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Investigating delay differential equations describing epidemic processes

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Introduction

In light of the COVID-19 pandemic and its significant influence on a global scale, we contribute in this manuscript to the rise of interest in infectious disease modeling. Contagious diseases have been for long a great threat to human populations, especially in areas with limited resources ([5]). The challenge of fighting these diseases and mitigating their expense on human life has brought significant attention to the field of epidemiology which is a rich and diverse interdisciplinary field that brings together experts from mathematics, epidemiology, computational physics, ecology, evolutionary biology, immunology, sociology, and public health ([22], [21]). It involves the use of mathematical modelling and the study of these models' aim is to understand the consequence of the assumptions about the infection process and how disease spreads through a population. By using the results, we can develop effective strategies to intervene and control dramatic outbreaks and predict their potential trajectory into the future. As such, these models have become an indispensable part of the fight against infectious diseases, helping us to mitigate their impact on our communities and improve the well-being of people worldwide ([4], [12]).

COVID-19

At the end of December 2019, the World Health Organization country office in China was notified of several cases of pneumonia of unknown etiology. These were the first cases of COVID-19 to soon later become a true catastrophe. In the three years since, COVID-19 has killed more than 7.3 million people worldwide ([14]). The virus was subsequently identified as a novel coronavirus known as SARS-CoV-2 which may have originated in bats ([15]), which are known to carry coronaviruses, and then transmitted to an intermediate animal host, possibly a pangolin, before jumping to humans. However, the exact pathway of transmission and the identity of the intermediate host are still under investigation. COVID-19 has experienced multiple waves of outbreaks and the emergence of different variants since the start of the pandemic ([24]). A wave is a period of increased transmission and cases of the disease, often followed by a decrease in cases. These waves can be caused by various factors, such as changes in public health measures, the introduction of new variants, and seasonal patterns ([23]). Some of the most well-known COVID-19 variants include:

- $AlphaVariant$ ($B.1.1.7$): First identified in the UK ([26]), this variant is thought to be more transmissible than the original strain.
- $BetaVariant$ ($B.1.351$): First identified in South Africa, this variant is thought to be less susceptible to some treatments and vaccines.
- $GammaVariant$ ($P.1$): First identified in Brazil, this variant is also thought to be more transmissible and may be less susceptible to some treatments and vaccines.
- $DeltaVariant$ ($B.1.617.2$): First identified in India, this variant is highly transmissible and has been associated with a surge in cases in many countries.

These variants ([23]) along with others, have led to concerns about their potential impact on public health and the effectiveness of current treatments and vaccines ([27]). Health officials and researchers are closely monitoring these variants and studying their characteristics and potential impact on the pandemic. Vaccines have also been developed and updated to be effective against these variants, and public health measures such as mask-wearing, social distancing, and testing continue to be important tools in controlling the spread of COVID-19 ([6]).

Overall, up until the finishing of this manuscript, the COVID-19 pandemic is still ongoing, and the daily confirmed cases continue to be reported at a high level ([25]), and its effects are likely to be experienced for years to come. Therefore continued mathematical modelling is essential for ongoing efforts to control.

Models structure

In the attempt of describing an epidemiological model, the choice of which compartments, transmission, and formulas associated with each arrow can all change, depending on the outbreak's biology and the modeler's focus. Yet the connection between equations stays consistent: one differential equation per compartment, each arrow showing flow into one compartment and/or flow out of another. Acronyms for epidemiology models are often based on the flow patterns between the compartments such as SEIR, SEIRS, SIR, SIRS, SEI, SEIS, SI, and SIS ([1], [17]). For example, in the SEIR model, newborns first become susceptible, then exposed in the latent period in which the disease is held, but not yet unleashed, then infectious, and then removed with permanent immunity. An SEIRS model would be similar, but the immunity in the R class would be imperfect so that individuals would regain their susceptibility after the temporary immunity has ended.

Abstract

This thesis is organized in a twofold structure, with two distinct sections that share similar themes. We introduce the basic compartmental models SIR and SEIR and their characteristics together with representing the threshold value $\mathscr{R}_0$ with the next generation matrix method outlined in [10], resp. in [11], and we are discussing how it relates to the parameters and structure of the SIR model then we are presenting a time delay definition, all in Chapter 1. Investigation of the model in [3] with taking care of its well-posedness in Chapter 2. Then its extension to a more realistic model incorporating a time delay and stability analysis of the equilibrium points in Chapter 3. Chapter 4 is devoted to a modification of the model proposed in [2] and its equilibrium points stability analysis followed by the analysis incorporating a time delay in Chapter 6.

Chapter 1

Compartmental Models

1.1 SIR Model

The basic paradigm for understanding the spread of infectious diseases, called (SIR), The model was first proposed in 1927 by Scot's epidemiologists *Anderson Kermack* and *William McKendrick* ([18]), it has become a very popular tool in epidemiology.

Susceptible **($S(t)$):** The total susceptible population at time t. Who are the people who are exposed to the disease at the time t?

Infected **($I(t)$):** These are individuals that are able to spread the disease to individuals in the S-class.

Recovered **(R):** Individuals that can neither spread the disease nor contract it again, Individuals may enter this compartment by isolation, infection prevention vaccinations, recuperation from infection with latent infection immunity, or disease-related death ([15]).

Each individual belongs to one and only one compartment at each time t. Transitions from $S \to I$ (infected) to $I \to R$ (recovered) may occur over time, and it is assumed that a recovered individual has immunity and cannot go back to being susceptible. The transit between one compartment to another is captured by a system of ODE's, including time t and the transfer rates between the compartments, which are independent parameters that make sense biologically. The derivatives in the ODE system express the evolution in the size of each compartment with respect to time t. We define the (SIR) model by the following ODE system:

$$\begin{cases} \frac{d}{dt}S(t) &= -\beta SI, \\ \frac{d}{dt}I(t) &= \beta SI - \alpha I, \\ \frac{d}{dt}R(t) &= \alpha I, \end{cases} \tag{1.1}$$

Providing visualization of the interactions between the compartments:

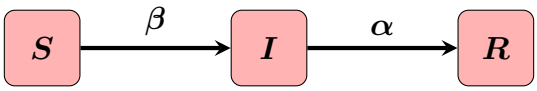

Figure 1.1: SIR model with 3 states

The following statements serve to define the system:

- β is used as the parameter describing the transmission rate that transfers individuals from S to I, and α is the recovery rate, describing the rate at which individuals go from I to R.
- The SI product serves for simulating chance meetings between the two compartments. Specifically, the term βSI can be utilized to model the transmission of a disease from infected individuals to susceptible ones.
- At a rate of αI per unit time, the population from the contaminated compartment transfers to the recovered compartment.

However, the ODE system cannot be analytically solved, quantitative methods are used to determine the behavior of the solutions.

If $S(0) < \alpha/\beta$ with $S(0)$ being the initial value of S, then $I' < 0$ and I, in this case, decreases to zero (no epidemy), but if $S(0) > \alpha/\beta$, I increases which implies epidemy presence.

The limit

$$\mathscr{R}_0 = \beta \frac{S(0)}{\alpha},$$

commonly called the "*Fundamental Reproduction Number*" represents the average number of new infections generated by a single infected individual in a susceptible population.

1.1.1 The Fundamental Reproduction Number

In a lot of epidemiology models, the outbreak of an infection in a fully susceptible population is possible only if $\mathscr{R}_0$ is larger than unity, which could be seen in the previous case. Therefore, $\mathscr{R}_0$ is often considered as the threshold value that determines whether an infection can invade and sustain itself in a new population. For this classic SIR model and a lot of other more complex models, the behavior of solutions almost completely depends on the threshold quantity $\mathscr{R}_0$, meaning that it determines when the local stability of the endemic-free equilibrium switches. It is worth noting that $\mathscr{R}_0$ is also referred to as the basic reproductive ratio or basic reproductive rate.

1.2 The SEIR Model

After introducing the simple compartmental model SIR, we continue by extending it to have another compartment called *Exposed* (E), getting reached by the *Suscebtible* population, as it is commonly recognized that, in general, when a susceptible individual contracts mumps for example, there is a long time lag until he or she becomes infectious or show symptoms. Thus this period in which E takes place is called latent. For a variety of diseases, this period takes typically from 2 to 18 days. So in the SEIR model that we are about to discuss, it is assumed that an exposed individual holds but does not transmit the disease. We will also add rates of natural births and deaths (unrelated to the disease) in a population. We propose an example of what a general SEIR model can look like:

$$\begin{cases} \frac{dS}{dt} &= \mu - \beta SI - \mu S, \\ \frac{dE}{dT} &= \beta SI - \sigma E - \mu E, \\ \frac{dI}{dt} &= \beta SI - \sigma E - \mu E, \\ \frac{dR}{dt} &= \gamma I - \mu R, \end{cases}$$

where μ, β, σ, and γ represent the natural birth/death rate (often assumed to be equal for simplicity reasons), transmission rate, incubation rate, and recovery rate, respectively.

1.2.1 The Next Generation Method

We use the Next Generation Method to find this model's basic reproduction number $\mathscr{R}_0$, The method is given by Diekmann et al. (1990) (cf. [10]) and van den Driessche and Watmough (2002) (cf. [11]) and is usually evaluated from observational data, which is often the most productive approach where there are large numbers of compartments. We first need to find the disease-free equilibrium point ($I = 0$) of the system by setting all derivatives equal to zero we can see that it is the point $(S, E, I, R) = (1, 0, 0, 0)$. We refer to it by C_0, $J_F(C_0)J_V^{-1}(C_0)$ denotes the next generation matrix such that F represents the rate of appearance of new infections in compartment I and J is the Jacobian. New infections can originate from $\frac{dS}{dt}$ and $\frac{dR}{dt}$ as these are called non-infection classes. From $\frac{dS}{dt}$, infection is only generated by the term βSI, hence we have $F_1 = \beta SI$. From, $\frac{dR}{dt}$ no new infection is generated, then $F_2 = 0$. As compartments $\frac{dE}{dt}$ and $\frac{dI}{dt}$ are the ones that pass infection through the population, we consider these dimensions when constructing the Jacobian it will then be:

$$J_F = \begin{bmatrix} \frac{\delta F_1}{\delta E} & \frac{\delta F_1}{\delta I} \\ \\ \frac{\delta F_2}{\delta E} & \frac{\delta F_2}{\delta I} \end{bmatrix}.$$

Substituting C_0:

$$J_F(C_0) = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}.$$

Here V represents the rate of transfer of individuals into and out of E and I by means other than infection originating outside of these classes. After attaching a minus sign we have $V_1 = (\mu + \sigma)E$, the transfer rate in E, and $V_2 = -\sigma E + (\mu + \gamma)I$ which is the transfer rate in I. We can now have the Jacobian for V:

$$J_V(C_0) = \begin{bmatrix} \frac{\delta V_1}{\delta E}(C_0) & \frac{\delta V_1}{\delta I}(C_0) \\ \\ \frac{\delta V_2}{\delta E}(C_0) & \frac{\delta V_2}{\delta I}(C_0) \end{bmatrix} = \begin{bmatrix} \mu + \sigma & 0 \\ \\ -\sigma & \mu + \gamma \end{bmatrix}.$$

Now we get $J_V^{-1}(C_0)$:

$$J_V^{-1}(C_0) = \frac{1}{(\mu + \sigma)(\mu + \gamma)} \begin{bmatrix} \mu + \gamma & 0 \\ \sigma & \mu + \sigma \end{bmatrix}.$$

Then finally:

$$J_F(C_0)J_V^{-1}(C_0) = \begin{bmatrix} \frac{\beta\sigma}{(\mu+\sigma)(\mu+\gamma)} & \frac{\beta(\mu+\sigma)}{(\mu+\sigma)(\mu+\gamma)} \\ 0 & 0 \end{bmatrix}.$$

The spectral radius of $J_F(C_0)J_V^{-1}(C_0)$ that is the eigenvalue with the largest absolute value is the basic reproduction number, in our case, we get:

$$\mathscr{R}_0 = \frac{\beta\sigma}{(\mu + \sigma)(\mu + \gamma)}.$$

1.3 The incorporation of a time delay in epidemiological models

The SIR and SEIR models are relatively simple compartmental models that consider a constant rate of disease transmission and recovery. Nowadays, mathematical models are more complex resulting from aiming to imitate reality. For example, including a large number of health classes such as pre-symptomatic, asymptomatic, and severely symptomatic individuals (see [29], for a model with much more epidemiological classes). However, in relation to the analysis of these models, one of the challenges is how to establish the stability of equilibrium points to learn more about the dynamic behavior of these models hence the disease. Other models are extended after the consideration that transmission and recovery rates can vary over time due to changes in the environment, behavior, and interventions. For example, some researchers have added age or spatial structure to the models to better capture the heterogeneity of the population and the effect of mobility on disease spread.

Another line of research concerns models that represent the dynamics of disease by systems of differential equations with a time delay, where a time delay is incorporated, to accurately provide a more detailed mechanism for the epidemic as people will stay in a specific compartment for an indefinitely long period of time, which leads to the definition of the delay between arriving and moving to another compartment. It is often reflecting the time taken for the holders of the disease to become infectious which is called the incubation period.

Chapter 2

A non-delayed model of coronavirus

In this second chapter, we discuss a model as presented in [3]. Let $N(t)$ denote the total population. It is divided into the following five classes of populations:

- $S(t)$: the *susceptible* class, the individuals who have not yet been exposed to the virus.
- $E(t)$: the *exposed* class refers to individuals who have come into contact with the virus but are still in the incubation period and cannot yet transmit the disease.
- $I(t)$: the *symptomatic infectious* class, individuals that manifest symptoms and can spread the disease.
- $A(t)$: the *asymptomatic infectious* class; those persons that can spread the disease without explicit symptoms.
- $R(t)$: the *removed class* includes the people who recovered from the disease.

Figure 2.1 illustrates the fundamental mechanisms that underlie the model. The model considers all potential interactions among the compartments previously described. It is imposed that susceptible individuals are recruited at the constant rate $\mathbf{\Lambda}$, and become infected by direct contact with an infectious individual at a rate β_I, which is scaled by a factor k to account for the possibility that the latter is asymptomatic. Finally, all human individuals are subject to natural mortality d_p. These considerations are incorporated in the first equation of the system (2.1). Individuals that contract the disease are accounted for in the second equation of (2.1). They become exposed, i.e., they cannot yet spread the virus, which needs an incubation period within the body of its hosts. The susceptible that were contaminated in the aforementioned two possible ways enter this class. People leave it by becoming infectious, and either showing symptoms, thereby migrating into class I, or not, therefore, finding themselves in class A. The progression rates into these two classes are ω_p and ω_p'. Furthermore, we assume that a fraction α becomes asymptomatic and $1-\alpha$ instead will manifest symptoms. The third equation models the symptomatic infectious, recruited from the exposed class at rate $(1-\alpha)\omega_p$ as described above. Furthermore, there could be asymptomatic individuals that become symptomatic at

rate ξ. They leave this class by either progressing to the recovered class at rate γ_p', or dying, naturally, or by causes related to the disease, at rate μ. The asymptomatic individuals modeled in the fourth equation appear from the exposed ones and leave the class by overcoming the disease at rate γ_p', dying naturally or by disease-related causes at rate ν, or eventually showing the symptoms, for which they migrate into class I. Recovered individuals are those that have healed from the disease. They are subject only to natural mortality. We assume that they have also become immune so that they are unaffected if they become in contact with the infectious.

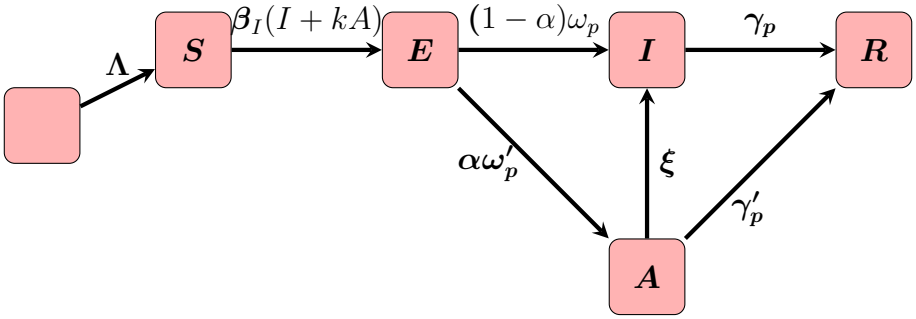

Figure 2.1: SEIR model with 4 states.

Taking into account the above considerations, the model dynamics is regulated by the following system of nonlinear ordinary differential equations:

$$\begin{cases} \dfrac{d}{dt}S(t) = \mathbf{\Lambda} - \beta_I S(t)[kA(t) + I(t)] - d_p S(t), \\ \dfrac{d}{dt}E(t) = \beta_I S(t)[kA(t) + I(t)] - (1-\alpha)\omega_p E(t) - \alpha\omega_p' E(t) - d_p E(t), \\ \dfrac{d}{dt}I(t) = (1-\alpha)\omega_p E(t) - (\gamma_p + d_p + \mu)I(t) + \xi A(t), \\ \dfrac{d}{dt}A(t) = \alpha\omega_p' E(t) - \left(d_p + \nu + \gamma_p'\right)A(t) - \xi A(t), \\ \dfrac{d}{dt}R(t) = \gamma_p I(t) + \gamma_p' A(t) - d_p R(t). \end{cases} \tag{2.1}$$

All the parameters are non-negative and their meaning is summarized in Table 2.

parameters	description
$\mathbf{\Lambda}$	susceptible recruitment rate
d_p	natural mortality
β_1	disease transmission rate
k	transmissible ratio
μ	disease-related mortality for infected
ν	disease-related mortality for asymptomatic
ω_p	progression rate from exposed to symptomatic
ω'_p	progression rate from exposed to asymptomatic
α	fraction of exposed that turn asymptomatic
ξ	progression rate from asymptomatic to symptomatic
γ_p	recovery rate from symptomatic infection
γ'_p	recovery rate from asymptomatic infection

Table 2.1: Model parameters and their meaning

2.0.1 Existence, uniqueness, and boundedness

The system (2.1) can be rewritten as

$$\dot{X}(t) = f(X(t)) \qquad (t > 0)$$

with $X = (S, E, I, A, R) \in \mathbb{R}^5$ where the non-linear operator f is defined in $\mathbb{R}^5$ by

$$f(S,E,I,A,R) = \begin{pmatrix} \mathbf{\Lambda} - \beta_I S(t)[kA(t) + I(t)] - d_p S(t) \\ \beta_I S(t)[kA(t) + I(t)] - (1-\alpha)\omega_p E(t) - \alpha\omega'_p E(t) - d_p E(t) \\ (1-\alpha)\omega_p E(t) - (\gamma_p + d_p + \mu) I(t) + \xi A(t) \\ \alpha\omega'_p E(t) - \left(d_p + \nu + \gamma'_p\right) A(t) - \xi A(t) \\ \gamma_p I(t) + \gamma'_p A(t) - d_p R(t) \end{pmatrix}.$$

In order to prove that the problem determined by (2.1) is well-posed, we introduce the compact region Ω defined by

$$\Omega = \left\{ X = (S, E, I, A, R) \in (\mathbb{R}_0^+)^5; 0 < S + E + I + A + R \leq M \right\}, \tag{2.2}$$

where

$$M := \max\left\{ N(0), \frac{\mathbf{\Lambda}}{d_p} \right\}.$$

The following theorem establishes the existence of global solutions to (2.1).

Theorem 1

For any $X_0 = (S_0; E_0; I_0; A_0; R_0) \in \Omega$, the Cauchy problem given by (2.1) and $X(0) = X_0$ admits a unique solution, denoted by $X(t; X_0)$, defined on $[0, +\infty[$, whose components are non-negative. Furthermore, the region Ω defined by (2.2) is positively invariant.

Proof:

From *Cauchy-Lipschitz's* theorem proved in [8], the existence and uniqueness of a maximal solution in time solution $X(t; X_0)$ to problem (2.1) starting from $X_0 \in \Omega$. The non-negativity of the components is guaranteed by the quasi-positivity of the non-linear operator f, which means that it satisfies the property:

$$\begin{cases} \dfrac{d}{dt}S(t)|_{S(t)=0} = \mathbf{\Lambda} > 0, \\ \dfrac{d}{dt}E(t)|_{E(t)=0} = \beta_I S(t)[kA(t) + I(t)] \geq 0, \\ \dfrac{d}{dt}I(t)|_{I(t)=0} = (1-\alpha)\omega_p E + \xi A \geq 0, \\ \dfrac{d}{dt}A(t)|_{A(t)=0} = \alpha\omega_p' E \geq 0, \\ \dfrac{d}{dt}R(t)|_{R(t)=0} = \gamma_p I + \gamma_p' A \geq 0. \end{cases} \tag{2.3}$$

It follows that the components of any solution $X(t; X_0)$ stemming from X_0 in Ω remain non-negative in future time. Finally, summing the five equations of system (2.1) leads to:

$$\frac{dN}{dt} + d_p N = \mathbf{\Lambda} - \nu A - \mu I \leq \mathbf{\Lambda}.$$

Solving the differential inequality gives:

$$N(t) \leq N(0)\exp(-d_p t) + \frac{\mathbf{\Lambda}}{d_p}[1 - \exp(-d_p t)] \leq M,$$

so that all subpopulations, being non-negative, are bounded as well. And the set Ω is positively invariant, Q.E.D.

2.1 Analyzing Equilibrium Points

The equilibrium points of the model are obtained by equating the right-hand side of system (2.1) to zero. The following equations hold:

$$\mathbf{\Lambda} - \beta_I S(kA + I) - d_p S = 0, \tag{2.4}$$
$$\beta_I S(t)(kA + I) - B_T E = 0, \tag{2.5}$$
$$(1-\alpha)\omega_p E - C_T I + \xi A = 0, \tag{2.6}$$
$$\alpha\omega_p' E - H_T A = 0, \tag{2.7}$$
$$\gamma_p I + \gamma_p' A - d_p R = 0. \tag{2.8}$$

where

$$B_T := (1-\alpha)\omega_p + \alpha\omega_p' + d_p,$$
$$C_T := \gamma_p + \mu + d_p,$$
$$H_T := \gamma_p' + \nu + \xi + d_p.$$

Adding (2.4) and (2.5) yields

$$S = \frac{\mathbf{\Lambda} - B_T E}{d_p}.$$

From (2.7) we get

$$A = \left(\frac{\alpha\omega_p'}{H_T}\right) E.$$

and by (2.6) we get

$$I = \left(\frac{(1-\alpha)\omega_p H_T + \alpha\omega_p'\xi}{C_T H_T}\right) E.$$

By (2.8) we get

$$R = \frac{\gamma_p}{d_p}\left(\frac{(1-\alpha)\omega_p H_T + \alpha\omega_p'\xi}{C_T H_T}\right) E + \frac{\gamma_p'}{d_p}\left(\frac{\alpha\omega_p'}{H_T}\right) E.$$

We have

$$SI = \frac{[(1-\alpha)\omega_p H_T + \alpha\omega_p'\xi][\mathbf{\Lambda}E - B_T E^2]}{d_p C_T H_T},$$

$$SA = \frac{\alpha\omega_p'[\mathbf{\Lambda}E - B_T E^2]}{d_p H_T} = \frac{\alpha\omega_p' C_T[\mathbf{\Lambda}E - B_T E^2]}{d_p H_T C_T}.$$

$$\begin{aligned}\beta_I S(t)(kA(t) + I(t)) =& \frac{\beta_I[\mathbf{\Lambda}E - B_T E^2][\alpha\omega_p' k C_T + (1-\alpha)\omega_p H_T + \alpha\omega_p'\xi]}{d_p H_T C_T} \\ =& \frac{\beta_I[\mathbf{\Lambda}E - B_T E^2][\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T]}{d_p H_T C_T},\end{aligned}$$

where $D_T = \xi + kC_T$. By replacing in (2.5) we get,

$$\frac{\beta_I[\mathbf{\Lambda}E - B_T E^2][\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T]}{d_p H_T C_T} - B_T E = 0,$$

then either, $E = 0$ or $\dfrac{\beta_I[\mathbf{\Lambda} - B_T E][\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T]}{d_p H_T C_T} - B_T = 0$. Thus,

$$\frac{\beta_I\mathbf{\Lambda}[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T] - \beta_I B_T E[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T]}{d_p H_T C_T} = B_T.$$

Hence,

$$\beta_I\mathbf{\Lambda}[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T] - \beta_I B_T E[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T] = B_T d_p H_T C_T.$$

Therefore,

$$\beta_I\mathbf{\Lambda}[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T] - B_T d_p H_T C_T = \beta_I B_T E[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T],$$

Moreover,

$$E = \frac{\beta_I\mathbf{\Lambda}[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T] - B_T d_p H_T C_T}{\beta_I B_T[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T]},$$

Consequently,

$$E = \frac{1}{B_T}\left[\mathbf{\Lambda} - \frac{d_p B_T C_T H_T}{\beta_I[(1-\alpha)\omega_p H_T + \alpha\omega_p' D_T]}\right],$$

with feasibility condition:

$$\left[\mathbf{\Lambda} > \frac{d_p B_T C_T H_T}{\beta_I[(1-\alpha)\omega_p H_T + \alpha\omega_p' D_T]}\right].$$

The system gives two equilibrium points:

- The coronavirus-free equilibrium $C_0 = (S_0, 0, 0, 0, 0,)$, with $S_0 = \frac{\mathbf{\Lambda}}{d_p}$.
- The full coronavirus endemic equilibrium $C^* = (S^*, E^*, I^*, A^*, R^*)$ where

$$
\begin{aligned}
S^* &= \frac{\mathbf{\Lambda} - B_T E^*}{d_p}, \\
E^* &= \frac{1}{B_T}\left(\mathbf{\Lambda} - \frac{d_p B_T C_T H_T}{\beta_I\left[(1-\alpha)\omega_p H_T + \alpha\omega_p' D_T\right]}\right), \\
I^* &= \left(\frac{(1-\alpha)\omega_p H_T + \alpha\omega_p'\xi}{C_T H_T}\right)E^*, \\
A^* &= \left(\frac{\alpha\omega_p'}{H_T}\right)E^* \\
R^* &= \frac{\gamma_p}{d_p}\left(\frac{(1-\alpha)\omega_p H_T + \alpha\omega_p'\xi}{C_T H_T}\right)E^* + \frac{\gamma_p'}{d_p}\left(\frac{\alpha\omega_p'}{H_T}\right)E^*.
\end{aligned}
$$

Remark 1

When $\alpha = 1$ and $\xi = 0$, we get the corona virus-symptomatic-infected-free equilibrium $C_I = (S_I, E_I, 0, A_I, R_I)$, which is the special case of C^*, where

$$
\begin{aligned}
S_I &= \frac{\mathbf{\Lambda} - B_T E_I}{d_p}, \\
E_I &= \frac{1}{B_T}\big(\mathbf{\Lambda} - \frac{d_p B_T C_T H_T}{\beta_I \omega_p' D_T}\big), \\
A_I &= \big(\frac{\omega_p'}{H_T}\big)E_I, \\
R_I &= \big(\frac{\gamma_p'\omega_p'}{d_p H_T}\big)E_I.
\end{aligned}
$$

The feasibility conditions are:

$$
\mathbf{\Lambda} > \frac{d_p B_T C_T H_T}{\beta_I \omega_p' D_T}, \quad \alpha = 1 \quad \text{and} \quad \xi = 0. \tag{2.9}
$$

2.2 The Basic Reproduction Number

The *basic reproduction number* $\mathscr{R}_0$ for system (2.1) is found using the next generation matrix method outlined in [11] to our model (2.1), the basic reproduction number can be computed by considering the new generation matrices F and V, that is, the matrices associated with the rate of appearance of new infections and the net rate out of the corresponding compartments, respectively, The reduced system of (2.1) may be written in compact form as $x' = F(x) - V(x)$ where $x = (E, I, A)$,

$$
F(E, I, A) = \begin{pmatrix} \beta_I S(I + kA) \\ 0 \\ 0 \end{pmatrix},
$$

and

$$
V(E, I, A) = \begin{pmatrix} (1-\alpha)\omega_p E + \alpha\omega_p' E + d_p E \\ -(1-\alpha)\omega_p E + (\gamma_p + d_p + \mu)I - \xi A \\ -\alpha\omega_p' E + (\gamma_p' + d_p + \nu)A + \xi A \end{pmatrix}.
$$

The Jacobian matrices of $F(X)$ and $V(X)$ at the disease-free equilibrium point C_0 are

$$J_F(C_0) = \begin{pmatrix} 0 & \beta_I S_0 & \beta_I S_0 k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$J_V(C_0) = \begin{pmatrix} B_T & 0 & 0 \\ -(1-\alpha)\omega_p & +C_T & -\xi \\ -\alpha\omega_p' & 0 & +H_T \end{pmatrix}$$

We find that

$$J_V^{-1}(C_0) = \begin{pmatrix} \frac{1}{B_T} & 0 & 0 \\ \frac{[(1-\alpha)\omega_p H_T + \alpha\omega_p' \xi]}{C_T B_T H_T} & \frac{1}{C_T} & \frac{\xi}{C_T H_T} \\ \frac{\alpha\omega_p'}{B_T H_T} & 0 & \frac{1}{H_T} \end{pmatrix}$$

The next-generation matrix is

$$-J_F(C_0)J_V^{-1}(C_0) = \begin{pmatrix} -\beta_I S_0 \frac{(1-\alpha)\omega_p H_T + \alpha\omega_p' D_T}{C_T B_T H_T} & -\frac{\beta_I S_0}{C_T} & -\frac{\beta_I S_0 D_T}{C_T H_T} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\mathscr{R}_0 = \rho\Big[(-J_F)(C_0)J_V^{-1}(C_0)\Big] = \max_{\lambda \in \mathrm{Spec}[(-J_F)(C_0)J_V^{-1}(C_0)]} |\lambda| = \beta_I \frac{\Lambda[(1-\alpha)\omega_p H_T + \alpha\omega_p' D_T]}{d_p C_T B_T H_T}.$$

We have the following theorem

Theorem 2

System (2.1) has the following equilibrium:

1. The coronavirus free equilibrium can be calculated as $C_0 = (\frac{\mathbf{\Lambda}}{d_p}, 0, 0, 0, 0)$ which exists always.

2. In addition, if $\mathscr{R}_0 > 1$ then system (2.1) admits another non-trivial equilibrium, in fact: when $\alpha = 1$ and $\xi = 0$, it is the corona virus-symptomatic-infected-free equilibrium $C_I = (S_I, E_I, 0, A_I, R_I)$. When either $\alpha \neq 1$ or $\xi \neq 0$, it is the full coronavirus endemic equilibrium $C^* = (S^*, E^*, I^*, A^*, R^*)$.

2.3 Stability of the equilibrium point

Letting

$$\begin{aligned} a_0 &= B_T C_T H_T, \\ a_1 &= H_T[B_T + C_T] + B_T C_T, \\ a_2 &= B_T + C_T + H_T, \\ b_0 &= (1-\alpha)\omega_p H_T + \alpha\omega_p' D_T, \\ b_1 &= (1-\alpha)\omega_p + k\alpha\omega_p', \\ \mathscr{R}_0 &= \beta_I \frac{\mathbf{\Lambda} b_0}{a_0 d_p}. \end{aligned}$$

2.3.1 Local Stability at C_0

Theorem 3

1. If $\mathscr{R}_0 < 1$ $(resp. \mathbf{\Lambda} < \frac{a_0 d_p}{\beta_I b_0})$ the coronavirus-free equilibrium $C_0 = (S_0, 0, 0, 0, 0)$ of the system (2.1) is stable.

2. If $\mathscr{R}_0 > 1$ $(resp. \boldsymbol{\Lambda} > \frac{a_0 d_p}{\beta_I b_0})$ the coronavirus-free equilibrium $C_0 = (S_0, 0, 0, 0, 0)$ of the system (2.1) is unstable.

Proof:

The characteristic equation of system (2.1) at C_0 is defined as

$$\det(\lambda I - L) = 0,$$

where L is the Jacobian matrix of system (2.1) at C_0. Then,

$$L = \begin{pmatrix} -d_p & 0 & -\beta_I \frac{\boldsymbol{\Lambda}}{d_p} & -k\beta_I \frac{\boldsymbol{\Lambda}}{d_p} & 0 \\ 0 & -B_T & \beta_I \frac{\boldsymbol{\Lambda}}{d_p} & k\beta_I \frac{\boldsymbol{\Lambda}}{d_p} & 0 \\ 0 & (1-\alpha)\omega_p & -C_T & \xi & 0 \\ 0 & \alpha\omega_p' & 0 & -H_T & 0 \\ 0 & 0 & \gamma_p & \gamma_p' & -d_p \end{pmatrix}$$

and the characteristic equation of system (2.1) at the coronavirus-free equilibrium C_0 is:

$$\begin{vmatrix} \boldsymbol{\lambda} + d_p & 0 & \beta_I \frac{\boldsymbol{\Lambda}}{d_p} & k\beta_I \frac{\boldsymbol{\Lambda}}{d_p} & 0 \\ 0 & \boldsymbol{\lambda} + B_T & -\beta_I \frac{\boldsymbol{\Lambda}}{d_p} & -k\beta_I \frac{\boldsymbol{\Lambda}}{d_p} & 0 \\ 0 & -(1-\alpha)\omega_p & \boldsymbol{\lambda} + C_T & -\xi & 0 \\ 0 & -\alpha\omega_p' & 0 & \boldsymbol{\lambda} + H_T & 0 \\ 0 & 0 & -\gamma_p & -\gamma_p' & \boldsymbol{\lambda} + d_p \end{vmatrix} = 0.$$

After calculation, this equation becomes

$$(\boldsymbol{\lambda} + d_p)^2(\boldsymbol{\lambda}^3 + a_2\boldsymbol{\lambda}^2 + s_1\boldsymbol{\lambda} + s_0) = 0. \tag{2.10}$$

Where

$$s_1 = a_1 - \beta_I \frac{\boldsymbol{\Lambda}}{d_p} b_1,$$
$$s_0 = a_0 - \beta_I \frac{\boldsymbol{\Lambda}}{d_p} b_0.$$

We need to prove that all roots of the characteristic equation (2.10) have negative real parts.

Note that $-d_p$ is a negative real root of the characteristic equation (2.10) and this equation reduces to,

$$P_1(\boldsymbol{\lambda}) := (\boldsymbol{\lambda}^3 + a_2\boldsymbol{\lambda}^2 + s_1\boldsymbol{\lambda} + s_0) = 0. \tag{2.11}$$

Using the *Routh–Hurwitz criterion* [38], we know that all roots of $P_1(\boldsymbol{\lambda})$ have negative real parts if, and only if, the coefficients of $P_1(\boldsymbol{\lambda})$ are strictly positive and $a_2 s_1 > s_0$. In this case,

$$\begin{aligned} s_1 &= a_1 - \beta_I \frac{\boldsymbol{\Lambda}}{d_p} b_1 = \frac{1}{b_0}[a_1 b_0 - \beta_I \frac{\boldsymbol{\Lambda}}{d_p} b_0 b_1] \\ &= \frac{1}{b_0}[H_T C_T b_0 + H_T B_T b_0 + B_T C_T b_0 + (s_0 - a_0) b_1] \\ &= \frac{1}{b_0}[H_T C_T b_0 + H_T B_T b_0 + \alpha\omega_p' D_T B_T C_T + s_0 b_1 - k\alpha\omega' a_0] \\ &= \frac{1}{b_0}[H_T C_T b_0 + H_T B_T[(1-\alpha)\omega_p H_T + \alpha\omega_p' D_T] + \alpha\omega_p' B_T C_T D_T + s_0 b_1 - k\alpha\omega' a_0] \\ &= \frac{1}{b_0}[H_T C_T b_0 + H_T B_T[(1-\alpha)\omega_p H_T + \alpha\omega_p'(\xi + kC_T)] + \alpha\omega_p' B_T C_T D_T + s_0 b_1 - k\alpha\omega' a_0] \\ &= \frac{1}{b_0}[H_T C_T b_0 + H_T^2 B_T (1-\alpha)\omega_p + \alpha\omega_p' H_T B_T \xi + \alpha\omega_p' B_T C_T D_T + s_0 b_1] > 0. \end{aligned}$$

And

$$\begin{aligned} s_0 &= a_0[1-\beta_I\frac{\mathbf{\Lambda}}{d_p a_0}b_0] \\ &= a_0[1-\mathscr{R}_0]. \end{aligned}$$

We have $a_2 > 0$, $s_1 > 0$ and if $\mathscr{R}_0 < 1$, $s_0 > 0$. Now, we only need to show the following statement: $a_2 s_1 > s_0$.

$$\begin{aligned} a_2 s_1 &= (B_T + C_T + H_T)\frac{1}{b_0}[H_T C_T b_0 + H_T^2 B_T(1-\alpha)\omega_p + \alpha\omega_p' H_T B_T \xi + \alpha\omega_p' B_T C_T D_T + s_0 b_1] \\ &= (B_T + C_T + H_T)[H_T C_T + \frac{1}{b_0}H_T^2 B_T(1-\alpha)\omega_p + \frac{1}{b_0}\alpha\omega_p' H_T B_T\xi + \frac{1}{b_0}\alpha\omega_p' B_T C_T D_T + \frac{s_0 b_1}{b_0}] \\ &= a_0 + \frac{1}{b_0}H_T^2 B_T^2(1-\alpha)\omega_p + \frac{1}{b_0}\alpha\omega_p' H_T B_T^2\xi + \frac{1}{b_0}\alpha\omega_p' B_T^2 C_T D + \frac{1}{b_0}s_0 b_1 B_T + \\ &+ (C_T + H_T)[H_T C_T + \frac{1}{b_0}H_T^2 B_T(1-\alpha)\omega_p + \frac{1}{b_0}\alpha\omega_p' H_T B_T\xi + \frac{1}{b_0}\alpha\omega_p' B_T C_T D_T + \frac{1}{b_0}s_0 b_1] \\ a_2 s_1 - s_0 &= a_2 s_1 - a_0[1-\mathscr{R}_0] \\ &= a_2 s_1 - a_0 + a_0\mathscr{R}_0 \\ &= \frac{1}{b_0}H_T^2 B_T^2(1-\alpha)\omega_p + \frac{1}{b_0}\alpha\omega_p' H_T B_T^2\xi + \frac{1}{b_0}\alpha\omega_p' B_T^2 C_T D + \frac{1}{b_0}s_0 b_1 B_T + a_0\mathscr{R}_0 + \\ &+ (C_T + H_T)[H_T C_T + \frac{1}{b_0}H_T^2 B_T(1-\alpha)\omega_p + \frac{1}{b_0}\alpha\omega_p' H_T B_T\xi + \frac{1}{b_0}\alpha\omega_p' B_T C_T D_T + \frac{s_0 b_1}{b_0}]. \end{aligned} \tag{2.12}$$

Thus $a_2 s_1 > s_0$. Then, according to the *Routh-Hurwitz criterion*, all the roots of the characteristic equation (2.11) have negative real parts. Therefore, the coronavirus-free equilibrium point C_0 is locally asymptotically stable under condition $\mathscr{R}_0 < 1$, $(resp.\, \mathbf{\Lambda} < \frac{a_0 d_p}{\beta_1 b_0})$.

For $\mathscr{R}_0 > 1.(resp.\ \mathbf{\Lambda} > \frac{a_0 d_p}{\beta_1 b_0})$, we have $P_1(0) = a_0(1-\mathscr{R}_0) < 0$, $\lim\limits_{\boldsymbol{\lambda}\to+\infty} P_1(\boldsymbol{\lambda}) = +\infty$. By the continuity of $P_1(\boldsymbol{\lambda})$, there exists at least one positive root of $P_1(\boldsymbol{\lambda}) = 0$. Thus, the infection-free equilibrium C_0 is unstable.

2.3.2 Local Stability at C^*

Since we can deduce the stability of the coronavirus symptomatic infected-free equilibrium C_I from the stability of the coronavirus endemic equilibrium C^* simply by taking $\alpha = 1$ and $\xi = 0$ in the latter, we now just analyze the coronavirus endemic equilibrium C^*.

Theorem 4

If $\mathscr{R}_0 > 1$ $(resp.\ \mathbf{\Lambda} > \frac{a_0 d_p}{\beta_1 b_0})$, the coronavirus endemic equilibrium C^* is locally asymptotically stable.

Proof:

The characteristic equation of system (2.1) at the coronavirus endemic equilibrium C^* is:

$$\begin{vmatrix} \boldsymbol{\lambda} + d_p + \beta_I(I^* + kA^*) & 0 & \beta_I S^* & \beta_I S^* k & 0 \\ -\beta_I(I^* + kA^*) & \boldsymbol{\lambda} + B_T & -\beta_I S^* & -\beta_I S^* k & 0 \\ 0 & -(1-\alpha)\omega_p & \boldsymbol{\lambda} + C_T & -\xi & 0 \\ 0 & -\alpha\omega_p' & 0 & \boldsymbol{\lambda} + H_T & 0 \\ 0 & 0 & -\gamma_p & -\gamma_p' & \boldsymbol{\lambda} + d_p \end{vmatrix} = 0,$$

in which B_T, C_T and H_T are same as before. From this, we get the characteristic polynomial:

$$P_2(\boldsymbol{\lambda}) = (\boldsymbol{\lambda} + d_p)[\boldsymbol{\lambda}^4 + (c_3 + d_3)\boldsymbol{\lambda}^3 + (c_2 + d_2)\boldsymbol{\lambda}^2 + (c_1 + d_1)\boldsymbol{\lambda} + (c_0 + d_0)], \tag{2.13}$$

where

$$
\begin{aligned}
c_3 &= d_p + B_T + C_T + H_T,\\
c_2 &= d_p(H_T + B_T + C_T) + B_T C_T + H_T B_T + H_T C_T,\\
c_1 &= d_p(B_T C_T + H_T B_T + H_T C_T) + H_T B_T C_T,\\
c_0 &= d_p H_T B_T C_T,\\
d_3 &= \beta_I(I^* + kA^*),\\
d_2 &= (B_T + C_T + H_T)\beta_I(I^* + kA^*) - \beta_I S^* b_1,\\
d_1 &= (B_T C_T + H_T B_T + H_T C_T)\beta_I(I^* + kA^*) - \beta_I S^*[b_0 + b_1 d_p],\\
d_0 &= H_T B_T C_T \beta_I(I^* + kA^*) - \beta_I S^* d_p b_0.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\beta_I S^* &= \beta_I \frac{\mathbf{\Lambda} - \left(\mathbf{\Lambda} - \frac{d_p B_T C_T H_T}{\beta_I\left((1-\alpha)\omega_p H_T + \alpha\omega_p' D_T\right)}\right)}{d_p},\\
&= \frac{d_p B_T C_T H_T}{d_p\left((1-\alpha)\omega_p H_T + \alpha\omega_p' D_T\right)},\\
&= \frac{a_0}{b_0},
\end{aligned}
$$

and $\beta_I(I^* + kA^*) = \beta_I \dfrac{b_0}{C_T H_T} E^* = d_p(\mathscr{R}_0 - 1)$. Thus,

$$
\begin{aligned}
c_3 &= d_p + a_2,\\
c_2 &= d_p a_2 + a_1,\\
c_1 &= d_p a_1 + a_0,\\
c_0 &= d_p a_0,\\
d_3 &= d_p(\mathscr{R}_0 - 1),\\
d_2 &= a_2 d_p(\mathscr{R}_0 - 1) - \frac{a_0}{b_0} b_1,\\
d_1 &= a_1 d_p(\mathscr{R}_0 - 1) - \frac{a_0}{b_0}\left[b_0 + b_1 d_p\right],\\
d_0 &= a_0 d_p(\mathscr{R}_0 - 1) - \frac{a_0}{b_0} d_p b_0.
\end{aligned}
$$

We have

$$
\begin{aligned}
c_0 + d_0 &= a_0 d_p(\mathscr{R}_0 - 1),\\
c_1 + d_1 &= \frac{d_p}{b_0}[a_1 \mathscr{R}_0 b_0 - a_0 b_1],\\
&= \frac{d_p}{b_0}[(1-\alpha)\omega_p H_T B_T(C_T(\mathscr{R}_0 - 1) + H_T) + H_T C_T b_0 + \alpha\omega_p' B_T(\xi H_T + D_T C_T \mathscr{R}_0)],\\
c_2 + d_2 &= a_2 d_p \mathscr{R}_0 + \frac{1}{b_0}[(1-\alpha)\omega_p H_T^2 B_T + H_T C_T b_0 + H_T B_T \alpha\omega_p' \xi + \alpha\omega_p' D_T B_T C_T],\\
c_3 + d_3 &= d_p + a_2 + d_p(\mathscr{R}_0 - 1) = a_2 + d_p \mathscr{R}_0.
\end{aligned}
$$

For $\mathscr{R}_0 > 1$. We have $c_0 + d_0 > 0$, $c_1 + d_1 > 0$, $c_2 + d_2 > 0$, $c_3 + d_3 > 0$. By the *Routh-Hurwitz criterion*, we only need to show the following statement:
$(c_1 + d_1)[(c_3 + d_3)(c_2 + d_2) - (c_1 + d_1)] > (c_0 + d_0)(c_3 + d_3)^2$, (the proof from [3] is in appendix A).
Then, according to the *Routh-Hurwitz criterion*, all the roots of the characteristic Equation (2.13) have negative real parts. Therefore, the coronavirus-free equilibrium point C^* is locally asymptotically stable under condition $\mathscr{R}_0 > 1$.

2.3.3 Local Stability at C_I

Theorem 5

If $\mathscr{R}_0 > 1$,$(resp.\mathbf{\Lambda} > \frac{a_0 d_p}{\beta_1 b_0})$, the coronavirus endemic equilibrium C_I is locally asymptotically stable.

Proof:

The result can be easily obtained from Theorem 4 by taking $\alpha = 1$ and $\xi = 0$.

2.4 Global stability when $\mathscr{R}_0 < 1$

Theorem 6

If $\mathscr{R}_0 < 1$, $(resp.\mathbf{\Lambda} < \frac{a_0 d_p}{\beta_I b_0})$, the disease-free equilibrium C_0 is globally asymptotically stable.

Proof:

Note that the four equations of (2.1) are independent of R, therefore, the last equation of (2.1) can be omitted without loss of generality. Let us consider the following Lyapunov functional,

$$V(t) = \frac{1}{2S_0}(S(t) - S_0)^2 + E(t) + \frac{B_T}{b_0}[H_T I(t) + D_T A(t)]. \tag{2.14}$$

By calculating the time derivative of V along the positive solution of system (2.1), we get

$$\frac{dV(t)}{dt} = \frac{1}{S_0}(S - S_0)S'(t) + E'(t) + \frac{B_T}{b_0}[H_T I'(t) + D_T A'(t)],$$

$$\frac{dV(t)}{dt} = \frac{1}{S_0}(S - S_0)[-\beta_I S(I + kA) - d_p(S - S_0)] + \beta_I S(I + kA) - B_T E +$$
$$+ \frac{B_T H_T}{b_0}[(1 - \alpha)\omega_p E - C_T I + \xi A] + \frac{B_T D_T[\alpha\omega_p' E - H_T A]}{b_0},$$

$$\frac{dV(t)}{dt} = -\frac{dp}{S_0}(S - S_0)^2 + \beta_I[2S - \frac{S^2}{S_0} - S_0](I + kA) + \beta_I S_0(I + kA)$$
$$- \frac{B_T H_T}{b_0}[C_T I + (D_T - \xi)A],$$

$$\frac{dV(t)}{dt} = -\frac{dp}{S_0}(S - S_0)^2 + \beta_I\Big[2S - \frac{S^2}{S_0} - S_0\Big](I + kA) + \Big[\beta_I S_0 - \frac{a_0}{b_0}\Big](I + kA),$$
$$= -\frac{dp}{S_0}(S - S_0)^2 - \beta_I\frac{(S - S_0)^2}{S_0}(I + kA) + \frac{\beta_I}{d_p}\left[\mathbf{\Lambda} - \frac{a_0 dp}{\beta_I b_0}\right](I + kA).$$

Since $(\mathscr{R}_0 < 1)$, resp. $\mathbf{\Lambda} < \frac{a_0 d_p}{\beta_I b_0})$, then, $\frac{dV(t)}{dt} \leq 0$ for all $(S, E, I, A) \in \mathbb{R}^4_+$. $\frac{dV(t)}{dt} = 0$ if and only if $(S, E, I, A) = (S_0, 0, 0, 0)$. Thus, the only invariant set contained in $\mathbf{R}^4_+$ is $\{(S_0, 0, 0, 0)\}$. Hence, *LaSalle's theorem* implies convergence of the solutions (S, E, I, A) to $(S_0, 0, 0, 0)$. From the last equation of (2.1) we can show obviously that R converges also to 0. Therefore, C_0 is globally asymptotically stable when $\mathscr{R}_0 < 1$.

Chapter 3

Delayed Model of Coronavirus

By presenting a compartmental model and incorporating time delay to provide a more detailed description of the transmission dynamics, it is precisely reflecting the time taken for the holders of the disease to become infectious which is called the incubation period, this time delay plays an important effect on the stability of the equilibrium points which by the end contribute to a more reliable forecasting. When adding a time delay τ to an ordinary differential equation (ODE) it becomes a delayed differential equation (DDE). The difference between them is that the derivatives of the DDE at any time t depend on the solution at prior times ([32]). Because of that, to solve a DDE we need to define how the dynamical system behaves when $\tau - t < t_0$, with t_0 being the starting time of the system. To describe the behavior of the system in the interval before t_0 an initial history function is used to specify the values of the solution set at that interval.

$$\begin{cases} \frac{d}{dt}S(t) = \mathbf{\Lambda} - \beta_I S(t-\tau)[kA(t-\tau) + I(t-\tau)] - d_p S(t), \\ \frac{d}{dt}E(t) = \beta_I S(t-\tau)[kA(t-\tau) + I(t-\tau)] - (1-\alpha)\omega_p E(t) - \alpha\omega_p' E(t) - d_p E(t), \\ \frac{d}{dt}I(t) = (1-\alpha)\omega_p E(t) - (\gamma_p + d_p + \mu)I(t) + \xi A(t), \\ \frac{d}{dt}A(t) = \alpha\omega_p' E(t) - \left(d_p + \nu + \gamma_p'\right)A(t) - \xi A(t), \\ \frac{d}{dt}R(t) = \gamma_p I(t) + \gamma_p' A(t) - d_p R(t), \end{cases} \tag{3.1}$$

where the state variables are subject to the initial conditions,

$$\begin{cases} S(\theta) = \psi_1(\theta) > 0, \\ I(\theta) = \psi_2(\theta) > 0, \quad \theta \in [-\tau, 0], \\ A(\theta) = \psi_3(\theta) > 0, \\ R(0) = R_0 \quad and \quad E(0) = E_0 \end{cases} \tag{3.2}$$

where $\psi = (\psi_1, \psi_2, \psi_3) \in C$ Here C denotes the Banach space $C([-\tau, 0])$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3_+$.

3.0.1 Existence and Uniqueness

Proposition 1

For any initial condition (3.2) defined in C, the system (3.1) has a unique positive solution on $[0, +\infty[$, denoted by $(S(t), E(t), I(t), A(t), R(t))$.

Proof:

From Hale and Verduyn Lunel [13], for each continuous initial condition (3.2), system (3.1) has a continuous maximal solution $(S(t), E(t), I(t), A(t), R(t))$. We can prove that this solution is bounded the same way as the previous model. Hence the solution is defined globally for $[0, +\infty[$.

3.1 Non-negativity of solutions

Proposition 2

The solutions $(S(t), E(t), I(t), A(t), R(t))$ of (3.1) are non-negative for all $t \geq -\tau$ with non-negative initial conditions (3.2).

Proof:

We have,

$$\begin{cases} \frac{d}{dt}S(t)|_{S(t)=0} = \mathbf{\Lambda} > 0, \\ \frac{d}{dt}E(t)|_{E(t)=0} = \beta_I S(t-\tau)[kA(t-\tau) + I(t-\tau)] \geq 0, \\ \frac{d}{dt}I(t)|_{I(t)=0} = (1-\alpha)\omega_p E + \xi A \geq 0, \\ \frac{d}{dt}A(t)|_{A(t)=0} = \alpha\omega_p' E \geq 0, \\ \frac{d}{dt}R(t)|_{R(t)=0} = \gamma_p I + \gamma_p' A \geq 0. \end{cases} \tag{3.3}$$

Therefore, any solution of system (3.1) is such that $(S(t), E(t), I(t), A(t), R(t)) \in (\mathbb{R}_0^+)^5$ for all $t \geq -\tau$.

3.2 Stability of the System's Equilibrium

The equilibrium points of the new system are the same as for the system with zero delays. This is because, by definition, all nearby trajectories will approach the stable equilibrium point asymptotically as $t \to +\infty$, so the delay will not have an effect on the equilibrium points.

Now, we prove some sufficient conditions for the local asymptotic stability of the disease free equilibrium, C_0, and the endemic equilibrium point, C^*, for any time delay $\tau \geq 0$.

Consider the following coordinate transformation:
$x_1(t) = S(t) - \bar{S}$, $x_2(t) = E(t) - \bar{E}$, $x_3(t) = I(t) - \bar{I}$, $x_4(t) = A(t) - \bar{A}$ and $x_5(t) = R(t) - \bar{R}$, where $(\bar{S}, \bar{E}, \bar{I}, \bar{A}, \bar{R})$ is any equilibrium of (3.1).
Then the linearized system of (3.1) takes the form

$$X'(t) = L_1 X(t) + L_2 X(t-\tau),$$

where $X(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))^T$. The characteristic equation at any equilibrium point $(\bar{S}, \bar{E}, \bar{I}, \bar{A}, \bar{R})$ is given by

$$\Delta(\lambda, \tau) := \det(\lambda I - L_1 - e^{-\lambda\tau} L_2) = 0 \tag{3.4}$$

with

$$L_1 = \begin{pmatrix} -d_p & 0 & 0 & 0 & 0 \\ 0 & -B_T & 0 & 0 & 0 \\ 0 & (1-\alpha)\omega_p & -C_T & \xi & 0 \\ 0 & \alpha\omega_p' & 0 & -H_T & 0 \\ 0 & 0 & \gamma_p & \gamma_p' & -d_p \end{pmatrix},$$

$$L_2 = \begin{pmatrix} -\beta_I(k\bar{A}+\bar{I}) & 0 & -\beta_I\bar{S} & -\beta_I k\bar{S} & 0 \\ \beta_I(k\bar{A}+\bar{I}) & 0 & \beta_I\bar{S} & \beta_I k\bar{S} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, (3.4) becomes

$$\begin{vmatrix} \boldsymbol{\lambda} + d_p + \beta_I(k\bar{A}+\bar{I})e^{-\boldsymbol{\lambda}\tau} & 0 & \beta_I\bar{S}e^{-\boldsymbol{\lambda}\tau} & k\beta_I\bar{S}e^{-\boldsymbol{\lambda}\tau} & 0 \\ -\beta_I(k\bar{A}+\bar{I})e^{-\boldsymbol{\lambda}\tau} & \boldsymbol{\lambda}+B_T & -\beta_I\bar{S}e^{-\boldsymbol{\lambda}\tau} & -k\beta_I\bar{S}e^{-\boldsymbol{\lambda}\tau} & 0 \\ 0 & -(1-\alpha)\omega_p & \boldsymbol{\lambda}+C_T & -\xi & 0 \\ 0 & -\alpha\omega_p' & 0 & \boldsymbol{\lambda}+H_T & 0 \\ 0 & 0 & -\gamma_p & -\gamma_p' & \boldsymbol{\lambda}+d_p \end{vmatrix} = 0,$$

3.2.1 Local Stability at C_0

The characteristic equation of system (3.1) at the coronavirus-free equilibrium C_0 is:

$$\begin{vmatrix} \boldsymbol{\lambda} + d_p & 0 & \beta_I S_0 e^{-\boldsymbol{\lambda}\tau} & k\beta_I S_0 e^{-\boldsymbol{\lambda}\tau} & 0 \\ 0 & \boldsymbol{\lambda}+B_T & -\beta_I S_0 e^{-\boldsymbol{\lambda}\tau} & -k\beta_I S_0 e^{-\boldsymbol{\lambda}\tau} & 0 \\ 0 & -(1-\alpha)\omega_p & \boldsymbol{\lambda}+C_T & -\xi & 0 \\ 0 & -\alpha\omega_p' & 0 & \boldsymbol{\lambda}+H_T & 0 \\ 0 & 0 & -\gamma_p & -\gamma_p' & \boldsymbol{\lambda}+d_p \end{vmatrix} = 0,$$

It is equivalent to

$$\boldsymbol{\lambda}^3 + a_2\boldsymbol{\lambda}^2 + a_1\boldsymbol{\lambda} + a_0 - \beta_I S_0 e^{-\boldsymbol{\lambda}\tau}[b_1\boldsymbol{\lambda} + b_0] = 0. \tag{3.5}$$

where a_0, a_1, a_2, b_0 and b_1 are defined in Chapter 2.

Theorem 7

If $\mathscr{R}_0 > 1$ the coronavirus-free equilibrium $C_0 = (S_0, 0, 0, 0, 0)$ of the system (3.1) is unstable.

Proof:

When $\mathscr{R}_0 > 1$, let

$$f(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^3 + a_2\boldsymbol{\lambda}^2 + a_1\boldsymbol{\lambda} + a_0 - \beta_I S_0 e^{-\boldsymbol{\lambda}\tau}[b_1\boldsymbol{\lambda} + b_0] = 0. \tag{3.6}$$

Then $f(0) = a_0(1 - \mathscr{R}_0) < 0$, $\lim_{\boldsymbol{\lambda}\to\infty} f(\boldsymbol{\lambda}) = +\infty$. By the continuity of $f(\boldsymbol{\lambda})$, there exists at least one positive root of $f(\boldsymbol{\lambda}) = 0$. Thus, the infection-free equilibrium C_0 is unstable if $\mathscr{R}_0 > 1$.

Theorem 8

If $\mathscr{R}_0 < 1$ the coronavirus-free equilibrium $C_0 = (S_0, 0, 0, 0, 0)$ of the system (3.1) is locally asymptotically stable.

Proof:

Let $\mathscr{R}_0 < 1$. We divide the proof into the non-delayed and delayed cases.

• Let $\tau = 0$ we get the same characteristic equation as for the no-delay model. Then the coronavirus-free equilibrium point is locally asymptotically stable, as shown in Chapter 2.

• Let $\tau > 0$. In this case, we will use Rouché's theorem to prove that all roots of the characteristic Equation (3.5) cannot intersect the imaginary axis, i.e., the characteristic

equation cannot have pure imaginary roots.
Suppose the contrary, that is, suppose there exists $y \in \mathbb{R}$ such that $\lambda = yi$ is a solution of (3.5). Replacing yi in we get that

$$-iy^3 + a_0 + ia_1y - a_2y^2 - S_0\beta_I e^{-i\tau y}\,(iyb_1 + b_0) = 0$$

then,

$$-iy^3 + a_0 + ia_1y - a_2y^2 - S_0\beta_I(\cos\tau y - i\sin\tau y)\,(iyb_1 + b_0) = 0$$

Separating real and imaginary parts, it follows that

$$\cos(y\tau) = \frac{-b_1y^4 + (b_1a_1 - b_0a_2)y^2 - b_0a_0}{\beta_I S_0(b_0^2 + b_1^2y^2)}$$
$$\sin(y\tau) = \frac{(b_0 - a_2b_1)y^3 + (a_0b_1 - a_1b_0)y}{\beta_I S_0(b_0^2 + b_1^2y^2)}$$

Replacing the above expressions in the following relation

$$\cos^2(y\tau) + \sin^2(y\tau) = 1$$

we get

$$\Big(-b_1y^4 + (b_1a_1 - b_0a_2)y^2 - b_0a_0\Big)^2 + \Big((b_0 - a_2b_1)y^3 + (a_0b_1 - a_1b_0)y\Big)^2 = {\beta_I}^2S_0^2(b_0^2 + b_1^2y^2)^2.$$

Hence, it follows that

$$b_1^2y^8 + \Big[b_0^2 + b_1^2(a_2^2 - 2a_1)\Big]\,y^6 + \Big[b_0^2(a_2^2 - 2a_1) + b_1^2(a_1^2 - 2a_0a_2) - b_1^4S_0^2{\beta_I}^2\Big]\,y^4 \quad (3.7)$$
$$+ \Big[b_0^2(a_1^2 - 2a_0a_2) + a_0^2b_1^2 - 2b_0^2b_1^2S_0^2{\beta_I}^2\Big]\,y^2 + a_0^2b_0^2 - b_0^4S_0^2{\beta_I}^2 = 0.$$

Let

$$K_1 = b_0^2 + b_1^2(a_2^2 - 2a_1),$$
$$K_2 = b_0^2(a_2^2 - 2a_1) + b_1^2(a_1^2 - 2a_0a_2) - b_1^4S_0^2{\beta_I}^2,$$
$$K_3 = b_0^2(a_1^2 - 2a_0a_2) + a_0^2b_1^2 - 2b_0^2b_1^2S_0^2{\beta_I}^2,$$
$$K_4 = a_0^2b_0^2 - b_0^4S_0^2{\beta_I}^2,$$

then (3.7) becomes

$$b_1^2y^8 + K_1y^6 + K_2y^4 + K_3y^2 + K_4 = 0 \quad (3.8)$$

We have
$a_2^2 - 2a_1 = B_T^2 + C_T^2 + H_T^2$,
$a_1^2 - 2a_0a_2 = H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2$
and $S_0\beta_I = \frac{\mathscr{R}_0a_0}{b_0}$ then

$$K_1 = b_0^2 + b_1^2(B_T^2 + C_T^2 + H_T^2) > 0$$
$$K_4 = a_0^2b_0^2(1 - \mathscr{R}_0^2) > 0 \quad (since \mathscr{R}_0 < 1)$$
$$K_2 = \frac{1}{b_0^2}\Big[b_0^4(B_T^2 + C_T^2 + H_T^2) + b_0^2b_1^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) - b_1^4{\mathscr{R}_0}^2a_0^2\Big]$$
$$K_3 = b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) + a_0^2b_1^2(1 - 2{\mathscr{R}_0}^2).$$

It remains to show that,
$K_3 > 0$ and $K_2 > 0$.

Lemma 1

Let $M = b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) - a_0^2b_1^2$ then $M > 0$.

Proof:

In fact,

$$\begin{aligned}
M &= b_0^2(H_T^2B_T^2) + b_0^2(C_T^2H_T^2 + B_T^2C_T^2) - a_0^2b_1^2\\
&= b_0^2(C_T^2H_T^2 + B_T^2C_T^2) + b_0^2(H_T^2B_T^2) - C_T^2H_T^2B_T^2b_1^2\\
&= b_0^2(C_T^2H_T^2 + B_T^2C_T^2) + H_T^2B_T^2[b_0^2 - C_T^2[(1-\alpha)\omega_p + k\alpha\omega_p']^2]\\
&> H_T^2B_T^2\Big[\Big((1-\alpha)\omega_pH_T + \alpha\omega_p'D_T\Big)^2 - C_T^2[(1-\alpha)\omega_p + k\alpha\omega_p']^2\Big]\\
&> H_T^2B_T^2\Big[(1-\alpha)^2{\omega_p}^2H_T^2 + \alpha^2{\omega_p'}^2D_T^2 + 2(1-\alpha)\alpha\omega_p\omega_p'H_TD_T - C_T^2(1-\alpha)^2{\omega_p}^2\\
&\quad - C_T^2k^2\alpha^2{\omega_p'}^2 - 2C_T^2k(1-\alpha)\alpha\omega_p\omega_p'\Big]\\
&> H_T^2B_T^2\Big[(1-\alpha)^2{\omega_p}^2(H_T^2 - C_T^2) + \alpha^2{\omega_p'}^2([C_Tk+\xi]^2 - C_T^2k^2)\\
&\quad + 2(1-\alpha)\alpha\omega_p\omega_p'(H_T[C_Tk+\xi] - C_T^2k)\Big].
\end{aligned}$$

In fact,

$$\begin{aligned}
M &= b_0^2(H_T^2B_T^2) + b_0^2(C_T^2H_T^2 + B_T^2C_T^2) - a_0^2b_1^2\\
&= b_0^2(C_T^2H_T^2 + B_T^2C_T^2) + b_0^2(H_T^2B_T^2) - C_T^2H_T^2B_T^2b_1^2\\
&= b_0^2(C_T^2H_T^2 + B_T^2C_T^2) + H_T^2B_T^2[b_0^2 - C_T^2[(1-\alpha)\omega_p + k\alpha\omega_p']^2]\\
&> H_T^2B_T^2\Big[\Big((1-\alpha)\omega_pH_T + \alpha\omega_p'D_T\Big)^2 - C_T^2[(1-\alpha)\omega_p + k\alpha\omega_p']^2\Big]\\
&> H_T^2B_T^2\Big[(1-\alpha)^2{\omega_p}^2H_T^2 + \alpha^2{\omega_p'}^2D_T^2 + 2(1-\alpha)\alpha\omega_p\omega_p'H_TD_T - C_T^2(1-\alpha)^2{\omega_p}^2\\
&\quad - C_T^2k^2\alpha^2{\omega_p'}^2 - 2C_T^2k(1-\alpha)\alpha\omega_p\omega_p'\Big]\\
&> H_T^2B_T^2\Big[(1-\alpha)^2{\omega_p}^2(H_T^2 - C_T^2) + \alpha^2{\omega_p'}^2([C_Tk+\xi]^2 - C_T^2k^2)\\
&\quad + 2(1-\alpha)\alpha\omega_p\omega_p'(H_T[C_Tk+\xi] - C_T^2k)\Big].
\end{aligned}$$

Since $H_T > C_T$, then we have, $M > 0$.

$$\begin{aligned}
K_3 &= b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) + a_0^2b_1^2(1 - 2{\mathscr{R}_0}^2)\\
&= b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) + a_0^2b_1^2(1 - {\mathscr{R}_0}^2 - {\mathscr{R}_0}^2)\\
&= b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) - a_0^2b_1^2{\mathscr{R}_0}^2 + a_0^2b_1^2(1 - {\mathscr{R}_0}^2)\\
&> b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) - a_0^2b_1^2\\
&= M > 0.
\end{aligned}$$

$$\begin{aligned}
K_2 &= \frac{1}{b_0^2}\Big[b_0^4(B_T^2 + C_T^2 + H_T^2) + b_0^2b_1^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) - b_1^4{\mathscr{R}_0}^2a_0^2\Big]\\
&= \frac{1}{b_0^2}\Big[b_0^4(B_T^2 + C_T^2 + H_T^2) + b_1^2[b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) - b_1^2{\mathscr{R}_0}^2a_0^2]\Big]\\
&> \frac{1}{b_0^2}\Big[b_0^4(B_T^2 + C_T^2 + H_T^2) + b_1^2[b_0^2(H_T^2B_T^2 + C_T^2H_T^2 + B_T^2C_T^2) - b_1^2a_0^2]\Big]\\
&> \frac{1}{b_0^2}\Big[b_0^4(B_T^2 + C_T^2 + H_T^2) + b_1^2M\Big]
\end{aligned}$$

Therefore $K_2 > 0$.

Since $K_1 > 0$, $K_2 > 0$, $K_3 > 0$ and $K_4 > 0$. This implies that the equation (3.5) has no

roots. This shows that (3.5) can not have a purely imaginary root.
Thus C_0 is locally asymptotically stable.

3.2.2 Local Stability at C^*

The characteristic equation of system (3.1) at the coronavirus endemic equilibrium C^* is:

$$\begin{vmatrix} \boldsymbol{\lambda} + d_p + \beta_I(I^* + kA^*)e^{-\boldsymbol{\lambda}\tau} & 0 & \beta_I S^* e^{-\boldsymbol{\lambda}\tau} & \beta_I S^* k e^{-\boldsymbol{\lambda}\tau} & 0 \\ -\beta_I(I^* + kA^*)e^{-\boldsymbol{\lambda}\tau} & \boldsymbol{\lambda} + B_T & -\beta_I S^* e^{-\boldsymbol{\lambda}\tau} & -\beta_I S^* k e^{-\boldsymbol{\lambda}\tau} & 0 \\ 0 & -(1-\alpha)\omega_p & \boldsymbol{\lambda} + C_T & -\xi & 0 \\ 0 & -\alpha\omega_p' & 0 & \boldsymbol{\lambda} + H_T & 0 \\ 0 & 0 & -\gamma_p & -\gamma_p' & \boldsymbol{\lambda} + d_p \end{vmatrix} = 0.$$

It is equivalent to

$$\boldsymbol{\lambda}^4 + c_3\boldsymbol{\lambda}^3 + c_2\boldsymbol{\lambda}^2 + c_1\boldsymbol{\lambda} + c_0 + e^{-\boldsymbol{\lambda}\tau}[d_3\boldsymbol{\lambda}^3 + d_2\boldsymbol{\lambda}^2 + d_1\boldsymbol{\lambda} + d_0] = 0, \tag{3.9}$$

where c_0, c_1, c_2, c_3, d_1, d_2, and d_3 are defined as in chapter 2.

Theorem 9

- Let $\tau = 0$. If $\mathscr{R}_0 > 1$, then the endemic equilibrium point C^* is locally asymptotically stable.
- If $\tau > 0$, the endemic equilibrium point C^* is locally asymptotically stable if
  $1 < \mathscr{R}_0 < 3$,
  and
  $F_2, F_4, F_6, d_1^2 - 2d_0d_2, d_2^2 - 2d_1d_3$ are strictly positive,
  where
  $$F_2 = c_1^2 - 2c_0c_2 + 2d_0d_2 - d_1^2,$$
  $$F_4 = c_2^2 + 2c_0 - 2c_1c_3 - d_1^2 + 2d_1d_3,$$
  $$F_6 = c_3^2 - 2c_2 - d_3^2.$$

Proof:

- Let $\tau = 0$, we get the same characteristic equation as for the no-delay model. Then The endemic equilibrium point is locally asymptotically stable, under condition $\mathscr{R}_0 > 1$, as shown in Chapter 2.

- Let $\tau > 0$.

If $i\omega(\omega > 0)$ is a solution of (3.9), then (3.9) becomes
$\omega^4 - c_2\omega^2 + c_0 - ic_3\omega^3 + ic_1\omega + e^{-i\omega\tau}[-i\omega^3 d_3 - d_2\omega^2 + i\omega d_1 + d_0] = 0.$
Separating real and imaginary parts, it follows that

$$\begin{cases} \omega^4 - c_2\omega^2 + c_0 + (d_0 - d_2\omega^2)\cos\omega\tau + (d_1\omega - d_3\omega^3)\sin\omega\tau = 0, \\ -c_3\omega^3 + c_1\omega + (d_1\omega - d_3\omega^3)\cos\omega\tau + (d_2\omega^2 - d_0)\sin\omega\tau = 0. \end{cases} \tag{3.10}$$

Thus

$$\begin{cases} \cos\omega\tau = \dfrac{(d_2 - c_3d_3)\omega^6 + (c_1d_3 - c_2d_2 + c_3d_1 - d_0)\omega^4 + (c_2d_0 - c_1d_1 + c_0d_2)\omega^2 - c_0d_0}{(d_2\omega^2 - d_0)^2 + (d_1\omega - d_3\omega^3)^2}, \\ \sin\omega\tau = \dfrac{d_3\omega^7 - (d_1 - c_3d_2 + c_2d_3)\omega^5 - (c_3d_0 - c_2d_1 + c_1d_2 - c_0d_3)\omega^3 - (c_0d_1 - c_1d_0)\omega}{(d_2\omega^2 - d_0)^2 + (d_1\omega - d_3\omega^3)^2}. \end{cases} \tag{3.11}$$

Replacing the above expressions in the "fundamental trigonometric formula",
we get

$$\Big(\omega^8 + c_0^2 + \omega^2 c_1^2 - 2\omega^6 c_2 + \omega^4 c_2^2 + 2\omega^4 c_0 - 2c_0 c_2 \omega^2 - 2c_1 c_3 \omega^4 + c_3^2 \omega^6$$
$$-d_0^2 - d_1^2 \omega^2 + 2d_0 d_2 \omega^2 - d_2^2 \omega^4 + 2d_1 d_3 \omega^4 - d_3^2 \omega^6\Big) \cdot \Big(d_0^2 + \omega^2 d_1^2 - 2\omega^2 d_0 d_2 + \omega^4 d_2^2$$
$$-2\omega^4 d_1 d_3 + \omega^6 d_3^2\Big) = 0.$$

Then

$$\Big[\omega^8 + (c_3^2 - 2c_2 - d_3^2)\omega^6 + (c_2^2 + 2c_0 - 2c_1 c_3 - d_2^2 + 2d_1 d_3)\omega^4$$
$$+(c_1^2 - 2c_0 c_2 + 2d_0 d_2 - d_1^2)\omega^2 + c_0^2 - d_0^2\Big] \cdot \Big[d_3^2 \omega^6 + (d_2^2 - 2d_1 d_3)\omega^4 + (d_1^2 - 2d_0 d_2)\omega^2 + d_0^2\Big] = 0.$$

Consequently

$$\Big[\omega^8 + F_6\omega^6 + F_4\omega^4 + F_2\omega^2 + c_0^2 - d_0^2\Big] \cdot \Big[d_3^2 \omega^6 + (d_2^2 - 2d_1 d_3)\omega^4 + (d_1^2 - 2d_0 d_2)\omega^2 + d_0^2\Big] = 0. \tag{3.12}$$

We have
$c_0^2 - d_0^2 = -d_p^2 a_0^2 (\mathscr{R}_0 - 3)(\mathscr{R}_0 - 1)$.
Under the assumption $1 < \mathscr{R}_0 < 3$ we get $c_0^2 - d_0^2 > 0$ and since F_2, F_4, F_6, $d_1^2 - 2d_0 d_2$, $d_2^2 - 2d_1 d_3$, are strictly positive. We conclude that the left hand-side of equation (3.12) is strictly positive, which implies that (3.9) does not have imaginary roots, and therefore C^* is locally asymptotically stable for any time delay $\tau > 0$.

Chapter 4

SIPR Model

A constant population model with four compartments is proposed, see Fig. 4.1. The compartments are defined as Susceptible (S), Infected (I), Infected symptomatic positive tested (P) and Recovered (R).

In Fig. 4.1, I is the infectious population compartment representing the population in the incubation stage, i.e., prior to the onset of symptoms.

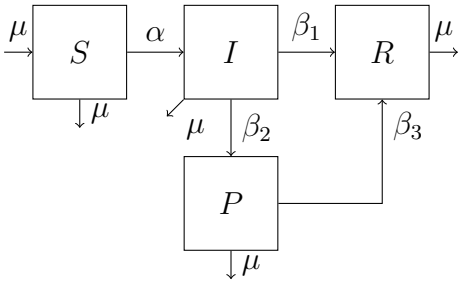

Figure 4.1: SIPR model with constant population.

The infected population can be asymptomatic or symptomatic, the incubation period is assumed to be 5.1 days. Infectiousness occurs 12 hours prior to the onset of the symptoms for the symptomatic. On the other hand, for those who are asymptomatic, the infectiousness is assumed to occur 4.6 days after infection. The average time between infection and infectiousness is 6.5 days.

Those who are asymptomatic or do not develop severe symptoms, i.e., cases which are neither tested nor documented are moved to the R (Recovered) compartment after a $1/\beta_1$ period [2].

The incubation period for those who are symptomatic is $1/\beta_2$. Once the infected individual is tested positive and the case is documented, it is moved to the P compartment, which consists of those patients with severe symptoms seeking medical attention, to assess the potential role of multiple preventive measures and strategies imposed. After a period $1/\beta_3$ the P population that recovers is moved to compartment R.

The possibility of temporary immunity for recovered individuals is also considered.

In addition, the population growth and death rates μ, including deaths due to COVID, are considered to be equal.

$$N = S(t) + I(t) + P(t) + R(t) \tag{4.1}$$

From the foregoing considerations, the SIPR mathematical model is given by,

$$\begin{aligned}
\frac{d}{dt}S(t) &= \mu - \alpha(1-\theta)S(t)I(t) - \mu S(t);\\
\frac{d}{dt}I(t) &= \alpha(1-\theta)S(t)I(t) - (\beta_1+\beta_2)I(t) - \mu I(t);\\
\frac{d}{dt}P(t) &= \beta_2 I(t) - \beta_3 P(t) - \mu P(t);\\
\frac{d}{dt}R(t) &= \beta_1 I(t) + \beta_3 P(t) - \mu R(t).
\end{aligned} \tag{4.2}$$

4.1 Non-negativity of solutions

Proposition 3

The solutions $(S(t), I(t), P(t), R(t))$ of (4.2) are non-negative for all $t \geq -\tau$ with non-negative initial conditions.

Proof:

We have,

$$\begin{cases}
\frac{d}{dt}S(t)|_{S(t)=0} = \mu > 0,\\
\frac{d}{dt}I(t)|_{I(t)=0} = 0,\\
\frac{d}{dt}P(t)|_{P(t)=0} = \beta_2 I(t) > 0,\\
\frac{d}{dt}R(t)|_{R(t)=0} = \beta_1 I(t) + \beta_3 P(t) > 0.
\end{cases} \tag{4.3}$$

In the next Section we show that model (4.2) has two equilibrium points: the disease-free and he endemic equilibrium.

4.2 Equilibrium Points

The equilibrium points of the model are obtained by equating the right-hand side of system (4.2) to zero:

$$\begin{aligned}
\mu - \alpha(1-\theta)S(t)I(t) - \mu S(t) &= 0;\\
\alpha(1-\theta)S(t)I(t) - (\beta_1+\beta_2)I(t) - \mu I(t) &= 0;\\
\beta_2 I(t) - \beta_3 P(t) - \mu P(t) &= 0;\\
\beta_1 I(t) + \beta_3 P(t) - \mu R(t) &= 0;
\end{aligned} \tag{4.4}$$

From the second equation, we obtain $I = 0$ or $S = \frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)}$.

If $I = 0$ we have $S = 1$, $P = 0$, and $R = 0$, from which the disease-free equilibrium,E_0, is given by $E_0 = (1, 0, 0, 0)$.

If $S = \frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)}$, we obtain from the first equation $I = \dfrac{\mu[1 - \frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)}]}{\beta_1+\beta_2+\mu}$, with feasibility condition $\dfrac{\alpha(1-\theta)}{\beta_1+\beta_2+\mu} > 1$. From the third equation $P = \frac{\beta_2 I}{\beta_3+\mu}$. And from the last equation

$R = \frac{\beta_1 I(t)+\beta_3 P(t)}{\mu} = \frac{(\beta_3+\mu)\beta_1+\beta_3\beta_2}{\mu(\beta_3+\mu)} I.$

Therefore $E^* = (\frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)}, \frac{\mu[1-\frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)}]}{\beta_1+\beta_2+\mu}, \frac{\beta_2 I}{\beta_3+\mu}, \frac{(\beta_3+\mu)\beta_1+\beta_3\beta_2}{\mu(\beta_3+\mu)} I)$

4.3 The Basic Reproduction Number

The basic reproduction number $\mathscr{R}_0$ for system (4.2) is found using the next generation matrix method. The reduced system of (4.2) may be written in compact form as $X' = F(X) - V(X)$ where $X = (I, P)$.

$$F(I,P) = \begin{pmatrix} \alpha(1-\theta)S(t)I(t) \\ 0 \end{pmatrix},$$

$$V(I,P) = \begin{pmatrix} (\beta_1+\beta_2)I(t) + \mu I(t) \\ -\beta_2 I(t) + \beta_3 P(t) + \mu P(t) \end{pmatrix}$$

The Jacobian matrices of $F(X)$ and $V(X)$ at the disease-free equilibrium point E_0 are

$$J_F(E_0) = \begin{pmatrix} \alpha(1-\theta) & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$J_V(E_0) = \begin{pmatrix} (\beta_1+\beta_2+\mu) & 0 \\ -\beta_2 & +(\beta_3+\mu) \end{pmatrix}$$

We find that

$$J_V^{-1}(E_0) = \begin{pmatrix} \frac{1}{\beta_1+\beta_2+\mu} & 0 \\ \frac{\beta_2}{(\beta_1+\beta_2+\mu)(\beta_3+\mu)} & \frac{-1}{\beta_3+\mu} \end{pmatrix}$$

The next generation matrix is

$$-J_F(E_0)J_V^{-1}(E_0) = \begin{pmatrix} \frac{\alpha(1-\theta)}{\beta_1+\beta_2+\mu} & 0 \\ 0 & 0 \end{pmatrix}$$

Thus

$$\mathscr{R}_0 = \rho\Big[-J_F(E_0)J_V^{-1}(E_0)\Big] = \max_{\lambda \in spec[-J_F(E_0)J_V^{-1}(E_0)]} |\lambda| = \frac{\alpha(1-\theta)}{\beta_1+\beta_2+\mu}.$$

We have the following theorem

Theorem 10

System (4.2) has the following equilibrium:

1. The corona virus-free equilibrium $E_0 = (S_0, 0, 0, 0) = (1, 0, 0, 0)$ which exists always.

2. In addition, if $\mathscr{R}_0 > 1$ then system (4.2) admits another non-trivial equilibrium,$E^* = (S^*, I^*, P^*, R^*)$. Where

$$S^* = \frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)},$$

$$I^* = \frac{\mu[1-\frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)}]}{\beta_1+\beta_2+\mu},$$

$$P^* = \frac{\beta_2 I^*}{\beta_3+\mu},$$

$$R^* = \frac{(\beta_3+\mu)\beta_1+\beta_3\beta_2}{\mu(\beta_3+\mu)} I^*.$$

4.4 Stability analysis

Now, we prove some sufficient conditions for the local asymptotic stability of the disease-free equilibrium, E_0, and the endemic equilibrium point, E^*.

The characteristic equation at any equilibrium point is given by

$$\Delta_1(\lambda) := det(\lambda I - M) = 0, \tag{4.5}$$

with

$$M = \begin{pmatrix} -\mu - \alpha(1-\theta)I & -\alpha(1-\theta)S & 0 & 0 \\ \alpha(1-\theta)I & -(\beta_1+\beta_2+\mu) + \alpha(1-\theta)S & 0 & 0 \\ 0 & \beta_2 & -(\beta_3+\mu) & 0 \\ 0 & \beta_1 & \beta_3 & -\mu \end{pmatrix}.$$

So, (4.5) becomes

$$\begin{vmatrix} \lambda+\mu+\alpha(1-\theta)I & \alpha(1-\theta)S & 0 & 0 \\ -\alpha(1-\theta)I & \lambda+(\beta_1+\beta_2+\mu)-\alpha(1-\theta)S & 0 & 0 \\ 0 & -\beta_2 & \lambda+\beta_3+\mu & 0 \\ 0 & -\beta_1 & -\beta_3 & \lambda+\mu \end{vmatrix} = 0. \tag{4.6}$$

4.4.1 Stability of E_0

Theorem 11

If $\mathscr{R}_0 < 1$, then the disease-free equilibrium E_0 is locally asymptotically stable.
If $\mathscr{R}_0 > 1$, then the disease-free equilibrium E_0 is unstable.

Proof:

From (4.6) the characteristic equation at the disease-free equilibrium $E_0(1,0,0,0)$ is given by,

$$\begin{vmatrix} \lambda+\mu & \alpha(1-\theta) & 0 & 0 \\ 0 & \lambda+(\beta_1+\beta_2+\mu)-\alpha(1-\theta) & 0 & 0 \\ 0 & -\beta_2 & \lambda+\beta_3+\mu & 0 \\ 0 & -\beta_1 & -\beta_3 & \lambda+\mu \end{vmatrix} = 0. \tag{4.7}$$

Therefore (4.7) becomes

$$P(\lambda) := (\lambda+\mu)^2(\lambda+\beta_3+\mu)[\lambda+\beta_1+\beta_2+\mu-\alpha(1-\theta)] = 0. \tag{4.8}$$

Let $\mathscr{R}_0 < 1$. We need to prove that all roots of the characteristic Equation (4.8) has negative real parts. It is easy to see that
$\lambda_1 = -\mu$, $\lambda_2 = -\beta_3 - \mu$ and
$\lambda_3 = -(\beta_1+\beta_2+\mu) + \alpha(1-\theta) = (\beta_1+\beta_2+\mu)(\mathscr{R}_0 - 1)$
are roots of Equation (4.8) and all of them are real negative roots. Therefore, E_0, is locally asymptotically stable , whenever $\mathscr{R}_0 < 1$.
Suppose now that $\mathscr{R}_0 > 1$. We know that the characteristic Equation (4.8) has two real negative roots $\lambda_1 = -\mu$, $\lambda_2 = -(\beta_3+\mu)$. Thus, we need to check if the remaining roots of

$$q(\lambda) := \lambda + \beta_1 + \beta_2 + \mu - \alpha(1-\theta) = 0 \tag{4.9}$$

have positive real roots. We have $q(0) = \beta_1+\beta_2+\mu-\alpha(1-\theta) = (\beta_1+\beta_2+\mu)(1-\mathscr{R}_0) < 0$ because we assume $\mathscr{R}_0 > 1$. And $\lim_{\lambda\to+\infty} q(\lambda) = +\infty$. Therefore, by continuity of $q(\lambda)$, there is at least one positive root of the characteristic equation (4.8). Therefore, we conclude that E_0 is unstable when $\mathscr{R}_0 > 1$. The proof is complete.

4.5 Stability of the endemic equilibrium point

Theorem 12

If $\mathscr{R}_0 > 1$, then the endemic equilibrium point E^* is locally asymptotically stable.

Proof:

The characteristic equation, computed at the endemic equilibrium E^*, is given by

$$P^*(\lambda) = (\lambda+\mu)(\lambda+\beta_3+\mu)\Big[(\lambda+\mu)(\lambda+A)+\alpha(1-\theta)[(\lambda+A)I^*-(\lambda+\mu)S^*]\Big] \quad (4.10)$$

where $A = \beta_1 + \beta_2 + \mu$.

Then, the equation (4.10) becomes

$$P^*(\lambda) = (\lambda+\mu)(\lambda+\beta_3+\mu)\Big[\lambda^2+\lambda(A+\mu)+\mu A+\alpha(1-\theta)[(\lambda(I^*-S^*)+AI^*-\mu S^*]\Big] \quad (4.11)$$

Looking at the roots of the characteristic Equation (4.11), it is easy to see that $\lambda_1 = -\mu$ and $\lambda_2 = -(\beta_3 + \mu)$ are real negative roots of (4.11). Considering the third term of the above equation, let

$$P_1^*(\lambda) := \lambda^2 + \lambda(A+\mu) + \mu A + \alpha(1-\theta)[(\lambda(I^* - S^*) + AI^* - \mu S^*]$$

Then,

$$P_1^*(\lambda) := \lambda^2 + \lambda[A + \mu + \alpha(1-\theta)(I^* - S^*)] + \mu A + \alpha(1-\theta)(AI^* - \mu S^*)$$

Using the Routh-Hurwitz criterion, we know that all roots of $P_1^*(\lambda)$ have negative real parts if, and only if, the coefficients of $P_1^*(\lambda)$ are strictly positive.

We have

$A + \mu + \alpha(1-\theta)(I^* - S^*) = \mu\mathscr{R}_0 > 0$

$\mu A + \alpha(1-\theta)(AI^* - \mu S^*) = A\mu(\mathscr{R}_0 - 1) > 0$ since $\mathscr{R}_0 > 1$.

Consequently when $\mathscr{R}_0 > 1$, then the endemic equilibrium point E^* is locally asymptotically stable.

Chapter 5

Delay SIPR Model

Now we present the second model's delayed version. Same as we did for our model, representing the time taken for the infection to run.

$$\begin{aligned}
\frac{d}{dt}S(t) &= \mu - \alpha(1-\theta)S(t-\tau)I(t-\tau) - \mu S(t);\\
\frac{d}{dt}I(t) &= \alpha(1-\theta)S(t-\tau)I(t-\tau) - (\beta_1+\beta_2)I(t) - \mu I(t);\\
\frac{d}{dt}P(t) &= \beta_2 I(t) - \beta_3 P(t) - \mu P(t);\\
\frac{d}{dt}R(t) &= \beta_1 I(t) + \beta_3 P(t) - \mu R(t);
\end{aligned} \tag{5.1}$$

5.1 Stability analysis

Now, we prove some sufficient conditions for the local asymptotic stability of the disease-free equilibrium, E_0, and the endemic equilibrium point, E^*.

The characteristic equation at any equilibrium point is given by

$$\Delta_1(\lambda,\tau) := det(\lambda I - M_1 - e^{-\lambda\tau}M_2) = 0 \tag{5.2}$$

with

$$M_1 = \begin{pmatrix} -\mu & 0 & 0 & 0\\ 0 & -(\beta_1+\beta_2+\mu) & 0 & 0\\ 0 & \beta_2 & -(\beta_3+\mu) & 0\\ 0 & \beta_1 & \beta_3 & -\mu \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -\alpha(1-\theta)I & -\alpha(1-\theta)S & 0 & 0\\ \alpha(1-\theta)I & \alpha(1-\theta)S & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, (5.2) becomes

$$\begin{vmatrix} \lambda+\mu+\alpha(1-\theta)Ie^{-\lambda\tau} & \alpha(1-\theta)Se^{-\lambda\tau} & 0 & 0\\ -\alpha(1-\theta)Ie^{-\lambda\tau} & \lambda+(\beta_1+\beta_2+\mu)-\alpha(1-\theta)Se^{-\lambda\tau} & 0 & 0\\ 0 & -\beta_2 & \lambda+\beta_3+\mu & 0\\ 0 & -\beta_1 & -\beta_3 & \lambda+\mu \end{vmatrix} = 0. \tag{5.3}$$

5.1.1 Stability of E_0

Theorem 13

If $\mathscr{R}_0 < 1$, then the disease-free equilibrium E_0 is locally asymptotically stable for any time-delay $\tau \geq 0$.
If $\mathscr{R}_0 > 1$, then the disease-free equilibrium E_0 is unstable for any time-delay $\tau \geq 0$.

Proof:

From (5.3) the characteristic equation at the disease-free equilibrium, E_0 is given by

$$P(\lambda) = (\lambda + \mu)^2 (\lambda + \beta_3 + \mu) [\lambda + \beta_1 + \beta_2 + \mu - \alpha(1-\theta)e^{-\lambda\tau}] = 0, \tag{5.4}$$

Let $\mathscr{R}_0 < 1$.
We divide the proof into the non-delayed and delayed cases.

- Let $\tau = 0$. We get the same characteristic equation as for the no-delay model (4.2). Then the coronavirus-free equilibrium point is locally asymptotically stable, as shown in Chapter 4.

- Let $\tau > 0$. and consider the third term of equation (5.4)

$$\lambda + \beta_1 + \beta_2 + \mu - \alpha(1-\theta)e^{-\lambda\tau} = 0 \tag{5.5}$$

In this case, If $j\omega$ ($\omega > 0$ and j is the complex identity element) is a solution of (5.5), then (5.5) becomes

$$j\omega + \beta_1 + \beta_2 + \mu - \alpha(1-\theta)e^{-j\omega} = 0.$$

Then, $j\omega + \beta_1 + \beta_2 + \mu - \alpha(1-\theta)[\cos(\omega\,\tau) - j\sin(\omega\,\tau)] = 0$.
Separating real and imaginary parts, it follows that;

$$\begin{cases} \omega + \alpha(1-\theta)\sin(\omega\,\tau) = 0, \\ \beta_1 + \beta_2 + \mu - \alpha(1-\theta)\cos(\omega\,\tau) = 0. \end{cases}$$

Then

$$\begin{cases} \sin(\omega\,\tau) = \frac{-\omega}{\alpha(1-\theta)}, \\ \cos(\omega\,\tau) = \frac{\beta_1+\beta_2+\mu}{\alpha(1-\theta)}. \end{cases}$$

By adding up the squares of both equations, and using the fundamental trigonometric formula, we obtain that

$$\omega^2 + (\beta_1 + \beta_2 + \mu)^2 = \alpha^2(1-\theta)^2,$$

which is equivalent to

$$\omega^2 = \alpha^2(1-\theta)^2 - (\beta_1 + \beta_2 + \mu)^2, \tag{5.6}$$

therefore $\omega^2 = (\beta_1 + \beta_2 + \mu)^2[\mathscr{R}_0^2 - 1]$. If $\mathscr{R}_0 < 1$, then we have $\omega^2 < 0$, which is a contradiction. Therefore, we have proved that whenever $\mathscr{R}_0 < 1$, the characteristic Equation (5.5) cannot have pure imaginary roots and the disease-free equilibrium E_0 is locally asymptotically stable, for any strictly positive time-delay τ.
Suppose now that $\mathscr{R}_0 > 1$. We know that the characteristic Equation (5.4) has two real negative roots $\lambda_1 = -\mu$, $\lambda_2 = -(\beta_3 + \mu)$. Thus, we need to check if the remaining roots of

$$q(\lambda) := \lambda + \beta_1 + \beta_2 + \mu - \alpha(1-\theta)e^{-\lambda\tau} = 0$$

have positive real roots. We have

$$q(0) = \beta_1 + \beta_2 + \mu - \alpha(1-\theta) = (\beta_1 + \beta_2 + \mu)(1 - \mathscr{R}_0) < 0$$

because we are assuming $\mathscr{R}_0 > 1$. And $\lim_{\lambda \to +\infty} q(\lambda) = +\infty$. Therefore, by continuity of $q(\lambda)$, there is at least one positive root of the characteristic equation (5.4). Hence, we conclude that E_0 is unstable when $\mathscr{R}_0 > 1$. The proof is complete.

5.2 Stability of the endemic equilibrium point

Theorem 14

Let $\tau = 0$. If $\mathscr{R}_0 > 1$, then the endemic equilibrium point E^* is locally asymptotically stable. When $\tau > 0$, the endemic equilibrium point E^* is locally asymptotically stable if the basic reproduction number $\mathscr{R}_0$ satisfies the following relations:

$$1 < \mathscr{R}_0 < 3 \tag{5.7}$$

and

$$N_2, N_3 > 0$$

where

$$N_2 = \frac{1}{\mathscr{R}_0^2 A^2}\Big[(\mathscr{R}_0 - 1)[2\mu^3 A(2\mathscr{R}_0^2 - 4\mathscr{R}_0 + 12\mu A^3 - \mu^4(\mathscr{R}_0 - 1)\mathscr{R}_0(\mathscr{R}_0 - 2)] + \\ + 2A^2\mu^2\mathscr{R}_0(3 - 2\mathscr{R}_0)\Big]$$

$$N_3 = \frac{1}{\mathscr{R}_0^2}(\mathscr{R}_0 - 1)\Big[A^2\mu^2(-\mathscr{R}_0 + 3) + \mu^4(-2\mathscr{R}_0^3 + 10\mathscr{R}_0^2 - 14\mathscr{R}_0 + 5) + 4A\mu^3(\mathscr{R}_0 - 2)^2\Big]$$

Proof:

The characteristic Equation, computed at the endemic equilibrium E^*, is given by

$$P^*(\lambda, \tau) = (\lambda + \mu)(\lambda + \beta_3 + \mu)\left[(\lambda + \mu)(\lambda + A) + \alpha(1 - \theta)[(\lambda + A)I^* - (\lambda + \mu)S^*]e^{-\tau\lambda}\right]$$

Then

$$P^*(\lambda, \tau) = (\lambda+\mu)(\lambda+\beta_3+\mu)\left[\lambda^2+\lambda(A+\mu)+\mu A+\alpha(1-\theta)[\lambda(I^*-S^*)+AI^*-\mu S^*]e^{-\tau\lambda}\right] \tag{5.8}$$

Let $\tau = 0$. In this case, the proof is the same as the proof of Theorem 12.

Let $\tau > 0$. By Rouche's theorem, we prove that all roots of the characteristic equation cannot intersect the imaginary axis, i.e., the characteristic equation cannot have pure imaginary roots. Suppose the opposite, i.e., that there exists $w \in \mathbb{R}$ such that $y = jw$ is a solution of (5.8). Replacing λ in the third term of (5.8), we get

$$-w^2+jw(A+\mu)+\mu A+\alpha(1-\theta)[\cos(\tau w)-j\sin(\tau w)][jw(I^*-S^*)+AI^*-\mu S^*] = 0.$$

Then,

$$\begin{cases} -w^2 + \mu A + \alpha(1-\theta)(AI^* - \mu S^*)\cos(\tau w) + \alpha(1-\theta)w(I^* - S^*)\sin(\tau w) = 0, \\ w(A+\mu) + \alpha(1-\theta)w(I^* - S^*)\cos(\tau w) - \alpha(1-\theta)(AI^* - \mu S^*)\sin(\tau w) = 0. \end{cases}$$

Then, separating real and imaginary parts, it follows that;

$$\begin{cases} \cos(\tau w) = \dfrac{w^2(AS^* - \mu I^*) - \mu A(AI^* - \mu S^*)}{\alpha(1-\theta)[(AI^* - \mu S^*)^2 + w^2(I^* - S^*)^2]}, \\ \sin(\tau w) = \dfrac{w(A^2 I^* - \mu^2 S^*) - w^3(I^* - S^*)}{\alpha(1-\theta)[(AI^* - \mu S^*)^2 + w^2(I^* - S^*)^2]}. \end{cases}$$

By adding up the squares of both equations and using the fundamental trigonometric formula, we obtain:

$$N_1 w^6 + N_2 w^4 + N_3 w^2 + N_4 = 0, \tag{5.9}$$

Where

$$N_1 = (I^* - S^*)^2,$$
$$N_2 = 2(A^2 I^* - \mu^2 S^*)(I^* - S^*) + (AS^* - \mu I^*)^2 - (I^* - S^*)^4\alpha^2(1-\theta)^2,$$
$$N_3 = (A^2 I^* - \mu^2 S^*)^2 - 2(AS^* - \mu I^*)\mu A(AI^* - \mu S^*) - 2(AI^* - \mu S^*)^2(I^* - S^*)^2\alpha^2(1-\theta)^2,$$
$$N_4 = \mu^2 A^2(AI^* - \mu S^*)^2 - (AI^* - \mu S^*)^4\alpha^2(1-\theta)^2.$$

Lemma 2

if $1 < \mathscr{R}_0 < 3$ then $N_4 \geq 0$.

Proof:

$$\begin{aligned}
N_4 &= \mu^2A^2(AI^* - \mu S^*)^2 - (AI^* - \mu S^*)^4\alpha^2(1-\theta)^2,\\
&= (AI^* - \mu S^*)^2\Big[\mu^2A^2 - (AI^* - \mu S^*)^2\alpha^2(1-\theta)^2\Big],\\
&= (AI^* - \mu S^*)^2\Big[\mu^2A^2 - \Big(\frac{\mu(\mathscr{R}_0-2)}{\mathscr{R}_0}\Big)^2A^2\mathscr{R}_0^2\Big],\\
&= (AI^* - \mu S^*)^2\Big[\mu^2A^2 - \Big(\frac{\mu^2(\mathscr{R}_0-2)^2}{\mathscr{R}_0^2}\Big)A^2\mathscr{R}_0^2\Big],\\
&= (AI^* - \mu S^*)^2\Big[\mu^2A^2[1 - (\mathscr{R}_0-2)^2\Big],\\
&= \mu^4A^2\frac{(\mathscr{R}_0-2)^2}{\mathscr{R}_0^2}(3-\mathscr{R}_0)(\mathscr{R}_0-1).
\end{aligned}$$

Therefore $N_4 \geq 0$ if and only if $1 < \mathscr{R}_0 < 3$.

$$N_2 = \frac{1}{\mathscr{R}_0^2A^2}\Big[(\mathscr{R}_0-1)[2\mu^3A(2\mathscr{R}_0^2 - 4\mathscr{R}_0 + 1) + 2\mu A^3 - \mu^4(\mathscr{R}_0-1)\mathscr{R}_0(\mathscr{R}_0-2)] + 2A^2\mu^2\mathscr{R}_0(-2\mathscr{R}_0+3)\Big]$$

$$N_3 = \frac{1}{\mathscr{R}_0^2}(\mathscr{R}_0-1)\Big[A^2\mu^2(-\mathscr{R}_0+3) + \mu^4(-2\mathscr{R}_0^3 + 10\mathscr{R}_0^2 - 14\mathscr{R}_0 + 5) + 4A\mu^3(\mathscr{R}_0-2)^2\Big]$$

Under the assumption $N_2 > 0$ $N_3 > 0$ and lemma (2), We conclude that the left-hand-side of equation (5.9) is strictly positive, which implies that (5.8) does not have imaginary roots, which implies that E^* is locally asymptotically stable for any time delay $\tau > 0$.

Chapter 6

Appendix

6.1 Appendix A

We provide the proof from our first model original paper [3] for the formula in Routh Hurwitz theorem: $(c_1+d_1)\Big[(c_3+d_3)(c_2+d_2)-(c_1+d_1)\Big] > (c_0+d_0)(c_3+d_3)^2$.
We have
$a_1 = B_T C_T + H_T\left(B_T + C_T\right)$
$b_0 = (1-\alpha)\omega_p H_T + \alpha\omega_p' D_T,$
$c_3 + d_3 = \beta_I\left(I^* + kA^*\right) + d_p + H_T + B_T + C_T > 0,$
$c_2 + d_2 = \left[\beta_I\left(I^* + kA^*\right) + d_p\right]\left(H_T + B_T + C_T\right) + a_1 - \Big[\alpha\omega_p' k + (1-\alpha)\omega_p\Big]\beta_I S^*,$
$c_1 + d_1 = \left[\beta_I\left(I^* + kA^*\right) + d_p\right] a_1 + H_T B_T C_T - \Big[\alpha\omega_p'\left(kd_p + D_T\right) + (1-\alpha)\omega_p\left(d_p + H_T\right)\Big]\beta_I S^*,$
$c_0 + d_0 = \left[\beta_I\left(I^* + kA^*\right) + d_p\right] H_T B_T C_T - d_p\Big[\alpha\omega_p' D_T + (1-\alpha)\omega_p H_T\Big]\beta_I S^*$
Thus
$c_3 + d_3 = \beta_I \frac{b_0}{H_T C_T} E^* + d_p + H_T + B_T + C_T > 0$
$c_0 + d_0 = \beta_I b_0 B_T E^* > 0$
$c_1 + d_1 = \frac{\beta_I b_0 a_1}{C_T H_T} E^* + d_p \dfrac{\left(B_T + C_T\right) H_T^2 (1-\alpha)\omega_p + H_T B_T \alpha\omega_p' \xi + C_T\left(B_T + H_T\right)\alpha\omega_p' D_T}{b_0} > 0$
$c_2 + d_2 = \beta_I a_2\left(\frac{b_0}{C_T H_T}\right) E^* + d_p a_2 + C_T H_T + B_T \dfrac{(1-\alpha)\omega_p H_T^2 + H_T \alpha\omega_p' \xi + \alpha\omega_p' C_T D_T}{b_0} > 0.$
Moreover

$$
\begin{aligned}
&(c_3+d_3)(c_2+d_2)-(c_1+d_1)=\left(\beta_I\left[\frac{b_0}{H_TC_T}\right]E^*+d_p+H_T+B_T+C_T\right)(c_2+d_2)\\
&-\beta_I b_0\left(\frac{C_TH_T+B_T\left(C_T+H_T\right)}{C_TH_T}\right)E^*\\
&-d_p\left(\frac{\left(B_T+C_T\right)H_T^2(1-\alpha)\omega_p+H_TB_T\alpha\omega_p'\xi+C_T\left(B_T+H_T\right)\alpha\omega_p'D_T}{[b_0]}\right)\\
&=\beta_I\left[\frac{b_0}{H_TC_T}\right]E^*(c_2+d_2)\\
&+\left(d_p+H_T+B_T+C_T\right)(c_2+d_2)\\
&-\beta_I\left[b_0\right]\left(\frac{C_TH_T+B_T\left(C_T+H_T\right)}{C_TH_T}\right)E^*\\
&-d_p\left(\frac{\left(B_T+C_T\right)H_T^2(1-\alpha)\omega_p+H_TB_T\alpha\omega_p'\xi+C_T\left(B_T+H_T\right)\alpha\omega_p'D_T}{[b_0]}\right)\\
&=\beta_I\left[\frac{b_0}{H_TC_T}\right]E^*(c_2+d_2)\\
&+\left(d_p+H_T+B_T+C_T\right)\beta_I\left(B_T+C_T+H_T\right)\left(\frac{b_0}{C_TH_T}\right)E^*\\
&+\left(d_p+H_T+B_T+C_T\right)\left[d_p\left(H_T+B_T+C_T\right)+C_TH_T\right]\\
&+\left(d_p+H_T+B_T+C_T\right)B_T\left(\frac{(1-\alpha)\omega_pH_T^2+H_T\alpha\omega_p'\xi+\alpha\omega_p'C_TD_T}{b_0}\right)\\
&-\beta_I\left[b_0\right]\left(\frac{C_TH_T+B_T\left(C_T+H_T\right)}{C_TH_T}\right)E^*\\
&-d_p\left(\frac{\left(B_T+C_T\right)H_T^2(1-\alpha)\omega_p+H_TB_T\alpha\omega_p'\xi+C_T\left(B_T+H_T\right)\alpha\omega_p'D_T}{[b_0]}\right)\\
&=\beta_I\left[\frac{b_0}{H_TC_T}\right]E^*(c_2+d_2)\\
&+\beta_I\left[\left(d_p+H_T+C_T\right)H_T+\left(d_p+C_T\right)C_T\right]\left(\frac{b_0}{C_TH_T}\right)E^*\\
&+\beta_I\left(d_p+H_T+B_T+C_T\right)B_T\left(\frac{b_0}{C_TH_T}\right)E^*\\
&+\left(H_T+B_T+C_T\right)\left[d_p\left(H_T+B_T+C_T\right)+C_TH_T\right]\\
&+d_p^2\left(H_T+B_T+C_T\right)\\
&+\left(H_T+B_T+C_T\right)B_T\left(\frac{(1-\alpha)\omega_pH_T^2+H_T\alpha\omega_p'\xi+\alpha\omega_p'C_TD_T}{b_0}\right).
\end{aligned}
$$

Then

$$\begin{aligned}
\left[(c_3+d_3)(c_2+d_2)-(c_1+d_1)\right](c_1+d_1) = \beta_I\left[\frac{b_0}{H_TC_T}\right]E^*(c_1+d_1)(c_2+d_2)\\
+\beta_I\left(d_p+H_T+C_T\right)H_T\left(\frac{b_0}{C_TH_T}\right)E^*(c_1+d_1)\\
+\beta_I\left(d_p+C_T\right)C_T\left(\frac{b_0}{C_TH_T}\right)E^*(c_1+d_1)\\
+\beta_I\left(d_p+H_T+B_T+C_T\right)B_T\left(\frac{b_0}{C_TH_T}\right)E^*(c_1+d_1)\\
+\left(H_T+B_T+C_T\right)\left[d_p\left(H_T+B_T+C_T\right)+C_TH_T\right](c_1+d_1)\\
+d_p^2\left(H_T+B_T+C_T\right)(c_1+d_1)\\
+\left(H_T+B_T+C_T\right)B_T\left(\frac{(1-\alpha)\omega_pH_T^2+H_T\alpha\omega_p'\xi+\alpha\omega_p'C_TD_T}{b_0}\right)(c_1+d_1)\\
= I_1+I_2+I_3+I_4+I_5+I_6+I_7,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \beta_I\left[\frac{b_0}{H_TC_T}\right]E^*(c_1+d_1)(c_2+d_2)\\
I_2 &= \beta_I\left(d_p+H_T+C_T\right)H_T\left(\frac{b_0}{C_TH_T}\right)E^*(c_1+d_1)\\
I_3 &= \beta_I\left(d_p+C_T\right)C_T\left(\frac{b_0}{C_TH_T}\right)E^*(c_1+d_1)\\
I_4 &= \beta_I\left(d_p+H_T+B_T+C_T\right)B_T\left(\frac{b_0}{C_TH_T}\right)E^*(c_1+d_1)\\
I_5 &= \left(H_T+B_T+C_T\right)\left[d_p\left(H_T+B_T+C_T\right)+C_TH_T\right](c_1+d_1)\\
I_6 &= d_p^2\left(H_T+B_T+C_T\right)(c_1+d_1)\\
I_7 &= \left(H_T+B_T+C_T\right)B_T\left(\frac{(1-\alpha)\omega_pH_T^2+H_T\alpha\omega_p'\xi+\alpha\omega_p'C_TD_T}{b_0}\right)(c_1+d_1).
\end{aligned}$$

Since

$$\begin{aligned}
(c_2+d_2) =&\beta_I\left(B_T+C_T+H_T\right)\left(\frac{b_0}{C_TH_T}\right)E^*\\
&+d_p\left(H_T+B_T+C_T\right)+C_TH_T+B_T\left(\frac{(1-\alpha)\omega_pH_T^2+H_T\alpha\omega_p'\xi+\alpha\omega_p'C_TD_T}{[b_0]}\right)\\
>&\beta_IH_T\left(\frac{b_0}{C_TH_T}\right)E^*,\\
(c_1+d_1) =&\beta_I\left[C_TH_T+B_T\left(C_T+H_T\right)\right]\left(\frac{b_0}{C_TH_T}\right)E^*\\
&+d_pB_T\left(\frac{H_T^2(1-\alpha)\omega_p+H_T\alpha\omega_p'\xi+C_T\alpha\omega_p'D_T}{[b_0]}\right)\\
&+d_pC_TH_T\\
>&\beta_I\left[C_TH_T+B_T\left(C_T+H_T\right)\right]\left(\frac{b_0}{C_TH_T}\right)E^*.
\end{aligned}$$

Therefore

$$\begin{aligned} I_1 &> \beta_I^3 \left(B_T + H_T\right) C_T H_T \left(\frac{b_0}{C_T H_T}\right)^3 E^{*3} \\ &> \beta_I^3 B_T C_T H_T \left(\frac{b_0}{C_T H_T}\right)^3 E^{*3} \end{aligned}$$

Moreover

$$\begin{aligned} I_2 >& \beta_I^2 \left(d_p + H_T + C_T\right)\left(B_T + H_T\right) C_T H_T \left(\frac{b_0}{C_T H_T}\right)^2 E^{*2} \\ >& \beta_I^2 B_T C_T H_T \left(d_p + H_T + C_T\right) \left(\frac{b_0}{C_T H_T}\right)^2 E^{*2} \\ I_4 >& \beta_I^2 B_T C_T \left(d_p + H_T + B_T + C_T\right)\left(B_T + H_T\right) \left(\frac{b_0}{C_T H_T}\right)^2 E^{*2} \\ >& \beta_I^2 B_T C_T H_T \left(d_p + H_T + 2B_T + C_T\right) \left(\frac{b_0}{C_T H_T}\right)^2 E^{*2} \\ I_5 >& \beta_I \left[C_T H_T + B_T\left(C_T + H_T\right)\right]\left(H_T + B_T + C_T\right) d_p \left(H_T + B_T + C_T\right) \left(\frac{b_0}{C_T H_T}\right) E^* \\ &+ \beta_I \left[C_T H_T + B_T\left(C_T + H_T\right)\right]\left(H_T + B_T + C_T\right) C_T H_T \left(\frac{b_0}{C_T H_T}\right) E^* \\ >& 2\beta_I B_T C_T H_T d_p \left(H_T + B_T + C_T\right) \left(\frac{b_0}{C_T H_T}\right) E^* \\ &+ \beta_I B_T C_T H_T \left[C_T H_T + B_T\left(C_T + H_T\right)\right] \left(\frac{b_0}{C_T H_T}\right) E^* \\ &+ \beta_I B_T C_T H_T \left(C_T + H_T\right)\left(H_T + C_T\right) \left(\frac{b_0}{C_T H_T}\right) E^* \\ I_6 >& \beta_I d_p^2 \left(H_T + B_T + C_T\right)\left(B_T + H_T\right) C_T \left(\frac{b_0}{C_T H_T}\right) E^* \\ >& \beta_I B_T C_T H_T d_p^2 \left(\frac{b_0}{C_T H_T} E^*\right). \end{aligned}$$

and

$$\begin{aligned}
I_7 >&\beta_I B_T \left(H_T + B_T + C_T\right)\left[C_T H_T + B_T\left(C_T + H_T\right)\right] \times \\
&\left(\frac{(1-\alpha)\omega_p H_T^2 + H_T \alpha\omega_p' \xi + \alpha\omega_p' C_T D_T}{[b_0]}\right)\left(\frac{b_0}{C_T H_T}\right) E^* \\
>&\beta_I B_T H_T \left(H_T + B_T + C_T\right)\left[C_T H_T + B_T\left(C_T + H_T\right)\right] \times \\
&\left(\frac{(1-\alpha)\omega_p H_T}{[b_0]}\right)\left(\frac{b_0}{C_T H_T}\right) E^* \\
&+ \beta_I B_T C_T \left(H_T + B_T + C_T\right)\left[C_T H_T + B_T\left(C_T + H_T\right)\right] \times \\
&\left(\frac{\alpha\omega_p' D_T}{b_0}\right)\left(\frac{b_0}{C_T H_T}\right) E^* \\
>&\beta_I B_T C_T H_T \left[C_T H_T + B_T\left(C_T + H_T\right)\right]\left(\frac{b_0}{C_T H_T}\right) E^* \\
&+ \beta_I B_T C_T H_T \left(H_T + B_T\right)\left[H_T + B_T\right]\left(\frac{(1-\alpha)\omega_p H_T}{[b_0]}\right)\left(\frac{b_0}{C_T H_T}\right) E^* \\
&+ \beta_I B_T C_T H_T \left(B_T + C_T\right)\left[C_T + B_T\right]\left(\frac{\alpha\omega_p' D_T}{b_0}\right)\left(\frac{b_0}{C_T H_T}\right) E^* \\
>&\beta_I B_T C_T H_T \left[C_T H_T + B_T\left(C_T + H_T\right)\right]\left(\frac{b_0}{C_T H_T}\right) E^* \\
&+ \beta_I B_T C_T H_T B_T^2 \left(\frac{b_0}{C_T H_T} E^*.
\end{aligned}$$

That is

$$\begin{aligned}
I_2 + I_4 >&2\beta_I^2 B_T C_T H_T \left(d_p + H_T + B_T + C_T\right)\left(\frac{b_0}{C_T H_T}\right)^2 E^{*2} \\
I_5 + I_6 + I_7 >&2\beta_I B_T C_T H_T d_p \left(H_T + B_T + C_T\right)\left(\frac{b_0}{C_T H_T}\right) E^* \\
&+ \beta_I B_T C_T H_T \left[B_T^2 + C_T^2 + H_T^2 + 2C_T H_T + 2B_T\left(C_T + H_T\right)\right]\left(\frac{b_0}{C_T H_T}\right) E^* \\
&+ \beta_I B_T C_T H_T d_p^2 \left(\frac{b_0}{C_T H_T}\right) E^* \\
=&\beta_I B_T C_T H_T \left(d_p + B_T + C_T + H_T\right)^2 \left(\frac{b_0}{C_T H_T}\right) E^*.
\end{aligned}$$

Hence

$$\begin{aligned}
\left((c_3 + d_3)(c_2 + d_2) - (c_1 + d_1)\right)(c_1 + d_1) &> I_1 + I_2 + I_4 + I_5 + I_6 + I_7 \\
&> (c_0 + d_0)(c_3 + d_3)^2.
\end{aligned}$$

Since

$$\begin{aligned}
(c_0 + d_0)(c_3 + d_3)^2 =&\beta_I b_0 B_T E^* \left(\beta_I \left[\frac{b_0}{H_T C_T}\right] E^* + d_p + H_T + B_T + C_T\right)^2 \\
=&\beta_I^3 B_T C_T H_T \left(\frac{b_0}{C_T H_T}\right)^3 E^{*3} \\
&+ 2\beta_I^2 B_T C_T H_T \left(d_p + H_T + B_T + C_T\right)\left(\frac{b_0}{C_T H_T}\right)^2 E^{*2} \\
&+ \beta_I B_T C_T H_T \left(d_p + H_T + B_T + C_T\right)^2 \left(\frac{b_0}{C_T H_T}\right) E^*.
\end{aligned}$$

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