begin{cases} \mathfrak{a}(u,w) = \sum_{j=1}^{k}\int_0^1 (u_j'w_j' + p_j u_j w_j)dx + \sum_{j=k+1}^{m}\int_0^\infty (u_j'w_j' + p_j u_j w_j)dx - \sum_{i,h=1}^{n} b_{ih}q_h r_i \\ D(\mathfrak{a}) := V \end{cases}$$

$$\Longrightarrow \mathfrak{a}(u,w) = \langle u,w\rangle_V + \sum_{j=1}^{k}\int_0^1 p_j u_j w_j dx + \sum_{j=k+1}^{m}\int_0^\infty p_j u_j w_j dx - \sum_{i,h=1}^{n} b_{ih}q_h r_i$$

We need now to show that V is complete with the norm $\|.\|_{\mathfrak{a}}$. It is sufficient to show that $\|.\|_{\mathfrak{a}}$ is equivalent to $\|.\|_V$ because we know that $(V = D(\mathfrak{a}), \|.\|_V)$ is complete. By definition, we have : $\|u\|_{\mathfrak{a}}^2 := \mathfrak{a}(u,u) + \|u\|_{\mathbf{E}_2}^2$

$$\begin{aligned} \Longrightarrow \|u\|_{\mathfrak{a}}^2 &= \sum_{j=1}^{k}\int_0^1 u_j'^2 dx + \sum_{j=k+1}^{m}\int_0^\infty u_j'^2 dx + \sum_{j=1}^{k}\int_0^1 u_j^2 dx + \sum_{j=k+1}^{m}\int_0^\infty u_j^2 dx \\ &\quad - \sum_{i,h=1}^{n} b_{ih}q_h q_i + \sum_{j=1}^{k}\int_0^1 p_j u_j^2 dx + \sum_{j=k+1}^{m}\int_0^\infty p_j u_j^2 dx, \\ &= \sum_{j=1}^{k}\int_0^1 (u_j'^2 + u_j^2)dx + \sum_{j=k+1}^{m}\int_0^\infty (u_j'^2 + u_j^2)dx + \sum_{j=1}^{k}\int_0^1 p_j u_j^2 dx \\ &\quad + \sum_{j=k+1}^{m}\int_0^\infty p_j u_j^2 dx - \sum_{i,h=1}^{n} b_{ih}q_h q_i, \\ &= \langle u,u\rangle_H - \sum_{i,h=1}^{n} b_{ih}q_h q_i + \sum_{j=1}^{k}\int_0^1 p_j u_j^2 dx + \sum_{j=k+1}^{m}\int_0^\infty p_j u_j^2 dx. \end{aligned}$$

Because $H^1(0,1)$ and $H^1(0,\infty)$ are continuously embedded in $C[0,1]$ and $C_0([0,\infty))$ respectively [1, Theorem 4.2], we have :

$$|q_i| \le \max\Big(\max_{1\leqslant j\leqslant k}\max_{x\in[0,1]}|u_j(x)|; \max_{k+1\leqslant j\leqslant m}\max_{x\in[0,\infty)}|u_j(x)|\Big),$$

$$\Longrightarrow |q_i| \le \max_{1\leqslant j\leqslant k}\max_{x\in[0,1]}|u_j(x)| + \max_{k+1\leqslant j\leqslant m}\max_{x\in[0,\infty)}|u_j(x)|),$$

$$\Longrightarrow |q_i| \le c_1(\max_{1\leqslant j\leqslant k}\|u_j\|_{H^1(0,1)}) + c_2(\max_{k+1\leqslant j\leqslant m}\|u_j\|_{H^1(0,\infty)}),$$

where c_1 and c_2 are two constant in $\mathbb{R}$. Now, take $c_3 := \max(c_1, c_2)$, then we have

$$|q_i| \le c_3\left(\sum_{j=1}^{k}\|u_j\|_{H^1(0,1)} + \sum_{j=k+1}^{m}\|u_j\|_{H^1(0,\infty)}\right),$$

By definition we have: $\sum_{j=1}^{k}\|u_j\|_{H^1(0,1)} =: \|u\|_{(H^1(0,1))^k}$ and

$$\sum_{j=k+1}^{m} \|u_j\|_{H^1(0,\infty)} =: \|u\|_{(H^1(0,\infty))^s},$$

$$\implies |q_i| \le c_3 \|u\|_H,$$

but since $\|u\|_H$ is equivalent to $\|u\|_V$, this implies that $\exists N \in \mathbb{R}$ such that $|q_i| \le N\|u\|_V$.

We have: $\|u\|_{\mathfrak{a}}^2 = \langle u, u\rangle_H - \sum_{i,h=1}^{n} b_{ih} q_h q_i + \sum_{j=1}^{k} \int_0^1 p_j u_j^2 dx + \sum_{j=k+1}^{m} \int_0^\infty p_j u_j^2 dx.$

Then we have :

1. $\exists M \in \mathbb{R} : \langle u, u\rangle_H \le M^2 \|u\|_V^2$
2. Set $P := \max_{j=1,\dots,m}\{p_j\} \implies \sum_{j=1}^{k} \int_0^1 p_j u_j^2 dx + \sum_{j=k+1}^{m} \int_0^\infty p_j u_j^2 dx \le P\|u\|_{\mathbf{E}_2}^2$. $H^2(0,1)$ and $H^2(0,\infty)$ are continuously embedded respectively in $L^2(0,1)$ and $L^2(0,\infty)$, therefore it follows that $\exists C \in \mathbb{R}$ such that:

$$\sum_{j=1}^{k} \int_0^1 p_j u_j^2 dx + \sum_{j=k+1}^{m} \int_0^\infty p_j u_j^2 dx \le C\|u\|_V^2.$$

3. $\exists N \in \mathbb{R} : -\sum_{i,h=1}^{n} b_{ih} q_h q_i \le \sum_{i,h=1}^{n} |b_{ih}||q_h||q_i| \le \sum_{i,h=1}^{n} |b_{ih}| N^2 \|u\|_V^2$.

$$\text{Set } B := \sum_{i,h=1}^{n} |b_{ih}| N^2 \implies -\sum_{i,h=1}^{n} b_{ih} q_h q_i \le B\|u\|_V^2.$$

All the parts are estimated from above by $\|u\|_V^2$, i.e

$$\exists Q \in \mathbb{R} : \|u\|_{\mathfrak{a}}^2 \le Q\|u\|_V^2$$

Since $-\sum_{i,h=1}^{n} b_{ih} q_h r_i$ and all p_j are non-negative,

$$\implies \|u\|_{\mathfrak{a}}^2 - \langle u, u\rangle_H \ge 0 \implies \|u\|_{\mathfrak{a}}^2 \ge M^2 \|u\|_V^2$$

$$\implies \|u\|_{\mathfrak{a}} \text{ is equivalent to } \|u\|_V \implies \mathfrak{a} \text{ is closed in } \mathbf{E}_2.$$

Continuity: Let $u, w \in V$ we have:

$$|\mathfrak{a}(u,w)| \le |\sum_{j=1}^{k} \int_0^1 u_j' w_j' dx + \sum_{j=k+1}^{m} \int_0^\infty u_j' w_j' dx| + \sum_{i,h=1}^{n} |b_{in}||q_h||r_i|$$

$$+|\sum_{j=1}^{k} \int_0^1 p_j u_j w_j dx + \sum_{j=k+1}^{m} \int_0^\infty p_j u_j w_j dx|$$

$\le |\langle u, w\rangle_V| + B\|u\|_V\|w\|_V + C\|u\|_V\|w\|_V \le \|u\|_V\|w\|_V + B\|u\|_V\|w\|_V + C\|u\|_V\|w\|_V$

$$= R\|u\|_V\|w\|_V$$

Where $R = (1+B+C)$ and we used the Cauchy-Schwarz inequality in the last step. Therefore the form is continuous. $\square$

Proposition 4.5

$(A, D(A))$ is densely defined, dissipative, self-adjoint, and generates an analytic contractions C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbf{E}_2$.

Proof. From Proposition 4.4 we have $\mathfrak{a}$ is DACC and symmetric, from Proposition 4.3, $(A, D(A))$ is the associated operator to $\mathfrak{a}$. Using Proposition1.20 we have $(A, D(A))$ being densely defined and dissipative. $\mathfrak{a}$ is symetric, hence $(A, D(A))$ is self adjoint from Proposition1.21.Using the fact that a densely defined self-adjoint operator is closed, it follows then from Theorem1.22 That $(A, D(A))$ generates an analytic contraction C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbf{E}_2$ $\square$

Corollary 4.6

The (ACP) is well-posed on $\mathbf{E}_2$.

Proof. The Proof follows from the previous proposition and Theorem 1.8. $\square$

These results proven previously are generalizations of the results found in [9, Section 2] to the case of a non-compact graph, this is to say that in the case of the compact part of G, by deleting the leads, the (ACP) and therefore the deterministic problem (4.2) is well-posed, in this compact case on the state space:

$$\mathcal{E} := \prod_{j=1}^{k} \mathbf{L}^2(0,1),$$

with the natural norm

$$\|u\|_{\mathcal{E}}^p := \sum_{j=1}^{k} \|u_j\|_{\mathbf{L}^2(0,1)}^2.$$

4.4 Well-posedness on $\mathbf{L}^p$

After establishing the well-posedness of the deterministic version (4.2), we proceed to the second step of our analysis. In Theorem 3.20, there are additional assumptions concerning the well-posedness not only in $\mathbf{L}^2$ spaces but also in $\mathbf{L}^p$ spaces. Therefore, in this section, our objective is to prove the existence and uniqueness of solutions for the cases of $\mathbf{L}^p$. To achieve this, we impose certain restrictions on the matrix associated with the non-local Kirchhoff law. By carefully selecting these restrictions, we can utilize established extension results that enable us to extend the well-posedness results from $\mathbf{L}^2$ spaces to $\mathbf{L}^p$ spaces. These extension results ensure that the solutions obtained in $\mathbf{L}^2$ also belong to $\mathbf{L}^p$ spaces and satisfy the necessary properties for well-posedness. Unfortunately, with our methodology for generalizing the results to the case of the non-compact graph G, a crucial part of the next result which concerns knowing the domain of the generators of the extended semigroup on the $\mathbf{L}^p$ spaces doesn't hold in the case of our non-compact graph G. Consider the following $\mathbf{L}^p$ state space :

$$\mathbf{E}_p := \prod_{j=1}^{k} \mathbf{L}^p(0,1) \times \prod_{j=k+1}^{m} \mathbf{L}^p(0,\infty) \quad p \in [1,\infty] \tag{4.6}$$

With the natural norms

$$\|u\|_{\mathbf{E}_p}^p := \sum_{j=1}^{k} \|u_j\|_{\mathbf{L}^p(0,1)}^p + \sum_{j=k+1}^{m} \|u_j\|_{\mathbf{L}^p(0,\infty)}^p, \quad u \in \mathbf{E}_p \quad p \in [1,\infty).$$

Let's pose some restrictions on the matrix M by assuming the followings :

- $M = (b_{ij})_{n\times n}$ is real, symmetric, and negative semi-definite ,
- M has positive off-diagonal: $i \neq k$, $b_{ik} \geq 0$,
- M is diagonally dominant : $\sum_{k\neq i} b_{ik} \leq -b_{ii}$, $i = 1, ..., n$.

Proposition 4.7
With the above assumptions , $(T_2(t))_{t\geq 0}$ extends to a family of contraction C_0-semigroup $(T_p(t))_{t\geq 0}$ on $\mathbf{E}_p$, for $1 \leq p < \infty$, and analytic for $p \in (1,\infty)$.

Proof. With the above assumption on the matrix and $\mathfrak{a}$ being symmetric and DACC, the extension follows from [13, Theorem 4.1] and [2, 7.2.2] since the arguments use were independent of the domain. □

With our current methods, we have reached the limit of our analysis concerning the extended semigroups $(T_p(t)_{t\geq 0})$. However, this is insufficient to address the issue of well-posedness in the stochastic problem. The reason for this lies in the fact that Theorem 3.20 imposes certain assumptions regarding the domain of the generators of the semigroups $(T_p(t))_{t\geq 0}$. In [13, Proposition 4.6], for the compact part of G, we do find information regarding the domain of the generators A_p of $(T_p(t))_{t\geq 0}$. They are given by:

$$D\left(A_p\right) := \left\{ u \in (\prod_{j=1}^{k} W^{2,p}(0,1)) \cap D(L) : MLu = Cu \right\}.$$

Where $W^{2,p}$ denotes the Sobolev space which consists of functions with weak derivatives up to order 2 that belong to the Lebesgue space $\mathbf{L}^p$.

However, the validity of the statement concerning the domains $D(A_p)$ remains uncertain for the non-compact graph G. This discrepancy arises from the proof presented in [13, Proposition 4.6], where an embedding inclusion is utilized between $\mathbf{L}^p$ spaces for different values of p. While this embedding holds for finite measure domains, such as $(0,1)$, it is not possible for the half-interval $(0,\infty)$, as it has an infinite measure. Thus, in the case of the non-compact graph G, our current understanding is limited, and we can only speculate about the generators A_p of the extended semigroups, for instance, by conjecturing that the generators A_p of the extended semigroups in the non-compact case, are given by :

$$D\left(A_p\right) := \left\{ u \in (\prod_{j=1}^{k} W^{2,p}(0,1)) \times (\prod_{j=k+1}^{m} W^{2,p}(0,\infty)) \cap D(L) : MLu = Cu \right\}.$$

This conjecture goes beyond mere superficiality and possesses substantial depth. The substantiveness arises from the investigation of a similar problem in [7], where the problem was explored on non-compact spaces using a different approach to establish well-posedness. Nevertheless, moving forward, we will solely now focus on the compact part of graph G. Although we will continue to refer to it as G for convenience, it is important to note that G has now become a compact graph, retaining the same vertices but with only k directed edges.

From now on, we will work on the state space

$$\mathcal{E} = \prod_{j=1}^{k} \mathbf{L}^2(0,1)$$

and,

$$\mathcal{E}_p = \prod_{j=1}^{k} \mathbf{L}^p(0,1), \tag{4.7}$$

with the natural norms

$$\|u\|_{\mathcal{E}_p}^p := \sum_{j=1}^{k} \|u_j\|_{\mathbf{L}^p(0,1)}^p.$$

All subsequent results from this point onward are derived exclusively from [9].

Proposition 4.8

With the above assumptions , $(T_2(t))_{t\geq 0}$ extends to a family of contraction C_0-semigroup $(T_p(t))_{t\geq 0}$ on $\mathcal{E}_p$, for $1 \leq p < \infty$, and analytic for $p \in (1,\infty)$. Moreover *the generator A_p of $(T_p(t))_{t\geq 0}$ is given by :*

$$\begin{cases} D\left(A_p\right) := \left\{ u \in (\prod_{j=1}^{k} W^{2,p}(0,1)) \cap D(L) : MLu = Cu \right\}, \\ \\ A_p := \begin{pmatrix} \frac{d^2}{dx^2} - p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{d^2}{dx^2} - p_m \end{pmatrix}_{(m\times m)} . \end{cases}$$

Corollary 4.9
The (ACP) is well-posed on $\mathcal{E}_p$ $\forall$ $p \in [1, \infty)$.

4.5 Semilinear stochastic system on a metric graph

In this section is mainly a presentation of already established results [9, Section 3]. As mentioned previously, one aspect of their analysis methodology is to investigate the well-posedness of a semilinear stochastic PDE with multiplicative noise on the compact graph G, considering the same vertex conditions as in the main problem (4.1), this general problem is given by the following equations :

$$\text{(SS:)} \begin{cases} \dot{u}_j(t,x) = & u_j''(t,x) - p_j(x)u_j(t,x) \\ & +f_j\left(t,x,u_j(t,x)\right) \\ & +g_j\left(t,x,u_j(t,x)\right)\frac{\partial w_j}{\partial t}(t,x), \quad t\in(0,\infty), x\in(0,1) \text{ if } j\in\{1,\dots,k\} \quad (a) \\ \\ u_j\left(t,v_i\right) = & u_\ell\left(t,v_i\right) =: q_i(t), \qquad t\in(0,T], \forall j,\ell\in\Gamma\left(v_i\right), i=1,\dots,n, \quad (b) \\ \\ [Mq(t)]_i = & -\sum_{j=1}^{m}\phi_{ij}u_j'\left(t,v_i\right), \qquad t\in(0,T], i=1,\dots,n, \quad (c) \\ \\ u_j(0,x) = & \mathrm{u}_j(x), \qquad x\in[0,1] \text{ if } j\in\{1,\dots,k\} \quad (d) \end{cases} \tag{4.8}$$

Where the equations b), c), and d) are the same as explained in the deterministic system (4.2). While equation a), represents a reaction-diffusion with potentials f_j and multiplicative noise g_j.

To address the well-posedness of this general case, they adopted a similar approach to what we did in the first three sections. Specifically, they reformulate the semilinear stochastic PDE (4.8) into an abstract semilinear stochastic PDE with multiplicative noise (3.3) :

$$\begin{cases} dU(t) = (AU(t) + F(t,U(t)))dt + G(t,U(t))dW_H(t), \quad t\in[0,T_0], \\ U(0) = u_0, \end{cases} \tag{4.9}$$

This reformulation requires us to construct the operators F, G, and W. Once these operators are constructed, their aim is to apply Theorem 3.20 by verifying that this theorem's assumptions also hold for the reformulated abstract stochastic problem. Throughout the construction of the operators F, G, and W, we simultaneously present their established results concerning the assumptions stated in Theorem 3.20, while for proofs we either provide a sketch, a brief explanation or refer to [9, Section 3].

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space equipped with a right continuous filtration $\mathbf{F} := (\mathcal{F})_{t\in[0,T]}$, where $T \in \mathbb{R}^{+}$.
We denote by $\mathcal{B}_I$ the Borel-sigma algebra generated by an interval $I \subset \mathbb{R}$, and by $\mathbb{Z}$ the set of integers.
By Theorem 3.20, if the solutions exist, they would belong to a Banach space B. In our case, the space B will be the space of continuous functions on the graph G :

$$B := (C(G), \|.\|_B),$$

where

$$\|u\|_B := \max_{j=1,\dots,k} \sup_{[0,1]} |u|,$$

and

$$C(G) := \left\{ \ u \in (C[0,1])^k : u_j\left(v_i\right) = u_l\left(v_i\right) \quad \forall j,l \in \Gamma\left(v_i\right), i = 1, ..., n \ \right\}.$$

We now define the functions f_j and g_j in (4.8) and impose certain assumptions on these functions. The subsequent technicalities follow a similar approach to [9, 3.3], and they are necessary to ensure that the assumptions of Theorem 3.20 hold for the operators we will construct.

1. **Assumptions on the functions** f_j :
   $\forall j \in \{1, ..., k\}$, the functions $f_j : [0,T] \times \Omega \times I \times \mathbb{R} \to \mathbb{R}$ are polynomial function of the form :

$$f_j(t,\omega,x,\eta) = -a_{j,2q+1}(t,\omega,x)\eta^{2q+1} + \sum_{l=0}^{2q} a_{j,l}(t,\omega,x)\eta^l, \tag{4.10}$$

$$\text{where } q \in \mathbb{Z} \text{ is fixed }, \eta \in \mathbb{R}, I = [0,1].$$

2. **Assumptions on the coefficients** $a_{j,l}$.

   (a) For almost all $\omega \in \Omega$, $\exists c, C \in \mathbb{R}$ where $0 < c \le C < \infty$ such that :

$$c \le a_{j,2q+1}(t,\omega,x) \le C, |a_{j,l}(t,\omega,x)| \le C, \tag{4.11}$$

$\forall j \in \{1,...,k\}, l \in \{0,2,...,2q\}, x \in I, t \in [0,T]$.

(b) $\forall j \in \{1,...,k\}, l \in \{0,2,...,2q\}, t \in [0,T]$, the functions

$$a_{j,l} : [0,T] \times \Omega \times I \to \mathbb{R}$$

are jointly measurable and adapted i.e

$$a_{j,l}(t,.) \text{ is } \mathcal{F}_t \otimes \mathcal{B}_I - measurable,$$

where $\mathcal{B}_I$ is the Borel sigma algebra generated by I, and $\mathcal{F}_t \otimes \mathcal{B}_I$ denotes the product of sigma algebras.

(c)

$$(a_{1,l}(t,\omega,.),...,a_{k,l}(t,\omega,.))^T \in B, \tag{4.12}$$

$$\forall l = 1,...,2q+1, t \in [0,T] \quad \text{almost all } \omega.$$

3. **Assumptions on the functions** g_j :

   (a) $\forall j \in \{1,...,k\}$, the functions $g_j : [0,T] \times \Omega \times I \times \mathbb{R} \to \mathbb{R}$, are locally Lipschitz continuous and of linear growth in the fourth variable, uniformly with respect to the first three variables.

   (b) $\forall j \in \{1,...,k\}$, the functions g_j are assumed to be jointly measurable and adapted, i.e $\forall j \in \{1,...,k\}$,, and $t \in [0,T]$, $g_j(t,.)$ is $\mathcal{F}_t \otimes \mathcal{B}_I \otimes \mathcal{B}_{\mathbb{R}} - measurable$.

4. $\forall j \in \{1,...,k\}$, the following notation $\frac{\partial w_j}{\partial t}$, denotes independent space-time white noises on I.

Remark 4.10

Unlike the other technical assumptions, assumption 2,(c) (4.12) is related to the structure of the graph, it is necessary, as will be seen later, this condition will play a role in showing that the original stochastic Allen-Cahen problem (4.1) is a special case of (4.8)

Being done with the above technicalities, now we focus on constructing the operators F, G, and W and simultaneously showing that the six assumptions of Theorem 3.20 hold.

Construction of A and assumptions 1,2 and 3 from Theorem 3.20

Let $\mathcal{E} := \mathcal{E}_p$ be the state space for $p \geq 2$, we set $(A, D(A)) := (A_p, D(A_p))$, where $(A_p, D(A_p))$ and $\mathcal{E}_p$ are defined as in Proposition 4.8 and 4.7. Let $\mathcal{E}^\theta$ denote the fractional space (see Section 3.2).

Proposition 4.11 (Assumption 1. of 3.20)
$(A, D(A))$ is densely defined, closed and sectorial on $\mathcal{E}$.

Proof. From Proposition 4.8, we see since $(A, D(A))$ is a generator of an analytic C_0-semigroup, then (A,D(A)) being densely defined and closed follows from Theorem 1.10, while Theorem 1.15 implies that $(A, D(A))$ is sectorial. □

Let

$$W_0(G) := \prod_{j=1}^{k} W_0^{2,p}(0,1),$$

where $W_0^{2,p}$ denotes the Sobolev space which consists of functions with weak derivatives up to order 2 that belong to the Lebesgue space $\mathbf{L}^p$, such that these functions and their first weak derivatives vanishes at the boundary.
Let's define now the maximal operator on $\mathcal{E}$ as follows:

$$D\left(A_{p,max}\right) := \left\{ u \in \left(\prod_{j=1}^{k} W^{2,p}(0,1) \cap D(L) \right) \right\},$$
$$A_{p,max} := \begin{pmatrix} \frac{d^2}{dx^2} - p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{d^2}{dx^2} - p_m \end{pmatrix}_{(m\times m)}. \tag{4.13}$$

Lemma 4.12

$$D(A_{p,max}) \cong W_0(G) \times \mathbb{R}^n. \tag{4.14}$$

$$B \cong (C_D[0,1])^m \times \mathbb{R}^n, \tag{4.15}$$

where

$$C_D[0,1] := \{u \in C[0,1] : u(0) = u(1) = 0\}.$$

Proof. The proof of these two lemmas follows from [9, Lemma 3.5, Lemma 3.6], the argument used for (4.14) is related to the result of [8, Lemma 1.2] which decomposes the domain of the operator $D(A_{p,max})$ as the directed sum of $\mathbb{R}^n$ and of $D(A_{p,max})$ restricted to its kernel which in this case will be $W_0(G)$, then due to a projection property, using the result of [16, Theorem 2.5] the direct sum will be equivalent to the direct product. □

Proposition 4.13 (Assumption 2. of 3.20)
For $\theta \in \left[0, \frac{1}{2}\right)$ we have continuous dense embedding : $\mathcal{E}^\theta \hookrightarrow B \hookrightarrow \mathcal{E}$.

Proof. Since $(A, D(A)))$ is a generator of a contraction C_0-semigroup, then by [2, Theorem 4.73] $(-A)$ has a bounded H^∞-calculus. Because $(-A)$ is sectorial and injective, then by [2, 4.53] $-A$ is BIP. Using [2, Proposition 4.4.10], we have :

$$\mathcal{E}^\theta = D((-A)^\theta) \cong [D(-A), \mathcal{E}]_\theta .$$

Here $[D(-A), \mathcal{E}]_\theta$ denotes the complex interpolation space, for more details concerning interpolation theory we refer to [17]. Concerning the "BIP" and H^∞-calculus properties we refer to [2, Section 4]. From Proposition 4.8 and 4.13, we have $D(A) \hookrightarrow D(A_{p,max})$. This implies :

$$\mathcal{E}^\theta \hookrightarrow [D(-A_{p,max}), \mathcal{E}]_\theta . \tag{4.16}$$

Notice that $\mathcal{E} \cong \mathcal{E} \times \{0_{\mathbb{R}^n}\}$, then by [17, Section 4.3.3]we have for $\frac{1}{2p} < \theta$:

$$[W_0(G) \times \mathbb{R}^n, \mathcal{E} \times \{0_{\mathbb{R}^n}\}]_\theta \hookrightarrow \left(\prod_{j=1}^{k} W_0^{2\theta,p}(0,1) \right) \times \mathbb{R}^n .$$

From (4.14), we have $D(-A_{p,max}) \cong W_0(G) \times \mathbb{R}^n$, using (4.16) we get :

$$\mathcal{E}^\theta \hookrightarrow \left(\prod_{j=1}^{k} W_0^{2\theta,p}(0,1) \right) \times \mathbb{R}^n . \tag{4.17}$$

Hence it follows that

$$\mathcal{E}^\theta \hookrightarrow C_0(G) \times \mathbb{R}^n . \tag{4.18}$$

By Lemma 4.15, this means for $\frac{1}{2p} < \theta$ we have :

$$\mathcal{E}^\theta \hookrightarrow B . \tag{4.19}$$

The embedding $B \hookrightarrow \mathcal{E}$ follows as in [9, Corollary 3,7] while the denseness of both the embeddings follows from [10]. □

Proposition 4.14 (Assumption 3. of 3.20)
Let $S := (T_p(t))_{t\geq 0}$ be the contraction C_0-semigroup analytic semigroup generated by $(A, D(A))$. Then S restricted to B i.e $S_{|B}$ is a contraction semigroup with dissipative generator $A_{|B}$.

Construction of F and assumption 4 and 5 :

Let F be defined as :

$$F : [0,T] \times \Omega \times B \to B$$

$$(t, \omega, u) \mapsto F(t, \omega, u)(s) := (f_1(t, \omega, u_1(s)), ..., f_k(t, \omega, u_k(s)))^T, \quad s \in [0,1],$$

which is well-defined thanks to assumption 2.(c) 4.12.

Proposition 4.15
F satisfies assumption 4. and assumption 5. from Theorem 3.20.

Proof. We present a sketch of the proof as explained in [9, Theorem 3.14].

1. F is locally-Lipshitz from B to B because of assumption 4.11,

2. from [11, Example 4,2], throughout some calculations, it is stated that there exists a positive $a \in \mathbb{R}$ such that :

$$f_j(t, \omega, x, \eta + \zeta) \cdot \operatorname{sgn}(\eta) \leq a\big(1 + |\zeta|^{2q+1}\big),$$

for all $(t, \omega, x) \in [0, T] \times \Omega \times I$ and any η, $\xi \in \mathbb{R}$,

3. Then by [5, Section 4.3], we have :

$$\langle F(t, \omega, u + v), u^* \rangle \leq a(1 + \|v\|)^N,$$

for all $t \in [0.T]$, $w \in \Omega$, $u, v \in B$ and $u^* \in \partial\|u\|$.
Since $A_{|B}$ is dissipative, it then follows that there exist a, $b \geq 0$ and $N \geq 1$ such that

$$\langle Au + F(t, \omega, u + v), u^* \rangle \leq a(1 + \|v\|)^N + b\|u\|,$$

for all $u \in D(A_{|B})$, $v \in B$ and $u^* \in \partial\|u\|$,

4. assumption 5. follows from [11, Example 4,5].

□

Construction of G and assumption 6 :

Let $H := \mathcal{E}_2$, the $\mathbf{L}^2$ state space , which is a Hilbert space. Define the multiplication operator $\Gamma : [0, T] \times B \to \mathcal{L}(H)$ as :

$$[\Gamma(t, u)]h := \begin{pmatrix} g_1(t, s, u_1(s)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_k(t, s, u_k(s)) \end{pmatrix} \cdot \begin{pmatrix} h_1(s) \\ \vdots \\ h_k(s) \end{pmatrix}, \quad s \in (0, 1),$$

where $u \in B$ and $h \in H$. Using the assumptions on the g_j functions we get that Γ maps into $\mathcal{L}(H)$.
We know from Proposition 4.8, $(A_2, D(A_2))$ is a generator on H. Take $k_G \in (\frac{1}{4}, \frac{1}{2})$.
By Proposition 4.13, we know that there exists two continuous embedding :

$$\mathcal{I} : \mathcal{E}_2^{k_G} \to \left(\prod_{j=1}^{k} W_0^{2\theta,2}(0, 1)\right) \times \mathbb{R}^n,$$

$$\mathcal{J} : \left(\prod_{j=1}^{k} W_0^{2\theta,2}(0,1) \times \right) \times \mathbb{R}^n \to \mathcal{E} \quad p \geq 2.$$

For $0 < \mathcal{V}$, we define G as :

$$(\mathcal{V} - A)^{-k_G} G(t,u)h := \mathcal{J}\mathcal{I}(\mathcal{V} - A_2)^{k_G} \Gamma(t,u)h, \quad u \in B, h \in H.$$

Proposition 4.16
For arbitrary $p \geq 2$ and $k_G \in (\frac{1}{4}, \frac{1}{2})$, we have the operator G defined previously, maps $[0,T] \times B$ into $\gamma(H, \mathcal{E}^{-k_G})$.

Proof. The proof follows from [9, Proposition 3.12]. However, it is worth mentioning that the main idea of the proof is to show that $\mathcal{J}$ is γ-radonifying for any $\frac{1}{4} < k_G$ which is proved in [18, Corollary 2.2] and then use the ideal property [19, Proposition 5.4] of γ-radonifying operators. □

Proposition 4.17
G satisfies assumption 6. from Theorem 3.20.

Construction of W and U_0:

$$W_H(t) := \begin{pmatrix} W_1(t) \\ \vdots \\ W_m(t) \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_1(0) \\ \vdots \\ u_m(0) \end{pmatrix}, \qquad t \in [0,T].$$

4.6 Well-posedness

Having constructed the operators A, F, G, and W, the semilinear stochastic problem on the graph G can be reformulated into an abstract stochastic problem (ASP) as in (4.9). The only things left now, are the well-posedness results for the (ASP), and for the original stochastic Allen-Cahen problem (4.1).

Theorem 4.18
Let A, F, G, and W be defined as constructed in the previous section. Then for arbitrary $4 < q$, and $u_0 \in \mathbf{L}^q(\Omega, \mathcal{F}_0, \mathcal{P}; B)$, the (ASP) admits a unique global mild solution $U \in \mathbf{L}^q(\Omega, C([0,T]); B)$

Proof. The proof follows from Theorem 3.20 since all the assumptions have been shown to be satisfied in the previous section. □

Now we are ready to handle the original problem (4.1). In (4.1), the functions f_j are of the following form :

$$f_j(t,\omega,x,\eta) = -\eta^3 + \beta_j^2 \eta, \qquad j \in \{1, \cdots, k\}.$$

Unfortunately, these nonlinear terms are not of the form (4.10).To handle this issue, we will introduce some notations which will take the anomaly in our functions f_j and put them instead in the functions p_j by making new functions $\tilde{p}_j$, and this transformation will work since the only condition we will need to be careful about is the non-negativity of the functions $\tilde{p}_j$. Set

$$\beta := \max_{j\in\{1,\dots,k\}} \beta_j,$$

$$f_j(t,\omega,x,\eta) := f(t,\omega,x,\eta) = -\eta^3 + \beta^2\eta, \qquad j \in \{1,\cdots,k\}.$$

Then from the following calculation :

$$\begin{aligned}
&u_j''(t,x) - p_j(x)u_j(t,x) + \beta_j^2 u_j(t,x) - u_j(t,x)^3 \\
&= u_j''(t,x) - \left(p_j(x) + \beta^2 - \beta_j^2\right) u_j(t,x) + \beta^2 u_j(t,x) - u_j(t,x)^3 \\
&= u_j''(t,x) - \tilde{p}_j(x)u_j(t,x) + \beta^2 u_j(t,x) - u_j(t,x)^3 \\
&= u_j''(t,x) - \tilde{p}_j(x)u_j(t,x) + f_j(t,\omega,x,u_j),
\end{aligned}$$

where we put $\tilde{p}_j(x) := \left(p_j(x) + \beta^2 - \beta_j^2\right)$. It is clear that new functions $\tilde{p}_j(x)$ are non-negative, and that our functions f_j are of the form (4.10), therefore we can reformulate our problem (4.1) into the equivalent form :

$$(\mathrm{AC})\begin{cases}
\dot{u}_j(t,x) = & u_j''(t,x) - p_j(x)u_j(t,x) \\
& +\beta_j^2 u_j(t,x) - u_j(t,x) \\
& +g_j\left(t,x,u_j(t,x)\right)\frac{\partial w_j}{\partial t}(t,x), \quad t\in(0,\infty), \begin{cases} x\in(0,1) \text{ if } j\in\{1,\dots,k\} \\ x\in(0,\infty) \text{ if } j\in\{k+1,\dots,m\}\end{cases} & (a) \\
u_j\left(t,v_i\right) = & u_\ell\left(t,v_i\right) =: q_i(t), \quad t\in(0,T], \forall j,\ell\in\Gamma\left(\mathrm{v}_i\right), i=1,\dots,n, & (b) \\
[Mq(t)]_i = & -\sum_{j=1}^m \phi_{ij} u_j'\left(t,v_i\right), \quad t\in(0,T], i=1,\dots,n, & (c) \\
u_j(0,x) = & \mathrm{u}_j(x), \quad \begin{cases} x\in[0,1] \text{ if } j\in\{1,\dots,k\} \\ x\in[0,\infty) \text{ if } j\in\{k+1,\dots,m\}\end{cases} & (d)
\end{cases} \tag{4.20}$$

This system (4.20) is clearly a special case of (4.8), and by Theorem 4.18, we have the well-posedness result of the equivalent system (4.20) to the original problem (4.1). Hence, we can conclude that the stochastic Allen-Cahen equations with multiplicative noise on the compact graph G (4.1) is well-posed in the sense of Theorem 4.18.

Chapter 5

Conclusion

The goal of this thesis was to examine the well-posedness of the stochastic Allen-Cahn equation on a non-compact graph. However, I encountered a challenge in determining the domains for the extended semigroups. It became evident that finding explicit domains using our method is not a straightforward task and requires extensive research as is mentioned in [15, Chapter 3].

Regarding the conjecture I proposed regarding the domains, one possible approach to proving it is true is by utilizing a different method that relies on cosine families, as demonstrated in [7]. As mentioned in the introduction, they successfully established well-posedness for the heat equation on non-compact graphs. One might question why we don't directly refer to their results and proceed to the stochastic case. However, the results in [7] are not directly applicable to our situation, as we consider the heat equation with a reaction term. Therefore, it is necessary to investigate whether we need to employ perhaps perturbation techniques to achieve similar outcomes. Additionally, we need to ascertain if the non-local vertex conditions mentioned in [7, 3.4] are equivalent to our vertex conditions.
Nonetheless, if we consider the matrix M in our problem to be diagonal, which would transform our non-local Kirchhoff vertex conditions into local δ-type vertex conditions, then we would not require further investigation since the handling of δ-type vertex conditions is already addressed in [7, 3.3]. Hence, we will leave this investigation for future exploration.

It is worth noting that if the conjecture holds true, the stochastic Allen Cahn equation would also be well-posed on a non-compact graph. The results from [9] for the compact case have been verified to apply to the non-compact scenario as well. However, due to the uncertainty surrounding the statement on the domains, it is not appropriate to continue presenting the worked results for the non-compact case. Instead, I have provided the established results for the compact case.

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Acronyms