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Diplomamunka címe: Representations and the Montréal Functor in the p-adic Langlands Programme

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p-adic Representations and the Montréal Functor

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Notations and Conventions

- The word "ring" means ring with identity. The category of rings is denoted by Ring.
- For an algebraic structure A (e.g. a group, algebra, module, ring, etc.) that has a topology on it, $K \leq_c A$ denotes a compact substructure of A. Similarly, $K \leq_o A$ denotes an open substructure of A. Finally, $K \leq_{c,o} G$ denotes a compact and open substructure.
- If $\mathcal{C}$ is a category, then the notation $A \in \mathcal{C}$ means that A is an object of $\mathcal{C}$.
- For any ring R, R-Mod denotes the category of left R-modules. R-mod denotes the category of *finitely generated* left R-modules.
- For any ring R, $R^\times$ denotes the group of units of R.
- For any ring R, $R[x_1, \ldots, x_k], R[[x_1, \ldots, x_k]], R((x_1, \ldots, x_k))$ denote the polynomial ring, the ring of formal power series, and the ring of formal Laurent series over R with indeterminates $x_1, \ldots x_k$, respectively.
- For any field K, $\overline{K}$ denotes some (fixed) algebraic closure of K.
- A p-adic number field is a field K which is a finite extension of $\mathbb{Q}_p$.

Chapter 1

Introduction and Notes

Consider a prime p, and a finite field extension $F/\mathbb{Q}_p$. The description of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ is a central problem in number theory, as it might give an insight on the absolute Galois group of $\mathbb{Q}$. One can say that a group is reasonably well understood once the category of its representations is described. The celebrated local Langlands programme does just that; for any fixed prime $\ell \neq p$ and $n \in \mathbb{N}$, it gives a bijection between certain irreducible $\overline{\mathbb{Q}_\ell}$-linear continuous representations of $\mathrm{Gal}(\mathbb{Q}_p/F)$ over n-dimensional $\overline{\mathbb{Q}_\ell}$-vector saces, and a class of "sufficiently well-behaved" representations of $\mathrm{GL}_n(F)$ on $\overline{\mathbb{Q}_\ell}$-vector spaces. There local Langlands program is a collection of statements (some of which are still conjectural) about these bijections. Besides the description of the absolute Galois groups, another reason to be interested in such a correspondence is that representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ come from adélic automorphic representations; and the description of such representations has immense applications in analytic number theory.

The case $\ell = p$ is much less understood. p-adic Banach space representations of $\mathrm{GL}_n(F)$ are considered instead of $\overline{\mathbb{Q}_p}$- representations. As described by Vigneras [1], for the $\ell \neq p$ case, the representations of $\mathrm{GL}_n(F)$ can be replaced by certain representations on ℓ-adic Banach spaces, hence this approach to the $p = \ell$ case is a generalization of the local Langlands correspondence. When $\ell = p$, one calls the statements about the mostly conjectural correspondence between such categories of representations, the p-adic local Langlands programme. The first (and so far, only) such correspondence was established by Colmez [2], for $n = 2$ and $F = \mathbb{Q}_p$. At the present, a similar correspondence is not known for any other n and F. Although several functors, from categories of n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ to p-adic Banach-representations of $\mathrm{GL}_n(F)$, were proposed (for example in [3], [4]), none of these are known to be equivalences of categories. Colmez described his results in a famous lecture in Montréal, hence his functor is sometimes called the Montréal functor; and it is customary to call its generalizations Montréal functors as well. An important distinction between the local and the p-adic Langlands setting is that the correspondence of Colmez is actually a functor, while the local Langlands correspondence is "only" a bijection, with remarkable properties.

The goal of this thesis is, on one hand, to give an overview of the general "setting" of the p-adic Langlands programme, in particular, to describe some of the basic properties of the numerous types of representations in the local and p-adic Langlands program. On the other hand, we describe the functor of Colmez, and deduce some of its properties (although not all of them; that would vastly exceed the scope of this text). We will mostly follow Colmez [2], but will eventually deviate, and instead employ a more algebraic approach, developed by Emerton [5].

Preliminaries

We start with an elementary observation. Suppose that $\phi_i : X_i \to Y_i$ are maps of topological spaces. If both ϕ_1 and ϕ_2 are continuous, then $\phi_1 \times \phi_2 : X_1 \times X_2$ is continuous as well. If both ϕ_1 and ϕ_2 are open, then $\phi_1 \times \phi_2 : X_1 \times X_2$ is open as well. Similar trivial topological statements will be used throughout the text.

Obviously, p-adic numbers and p-adic number fields (i.e. finite extensions of $\mathbb{Q}_p$) are the fundamental objects in this text. These are non-Archimedean local fields of characteristic 0. The ring of integers is always a maximal compact subgroup (the set with absolute value at most 1), and a discrete valuation ring, with finite residue field. With a few exceptions, we will use algebraic methods, and only rarely depend on analytical techniques (but the topology on these rings is still used).

Let $G = \mathrm{GL}_n(L)$ for some p-adic number field L, and let $\mathcal{O}_L$ be the ring of integers of L, $\mathrm{GL}_n(\mathcal{O}_L)$ the maximal compact subgroup of G Then we have the following decomposition:

Proposition 1.0.1 (Iwasawa decomposition)**.** *$G = UK$, where U is the subgroup of upper triangular matrices, and $K = \mathrm{GL}_n(\mathcal{O}_L)$.*

Abelian Categories

An Abelian category is an additive category in which every morphism has a kernel and a cokernel and every monomorphism and every epimorphism is normal.

Fact 1.0.2. *For any ring R, the category of R-modules is Abelian.*

On multiple occasions in this text, it will be necessary to prove that some full subcategory of an Abelian category (usually that of certain representations) is Abelian. The following lemma is the main tool of these proofs.

Lemma 1.0.3. *Let $\mathcal{S}$ be a full subcategory of an Abelian category $\mathcal{A}$. If for all $A, B \in \mathrm{Ob}\,\mathcal{S}$, and $f \in \mathrm{Hom}_{\mathcal{S}}(A, B)$*

1. *the zero object is in $\mathrm{Ob}\,\mathcal{S}$.*
2. *$A \oplus B \in \mathrm{Ob}\,\mathcal{S}$*
3. *$\ker f$ and $\operatorname{coker} f \in \mathrm{Ob}\,\mathcal{S}$*

then $\mathcal{S}$ is an Abelian category.

For a proof, see [6, Proposition 5.92].

Coherent Rings

Let R be a (not necessarily commutative) ring with identity, $M \in R$-Mod. We say that M is finitely generated, if there exists a surjective homomorphism from R^n to M. We say that M is finitely presented, if there exists an exact sequence $R^m \to R^n \to M \to 0$. Informally, finitely presented modules are the finitely generated modules which can be defined using only finitely many relations.

Definition 1.0.4. We say that a ring R with identity is left-coherent, if every finitely generated left ideal of R is in fact finitely presented. We can similarly define the right coherence of rings.

One usually prefers to work with some finiteness condition; a common restriction is that the rings should be Noetherian. In a sense, coherent rings are a natural enlargening of the category of Noetherian rings. With the following few propositions, we wish to compare left-coherent rings to left-Noetherian rings.

Proposition 1.0.5. *Any left-Noetherian ring is left-coherent.*

Proof. A finitely generated left-ideal of R, $I \leq_R R$ is just an exact sequence $R^n \to I \to 0$. R^n is a left-Noetherian module over R, implying that all of its submodules are finitely generated. In particular, $\ker R^n \to I$ is finitely generated, hence I is finitely presented. □

By the same argument, any finitely generated module over a Noetherian ring is finitely presented as well.

Proposition 1.0.6. *Let R be a ring. Then the following are equivalent.*

1. *R is left-Noetherian.*
2. *The category R-mod of finitely generated left modules over R is Abelian.*

Proposition 1.0.7. *Let R be a ring. Then the following are equivalent.*

1. *R is left-coherent.*
2. *The category $R\text{-mod}^{\mathrm{fp}}$ of finitely presented left modules over R is Abelian.*

Althought the definition of left-coherence seems technical, it is "the best" one can get when working with non-Noetherian rings. In particular, proposition 1.0.7 can be an extremely useful property.

Flatness

Definition 1.0.8. Let R be a ring, $M \in \text{Mod-}R$ a right module. We say that M is flat, if the functor $M \otimes_A - : R\text{-Mod} \to \text{Ab}$ is exact.

If $\varphi : R \to S$ is a ring homomorphism, we say that φ is flat, if the functor $S \otimes_R - : R\text{-Mod} \to S\text{-Mod}$ is exact.

Note that since S is itself an S-R bimodule, its tensor product with any left R-module (i.e. any R-$\mathbb{Z}$ bimodule) is an S-$\mathbb{Z}$ bimodule, hence a left S-module. This shows that the functor $S \otimes_R -$ indeed has an image in S-Mod. Furthermore, since any exact sequence of Abelian groups that consists of left S-modules and left S-module homomorphisms is exact as a sequence of S-modules, we have that $\varphi : R \to S$ is flat if and only if S is a flat right module over R.

Suppose that R is a ring, and $F : R \to R$ is a ring-endomorphism. Then R has an R-R-bimodule structure, where multiplication from the left is just the usual multiplication of R, but multiplication by right is defined by F; $r' \circ r \stackrel{\text{def}}{=} r' \cdot F(r)$. For any left R-module $M \in R\text{-Mod}$, the tensor product $F^*M \stackrel{\text{def}}{=} (R, \circ) \otimes_R M$ is then a left R-module with the usual multiplication of R: $r \cdot (r' \otimes m) \stackrel{\text{def}}{=} (rr') \otimes m$. The flatness condition on F means precisely that $M \mapsto F^*M$ is an exact functor. Note that in F^*M we have that $F(r) \otimes m = 1 \otimes rm$.

Pontryagin Dual of Modules

Let A be a locally compact, commutative topological group, then the Pontryagin dual $A^\vee$ is the set of continuous group homomorphisms from A to $\mathbb{T} = S^1 \leq \mathbb{C}^\times$, equipped with the compact open topology. Pontryagin duality is a contravariant endofunctor on the category of locally compact Abelian groups which is, in fact a contravariant equivalence $LCA \to LCA^{op}$. It sends projective limits to injective limits, and maps any compact group to a discrete group and vice versa.

If A is equipped with a continuous group action of some group G from the left, then $A^\vee$ has two possible group actions (both continuous), one from the left, the other from the right. Let $\mu \in A^\vee$.

1. $(g\mu)(a) \stackrel{\text{def}}{=} \mu(g^{-1}a)$ or
2. $(\mu g)(a) \stackrel{\text{def}}{=} \mu(ga)$.

Throughout this text, we will use the **second** definition, which is more common in the literature.

If A happens to be a (not necessarily topological) left R-module for some ring R, then, and if R is not commutative, the only reasonable way to equip A with the structure of a topological R module is to act with R from the right: $(\mu r)(a) \stackrel{\text{def}}{=} \mu(ra)$. The problem with the convention we just fixed for group actions on Abelian groups is that if A is a G-representation over a ring R, the action of G on the dual group is from the left, but the action of R on the dual group is from the right. For commutative R, this problem is obviously not present, since any left module is a right module as well. To avoid this problem, we simply require R to be commutative whenever the dual of a module over $R[G]$ is considered. To summarize, if R is a commutative ring, G is a group, then if $M \in R[G]\text{-Mod}$, then $M^\vee \in R[G]\text{-Mod}$ via the inverse action: if $\mu \in M^\vee$, then $(rg)\mu(m) \stackrel{\text{def}}{=} \mu(rg^{-1}m)$.

Chapter 2

p-adic Representations

Representations of locally profinite groups are the main objects of the "GL_n" side of the local Langlands correspondence and the p-adic Langlands correspondence. As the work of Colmez ([2]), which we cover in chapter 3, almost entirely handles the GL_n side, these representations are actually the main objects we consider throughout this thesis.

In this chapter, we introduce the various types of representations that shall be used in, or are related to the description of the Montréal functor. We will also highlight a key difference between representations over p-adic and ℓ-adic modules with $\ell \neq p$.

2.1 Locally Profinite Groups

The most general type of topological group that occurs in the p-adic Langlands programme is a locally profinite group.

Definition 2.1.1. We say that a topological group G is profinite, if it is the inverse limit of finite topological groups.

Although this definition seems restrictive, the following proposition shows that such groups are actually quite general among totally disconnected groups.

Proposition 2.1.2. *A topological group is profinite if and only if it is Hausdorff, compact, and totally disconnected.*

This proposition trivially implies that closed subgroups, quotients, and finite products of profinite groups are again profinite. In particular:

Lemma 2.1.3. *Let $K = \varprojlim K/H$ be a profinite group. Then any closed subgroup C of K is again profinite, and $C = \varprojlim C/(H \cap C)$.*

The prime example of a profinite group is the ring of p-adic integers, $\mathbb{Z}_p$. We call a profinite group pro-p, if it is the inverse limit of finite subgroups, each of which has order p^n for some prime number $p \in \mathbb{N}$. The ring of integers $\mathcal{O}_K$ for any p-adic number field K is a pro-p group.

Proposition 2.1.4. *If G is a profinite group, and $1 \in G$ is the identity element, then one can choose a neighbourhood basis of 1 consisting of open normal subgroups.*

Note that since G is compact, any open subgroup is of finite index. In general, any open subgroup is closed, hence in a Hausdorff compact topological group, any open subgroup is actually compact. We obtain:

Corollary 2.1.5. *In a profinite group G, 1 admits a neighbourhood basis consisting of compact, open subgroups.*

If G is compact, Hausdorff, then the converse of this statement holds as well.

The non-compact generalization of a profinite group is a locally profinite group:

Definition 2.1.6. We say that a topological group G is locally profinite, if it has a profinite subgroup.

Trivially, a locally profinite group is profinite if and only if it is compact. From the properties of profinite groups, the following proposition is more or less trivial.

Proposition 2.1.7. *Let G be a topological group. Then the following are equivalent:*

1. *G is locally profinite.*
2. *G is locally compact, Hausdorff, and totally disconnected.*
3. *$1 \in G$ admits a neighbourhood basis consisting of compact open subgroups.*

An example of a locally profinite group is $(\mathbb{Q}_p, +)$; condition 1. is trivially satisfied with $H = \mathbb{Z}_p$. If K is a p-adic number field, then, since $\mathcal{O}_K$ is profinite, K is locally profinite.

The last big family of groups we consider are the p-adic Lie groups.

Definition 2.1.8. We say that a group G is a p-adic Lie group, if it is topological group that is also an analytic manifold over $\mathbb{Q}_p$.

Proposition 2.1.9. *Any p-adic Lie group is locally profinite.*

We will mostly work with the group $\mathrm{GL}_n(K)$ for some p-adic number field K. It is clearly a p-adic Lie group, hence profinite. A neighbourhood basis of 1 in $\mathrm{GL}_n(K)$ is $\{K_k\}_{k\in N}$, where $K_k = 1 + p^k\mathbb{Z}_p^{n\times n}$. We will refer to this family of subgroup as the "standard" neighbourhood basis of 1 in $\mathrm{GL}_n(L)$. A locally profinite group that admits a compact and open pro-p is called a locally pro-p group. The standard neighbourhood basis of 1 shows that any p-adic number field is locally pro-p. It is important to note that for a locally pro-p group, not *all* compact open subgroups are necessarily pro-p. For example, the standard neighbourhood basis of $\mathrm{GL}_n(\mathbb{Q}_p)$ consists of pro-p, compact open subgroups (showing that $\mathrm{GL}_n(\mathbb{Q}_p)$ is locally pro-p); but $\mathrm{GL}_n(\mathbb{Z}_p)$, which is the maximal compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ is not pro-p.

2.2 Smooth Representations

The most general type of representation we consider is a smooth representation. We allow representations over arbitrary base rings instead of just fields; this is necessary to work with representations over $\mathbb{Z}_p$, $\mathcal{O}_K$ (for some p- adic number field K), or any finite quotient of these rings.

Definition 2.2.1. For any group G, a representation of G over a ring R is a pair (π, M), where $M \in R$-Mod, and $\pi : G \to \mathrm{Aut}_R(M)$ is a group homomorphism. Given two representations $(\pi_1, M_1), (\pi_2, M_2)$, we say that a morphism of modules $f : M_1 \to M_2$ is a G-equivariant map, if $\forall g \in G$, $\forall m \in M_1$: $\pi_2(g)f(m) = f(\pi_1(g)m)$. A representation is called irreducible, if the only submodules $N \leq M$ that satisfy $\pi(G)N \subset N$ are $N = M$ and $N = 0$. For a subset $U \subseteq G$, we use the notation $M^U = \{m \in M \mid \forall u \in U\colon \pi(u)m = m\}$ for the submodule of U-invariant elements. For some $m \in M$, $\mathrm{Stab}_G(m) = \{g \in G \mid \pi(g)m = m\}$ is the stabilizer of m.

Definition 2.2.2. The representations of G over R form a category, with the morphisms being the G-equivariant maps. We denote this category by $\mathrm{Rep}_R(G)$.

Proposition 2.2.3. $\mathrm{Rep}_R(G)$ *is equivalent to* $R[G]$-Mod.

Here, $R[G]$ is the group ring of G with coefficients in R. The proof of this proposition is straightforward. It implies, in particular, that $\mathrm{Rep}_R(G)$ is an Abelian category.

Let $Z(G)$ denote the center of G. We say that a representation $(\pi, M) \in \mathrm{Rep}_R(G)$ is of *central character*, if there exists a multiplicative character $\delta_Z : Z \to R^\times$, such that $\pi(g)m = \delta(g)m$ for all $g \in Z(G)$. Note that we do not require the central character to be unique.

Suppose now that G is a topological group. We only wish to consider representations that satisfy certain continuity properties. To make this precise, for a representation $(\pi, M) \in \mathrm{Rep}_R(G)$, one can consider the map $\Phi_\pi : G \times M \to M$, $(g, m) \mapsto \pi(g)m$.

Definition 2.2.4. If $(\pi, M) \in \mathrm{Rep}_R(G)$ is a representation of G over R, where M is equipped with a topology, then we say that (π, M) is a *continuous representation*, if the map $\Phi_\pi : G \times M \to M$ is continuous.

If $(\pi, M) \in \mathrm{Rep}_R(G)$ is any representation of G over R (and M is not equipped with a topology), then we call (π, M) a *smooth representation*, if $\Phi_\pi : G \times M \to M$ is continuous with respect to the discrete topology on M (i.e. (π, M) is a continuous representation w.r.t. the discrete topology).

Notice that for a smooth representation, each of the maps $\varphi_m : G \to M$, $g \mapsto \pi(g)m$ satisfies that the preimage of any set is open. In particular, if we were to give M any other topology, these maps would still remain continuous. Hence a smooth representation is "as continuous as possible". This justifies the term "smooth". Note that if $H \leq G$ is any subgroup, then the restriction of a smooth representation from G to H is again smooth: indeed, $H \times M \to M$ is just the composition $H \times M \to G \times M \to M$, which is continuous. We emphasize that even if R is a topological ring, the continuity or smoothness of a G-representation does not depend on this topology.

Proposition 2.2.5. *Let G be any topological group, $(\pi, M) \in \mathrm{Rep}_R(G)$ for some ring $R \in \mathrm{Ring}$. Then the following are equivalent:*

a) (π, M) is smooth.

b) For any $m \in M$, the map $\varphi_m : G \to M$, $g \mapsto \Phi_\pi(g, m) = \pi(g)m$ is locally constant, i.e. for any $g \in G$, there exists an open neighbourhood $g \in U$ such that $\forall x, y \in U : \pi(x)m = \pi(y)m$.

Proof. a) $\implies$ b): The continuity of Φ_π implies that each φ_m is continuous: $\varphi_m = \Phi_\pi \circ i_m$, where $i_m : G \to G \times M$, $g \mapsto (g, m)$ is a continuous function. Hence φ_m is the composition of continuous functions. In particular, the preimage of any singleton set by φ_m is open. But then for any $h \in G$, the set $P = \varphi_m^{-1}(\{\varphi_m(h)\})$ is open, contains h, and, by definition, φ_m is constant on P.

b) $\implies$ a): By our assumption, φ_m is locally constant, hence for any point $x \in \varphi_m^{-1}(n)$, there is an open neighbourhood U of x in G such that $U \subseteq \varphi^{-1}(n)$. This implies that $\varphi_m^{-1}(n)$ is open, since for any point $x \in \varphi_m^{-1}(n)$, there is an open neighbourhood of x that is fully contained in $\varphi_m^{-1}(n)$. It follows that φ_m is continuous for any m. But $\Phi_\pi^{-1}(n) = \{(g, m) \in G \times M \mid \pi(g)m = n\} = \bigcup_{m \in M} \{g \in G \mid \pi(g)m = n\} \times \{m\} = \bigcup_{m \in M} \varphi_m^{-1}(n) \times \{m\}$, which is a union of finite products of open sets, hence itself open. The continuity of Φ_π follows. □

The above proposition implies that π is constant on each connected component of G. This shows that one can only expect good properties of smooth representations only if G is totally disconnected. It is furthermore reasonable to require G to be Hausdorff and locally compact; hence the groups of interest are precisely the locally profinite groups. We shall see that smooth representations of locally profinite groups indeed have nice properties.

Proposition 2.2.6. *Suppose that G is a locally profinite group, $(\pi, M) \in \mathrm{Rep}_R(G)$ for some ring R. Then the following are equivalent:*

a) (π, M) is smooth.

b) $\forall m \in M$: $g \mapsto \pi(g)m$ is locally constant.

c) $\forall m \in M \ : \exists H \leq_o G$ open subgroup s.t. $m \in M^H$.

d) $\forall m \in M \ : \exists H \leq_{o,c} G$ open compact subgroup s.t. $m \in M^H$.

e) For any $m \in M$, $\mathrm{Stab}_G(m) \leq_o G$ is an open (hence closed) subgroup.

f) $M = \bigcup_{H \leq_o G} M^H$, where the union is taken over all open subgroups of G.

g) $M = \bigcup_{H \leq_{o,c} G} M^H$, where the union is taken over all compact open subgroups of G.

Proof. We have seen a) $\iff$ b) in general.

a) $\implies$ e): We have that $\varphi_m : g \mapsto \pi(g)m$ is continuous, hence for all $n \in M$: $\varphi_m^{-1}(n)$ is open. Then $\mathrm{Stab}_G(m) = \{g \in G \mid \pi(g)m = m\} = \varphi_m^{-1}(m)$ is also open.

e) $\implies$ d): We have a neighbourhood basis of e consisting of compact open sugroups, hence (since $\mathrm{Stab}_G(m)$ is open) $\exists K$ compact open, such that $K \subseteq \mathrm{Stab}_G(m)$. This compact open subgroup then stabilizes m.

d) $\implies$ c): trivially.

c) $\implies$ b): We need that for any $m \in M$, φ_m is locally constant. Fix some $h \in G$. Then for any $s \in U$: $\varphi_m(hs) = \pi(hs)m = \pi(h)\pi(s)m = \pi(h)m = \varphi_m(h)$. Since $1 \in U$, we have that $h \in hU$. Multiplication with h is a homeomorphism of G, hence hU is open. By the previous argument, we have seen that φ_m is constant on hM. This shows that φ_m is locally constant.

f) and g) are just reformulations of c) and d). □

Definition 2.2.7. The full subcategory of $\mathrm{Rep}_R(G)$ with objects being the smooth representations of G over R is denoted $\mathrm{SRep}_R(G)$.

Proposition 2.2.8. *If G is locally profinite, then $\mathrm{SRep}_R(G)$ is an Abelian category.*

Proof. We will use lemma 1.0.3. Clearly, 0 is a smooth representation. For $(\pi, M), (\rho, N)$ smooth representations, the direct sum $(\pi \oplus \rho, M \oplus N)$ is smooth, since $\Phi_{\pi\oplus\rho} : (g, (m, n)) \mapsto (\pi(g)m, \rho(g)n)$ is continuous. This can be seen by considering the preimage of a singleton (m, n): $P \overset{\text{def}}{=} \{(g, a, b) : \pi(g)a = m \wedge \rho(g)b = n\}$. But then $P = \big(\Phi_\pi^{-1}(m) \times N\big) \cap \big(\Phi_\rho^{-1}(n) \times M\big)$, as a subset of $G \times M \times N$. This shows that P is open. For an equivariant morphism $f : (\pi, M) \to (\rho, N)$, let us denote the kernel of f by (τ, K). Then $K \leq M$ is a submodule. If $k \in K \subseteq M$ is fixed by an open subgroup U: $k \in M^U$, then it is fixed by U as an element of K as well: $k \in K^U$. By 2.2.6, (τ, K) is smooth. Let $(\alpha, C) = \operatorname{coker} f$, then $C = N/\operatorname{im} f$. If $n \in N$ is fixed by some open sugroup $U \leq_o G$, then in C, we have $U(n + \operatorname{im} f) = Un + U \operatorname{im} f = n + \operatorname{im} f$, which shows that $n + \operatorname{im} f \in C^U$. By 2.2.6, C is smooth. □

Maschke's theorem is a useful tool when working with representations in general. The following is a smooth version of Maschke's theorem. The restriction $|K : N| \cdot 1_R \in R^\times$ will turn out to be problematic in the p-adic Langlands setting, as one usually has pro-p groups and representations over $\mathcal{O}_K$, where $\mathcal{O}_K$ is a ring of integers of a p-adic number field.

Lemma 2.2.9 (Maschke's theorem)**.** *Let G be any group, R any ring with identity. Then $R[G]$ is semisimple if and only if the following conditions hold:*

i) *R is semisimple.*
ii) *G is finite.*
iii) $|G| \cdot 1_R \in R^\times$.

For a proof, see for example the book [7] of Milies and Sehgal on group rings.

Theorem 2.2.10 (Smooth Maschke's theorem for profinite groups)**.** *Suppose that K is a profinite group, $R \in \mathrm{Ring}$ a semisimple ring. Furthermore assume, that $1 \in K$ has a neighbourhood basis consisting of open normal subgroups $N \lhd_o K$, satisfying $|K : N| \cdot 1_R \in R^\times$. Then any smooth representation $(\pi, M) \in \mathrm{SRep}_R(K)$ is semisimple.*

Proof. Let $(\pi, M) \in \mathrm{SRep}_R(K)$. We fix $v \in M$. Let M_v be the R-module generated by the K-orbit $K \cdot v$ of v; it is then an $R[K]$-submodule of M, i.e. a subrepresentation. (π, M) is a smooth representation $\implies$ by 2.2.6, $\exists U \leq_o K$ open subgroup, such that $v \in M^U$. By the assumption, $\exists N \lhd_o K$ such that $|K : N|$ is invertible in R, and $N \subseteq U$. Hence $v \in M^N$. But then $M_v \subseteq M^N$, and we have a K/N-action on M_v; indeed, if $k_1^{-1} \cdot k_2 \in N$, then for all $w \in M_v$, $k_2 w = (k_1 \cdot n)w = k_2 w$ (for some $n \in N$).

$|K : N|$ is invertible in R $\implies$ by the general Maschke's theorem 2.2.9, $R[K/N]$ is semisimple. All modules over a semisimple ring are semisimple $\implies$ M is semisimple as an $R[K/N]$-module, i.e. $M_v = \bigoplus_{i \in I} S_i$ where each S_i is a simple K/N-representation. But then S_i is a simple K-representation as well. We can prove this by contradiction: if $W \leq S_i$ was any K-subrepresentation, then $KW \subseteq W \implies \forall k \in K : kW \subseteq W \implies (kN)W \subseteq W \implies W$ is a K/N subrepresentation of S_i. If W is the zero representation, over K/N, then it is the zero representation over K as well. This shows that W is a nontrivial K/N subrepresentation of S_i, which contradicts the fact that S_i is simple.

We obtained that M_v is semisimple over $R[K]$. But

$$M = \sum_{v \in M} M_v = \sum_{v \in M} \bigoplus_{i \in I(v)} S_i^{(v)}.$$

This shows that M is a sum of a family of simple submodules, i.e. is semisimple. □

Corollary 2.2.11 (Smooth Maschke's theorem for pro-p groups)**.** *Let K be a pro-p group, $R \in$ Ring a semisimple ring such that $p \cdot 1_R \in R^\times$. Then any smooth representation of K over R is semisimple.*

Proof. Since K is profinite, $1 \in K$ has a neighbourhood basis consisting of open normal subgroups. Since K is pro-p, the index of all of these subgroups is a power of p. Since p is invertible in R, so is p^k for all positive integers k. Hence we can apply 2.2.10. □

Corollary 2.2.12. *Let K be a pro-p group, $\mathbb{F}$ a field with $\operatorname{char} \mathbb{F} \neq p$. Then any smooth K-representation over $\mathbb{F}$ is semisimple.*

Once again, let K be a p-adic number field, with ring of integers $\mathcal{O}_K$, and G some pro-p (or locally pro-p) group. This corollary shows, that even though semisimplicity can not be deduced in the case of $\mathcal{O}_K$ coefficients, it can for coefficients in K.

The semisimplicity of smooth representations can be used to deduce that a certain important subcategory of smooth representations is Abelian. In the case of $\mathcal{O}_K$, a completely different apporach is needed; for which we need to introduce Iwasawa algebras.

2.3 Iwasawa Algebras

We need the notion of an Iwasawa algebra to properly handle admissible representations in rings with characteristic p; in particular in mod p representations (which are needed to tackle unitary L-Banach space representations). Furthermore, one of the ideas needed to establish the functor of Colmez is to associate a representation of a formal power series ring to a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation; the intermediate step of this association is an Iwasawa algebra.

Definition 2.3.1. Let G be a compact topological group, R a topological ring. Then $R[[G]]$ denotes the *completed group algebra* of G with coefficients in R:

$$R[[G]] = \varprojlim_{U \in ON(G)} R[G/U]$$

where $ON(G)$ is the set of open normal subgroups of G. The topology on $R[[G]]$ is given by the inverse limit of the topologies on $R[G/U]$.

Note that since G is compact, open normal subgroups are of finite index. The quotient groups with the quotient topology are discrete. We define the topology on $R[G/U]$ to be the quotient topology given by $R^{|G/U|} \to R[G/U]$. This turns $R[G/U]$ into a topological ring, and a topological R-module (where the topologies coincide). The arrows of the projective system defining the limit are simply given by $R[G/V] \to R[G/U]$; $r(g+V) \mapsto r(g+U)$, whenever $V \leq U$. The inverse limit of any inverse system of topological spaces exists, since Top is complete; and one can endow it with a natural ring structure to turn it into a topological ring. In the case when $R = \mathcal{O}_K$ for a p-adic number field K, and G is a profinite group, the completed group algebra $\mathcal{O}_K[[G]]$ is called an Iwasawa algebra. When G is profinite, open normal subgroups become "arbitrarily small", hence one can expect that the completed group algebra contains all information about G. Also, if G is profinite, a useful property of $R[[G]]$ is that it encodes the topology of both R and G (unlike the group ring $R[G]$, which in general does not have a topology that respects the topology of G). The precise formulation of this statement is the following proposition:

Proposition 2.3.2. *If G is profinite, then*

a) $R[[G]] = \varprojlim_{U \leq_o G} R[G/U]$ *where U runs on all open subgroups, as a topological space.*

b) *For any topological ring R, $R[[G]] = \varprojlim_{K \leq_{c,o} G} R[[G/K]]$ where K runs on all compact open normal subgroups.*

c) *If M is a topological module over $R[[G]]$, then M is naturally a continuous G-representation over R, and*

d) *M is naturally a topological module over R.*

Proof. $1 \in G$ has a neighbourhood basis consisting of compact open subgroups, and any open subgroup of a profinite group is compact. This immediately implies the first two propositions. d): the embedding $i_R : R \hookrightarrow R[[G]]$, which maps r to the element rI, where I is the inverse limit of $1 + U$, is continuous, since the individual maps $R \to R[G/U]$ are trivially continuous (and i_R is the map given by the universal property of the limit in the category Top). $R \times M \to M$ is just the composition of $i_R \times id$ and $R[[G]] \times M \to M$, which are both continuous maps.

c): We have a map $i_G : G \to R[[G]]$ given by $g \mapsto l_g$, where l_g is the inverse limit of the elements $1_R \cdot (g + U)$. The maps $G \to G/U;\ g \mapsto g + U$ are continuous by the definition of the profinite topology. But G/U is discrete, hence $G/U \to R[G/U]$ is continuous. The map i_G is exactly the map given by the universal property of the inverse limit in Top (applied to the limit of the topological spaces $R[G/U]$); this shows that i_G is continuous. But then $G \times M \to M$ is just the composition of $i_G \times id$ and $R[[G]] \times M \to M$, which are both continuous maps. We still need that $G \times M \to M$ is actually a representation of G. It is enough to show that there is a ring homomorphism $R[G] \to R[[G]]$, that induces i_G. $rg \mapsto \varprojlim r(g + U)$ is clearly such homomorphism of rings.

□

In the case $R = \mathcal{O}_K$, we have that

$$\mathcal{O}_K[R] \simeq \varprojlim_{k \in \mathbb{N}, U \leq_o G} O_K/\mathfrak{p}^k[G/U] \tag{2.1}$$

where $\mathfrak{p}_K$ is the unique maximal ideal of $\mathcal{O}_K$. We shall now turn our interest to Iwasawa algebras with coefficient ring $\mathcal{O}_K$ or K, and G profinite.

Proposition 2.3.3. *Let G be any compact group, $R \in$ Ring a topological ring.*

1. *If R is compact, then $R[[G]]$ is compact.*
2. *If R is Hausdorff, then $R[[G]]$ is Hausdorff.*

Proof. If R is compact, then so is $R[G/U]$ for any open subgroup of G, since the topology on $R[G/U]$ comes from the direct product topology of distinct copies of R. The inverse limit of compact spaces is compact, since it is a closed subset of the product $\prod_{i \in I} K_i$. Similarly: if R is Hausdorff, then so is $R[G/U]$. The inverse limit of Hausdorff spaces is trivially Hausdorff. □

Theorem 2.3.4. *Let G be a compact p-adic Lie group. Then*

1) *$\mathcal{O}_K[[G]]$ is left and right Noetherian.*

2) *$K[[G]]$ is left and right Noetherian.*

This theorem was proved by Lazard in [8].

Corollary 2.3.5. *If G is a p-adic Lie group, then the category of finitely generated modules over $\mathcal{O}_K[[G]]$ (resp. $K[[G]]$) is Abelian.*

The following proposition is the basis of the functor of Colmez. Both the theorem and the proof are standard. For $K = \mathbb{Q}_p$, this proposition was first formulated by Serre, in a Seminaire Bourbaki lecture on Iwasawa's results in 1959.

Proposition 2.3.6. *Let $K/\mathbb{Q}_p$ a finite field extension, $\mathcal{O}_K$ the ring of integers of K. Then*

$$\mathcal{O}_K[[\mathbb{Z}_p]] \simeq O_K[[T]]$$

where $\mathcal{O}_K[[T]]$ is the formal power series ring of $\mathcal{O}_K$ in one variable T. The isomorphism is explicitly given by

$$1_{\mathbb{Z}_p} \mapsto 1 + T.$$

Note that one classically states the theorem for some multiplicative topological group Γ that is isomorphic to the additive topological group $\mathbb{Z}_p$.

Proof. Consider the map

$$\phi_n : \mathcal{O}_K[\mathbb{Z}_p/p^n\mathbb{Z}_p] \to \mathcal{O}_K[T]/((1+T)^{p^n}-1)\mathcal{O}_K[T],$$
$$1+p^n\mathbb{Z}_p \mapsto 1+T+((1+T)^{p^n}-1)\mathcal{O}_K[T].$$

We claim that ϕ_n is an isomorphism of topological rings. As modules, both sides are free $\mathcal{O}_K$ modules of rank p^n. The "variable" on both sides generates the module together with $\mathcal{O}_K$, and is of rank p^n. We obtain that the two sides are isomorphic as topological modules. ϕ_n is clearly a ring homomorphism. All of the ϕ_n are isomorphisms. We have compatible (via ϕ_n) inverse systems of $(\mathcal{O}_K[\mathbb{Z}_p/p^n\mathbb{Z}_p])_{n\in\mathbb{N}}$ and $\left(\mathcal{O}_K[T]/((1+T)^{p^n}-1)\mathcal{O}_K[T]\right)_{n\in\mathbb{N}}$. This shows the desired isomorphism. □

Corollary 2.3.7. *$\mathcal{O}_K/\varpi_K^n\mathcal{O}_K[[\mathbb{Z}_p]] \simeq \mathcal{O}_K/\varpi_K^n\mathcal{O}_K[[T]]$ as topological rings. The images of the reduction maps are precisely the two sides of the isomorphism in the statement.*

Proof. The reduction mod ϖ^n is a surjective map both on $\mathcal{O}_K[[\mathbb{Z}_p]]$ and on $\mathcal{O}_K[[T]]$; and the kernels correspond to each other via the isomorphism of proposition 2.3.6. □

The following proposition is extremely important for our later discussion; it implies that $R[[H]] \to R[[G]]$ is a flat morphism of rings.

Proposition 2.3.8. *Let G be a profinite group, R a compact, Hausdorff topological ring, $H \leq_{c,o} G$ a compact and open subgroup of G. Then $R[[H]]$ is a closed subalgebra of $R[[G]]$. Furhtermore, if we denote a set of coset representatives of G/H by $g_1,\ldots,g_n$, then $R[[G]] = \bigoplus_{i=1}^n g_iR[[H]]$.*

Proof. G is profinite, hence it has a neighbourhood basis of 1 consisting of open normal subgroups. This basis then gives a neighbourhood basis of 1 in H. But then this basis is cofinal in the system of open subgroups of H, hence $R[[H]] = \varprojlim_{U\leq_o G, U\subseteq H} R[H/U]$. Similarly, $R[[G]] = \varprojlim_{U\leq_o G, U\subseteq H} R[G/U]$. The first inverse system is injected in the second ("pointwise"), hence by the left exactness of the inverse limit functor, we get an embedding $R[[H]] \hookrightarrow R[[G]]$. By proposition 2.3.3, $R[[G]]$ is Hausdorff, and $R[[H]]$ is a compact subspace of $R[[G]]$, which shows that $R[[H]]$ is closed.

For the second statement; if N is an open subgroup of G, contained in H, then $R[G/N] = \bigoplus_{i=1}^n g_iR[H/N]$, because the cosets of H in G are disjoint. This decomposition is compatible with the inverse system that defines the completed group algebra. Since G is profinite, the open subgroups contained in H are cofinite in the inverse system of open subgroups of G. The proposition follows. □

Proposition 2.3.9. *Let G be a profinite group. Then the natural homomorphism $\mathcal{O}_K[G] \hookrightarrow \mathcal{O}_K[[G]]$, is injective with a dense image. Furthermore, composing this map with the embedding $G \hookrightarrow \mathcal{O}_K[G]$, $g \mapsto 1_{\mathcal{O}_K}g$, gives a continuous embedding $G \to \mathcal{O}_K[[G]]$ of topological spaces (in particular, G is homeomorphic to its image).*

A proof can be found for example in [9]. Note that the embedding $G \hookrightarrow \mathcal{O}_K[[G]]$; $g \mapsto 1_{\mathcal{O}_K}g$ has an image in $\mathcal{O}_K[[G]]^\times$, and $G \hookrightarrow \mathcal{O}_K[[G]]^\times$ is hence a continuous embedding of topological groups.

Lemma 2.3.10. *Let G be a profinite group, R a compact, Hausdorff topological ring, $H \leq_{c,o} G$ a compact and open subgroup of G, and let $M \in R[[G]]$-Mod. Then M is finitely generated over $R[[G]]$ if and only if M is finitely generated over $R[[H]]$.*

Proof. By the first statement of proposition 2.3.8, $R[[H]]$ is a closed subalgebra of $R[[G]]$. It follows, that if M is finitely generated over $R[[H]]$, then M is finitely generated over $R[[G]]$. For the converse, by the second statement of 2.3.8, we have that $R[[G]] = \bigoplus_{i=1}^n g_iR[[H]]$, for some $g_i \in G$. But then if M is finitely generated over $R[[G]]$, then the finitely many generators of M, multiplied by the elements g_i, give a finite generating set of M as an $R[[H]]$-module. □

We now consider a version of Nakayama's lemma applicable to profinite modules over compact rings. For any ring R, its Jacobson radical is denoted by $J(R)$. The following theorem in this precise form was given by Balister and Howson (who pointed out an error in earlier proofs), in [10].

Definition 2.3.11. Let R be a topological ring, I a left ideal of R. We write $I^n \to 0$, if for any open neighbourhood U of 0 in R, there is an $n \in \mathbb{N}$ such that $I^n \subseteq U$.

Theorem 2.3.12 (Nakayama Lemma). *Let Λ be a compact topological ring, I a left ideal with $I^n \to 0$ in the sense that there exists a neighbourhood basis U_n of . Let X be a profinite topological module. Then*

1. *If $IX = X$, then $X = 0$.*
2. *If I is a two-sided ideal, X/IX is finitely generated as a Λ/I-module, then X is finitely generated as a Λ-module.*

We wish to apply theorem 2.3.12 for the augmentation ideal an Iwasawa algebra.

Now we prove that the Jacobson radical of the Iwasawa algebra $\mathcal{O}_K[[G]]$ contains the augmentation ideal. This, together with the Nakayama Lemma will be an essential tool to handle admissible representations in characteristic p.

Definition 2.3.13. Let G be any group, R a ring. The augmentation ideal of $R[G]$ is is the kernel of the ring homomorphism $Aug : \sum r_i g_i \mapsto \sum r_i$. The augmentation ideal is denoted by I_G.

It is straightforward to show that I_G is generated (as a two-sided ideal, or even as a left-ideal) by elements of the form $g - 1_G$, where g is any element of G. If F is a field, since $F[G]/\ker Aug \simeq \operatorname{im} Aug = F$, we have that I_G is a maximal two-sided ideal of $F[G]$ (in fact, I_G is a maximal F-submodule of $F[G]$).

The following is a well-known lemma in the theory of group rings.

Lemma 2.3.14. *Leg G be a finite p-group, and F a field with $\operatorname{char} F = p$. Then I_G is a nil ideal (i.e. every element of I_G is nilpotent).*

Proposition 2.3.15. *Let G be a finite p-group, F a field with $\operatorname{char} F = p$. Then $J(F[G]) = I_G$.*

Proof. In non-commutative rings, like $F[G]$, the Jacobson radical does not necessarily contain all nilpotent elements, however it does contain all nil two-sided ideals. Indeed, if I is a nil ideal of a ring R, then we need that for any $i \in I$, $1 - sir \in R^\times$. Since I is a two-sided ideal, it is enough to show that $1 - i \in R^\times$. $i^n = 0$ for some n. But then $1 + i + \dots i^{n-1}$ is an inverse of $1 - i$. The claim follows from lemma 2.3.14. □

We now return to the case of $\mathcal{O}_K$ coefficients. The residue field is denoted by $k = \mathcal{O}_K/\varpi\mathcal{O}_K$.

Corollary 2.3.16. *Let G be a finite p-group. Then $J(\mathcal{O}_K[G]) = (I_G, \varpi)$.*

Proof. We have a surjection $\pi : \mathcal{O}_K[G] \to k[G]$. By proposition 2.3.15, I_G^k is the unique maximal left ideal of $k[G]$. The image of any maximal left ideal of $\mathcal{O}_K[G]$ is then either I_G^k or $k[G]$. The only maximal left ideal of $\mathcal{O}_K[G]$ with image I_G^k is (ϖ, I_G^k). If $\pi(N) = k[G]$, then $(N, \varpi) = \mathcal{O}_K[G]$, i.e. $\varpi \notin N$, for any maximal ideal N. But for such N, $1 \in \pi(N)$, i.e. $\exists a \in \mathcal{O}_K$ such that $a \cdot 1 \in N$. Here $a \notin \varpi\mathcal{O}_K$, since then $\varpi \in N$ would be the case; hence a is a unit. But then N contains a unit of $\mathcal{O}_K[G]$. This is a contradiction. □

Definition 2.3.17. Let G be a profinite group. Consider the the projection map

$$\mathcal{O}_K[[G]] \to \mathcal{O}_K[G/G] \xrightarrow{\simeq} \mathcal{O}_K. \tag{2.2}$$

The augmentation ideal $\overline{I_G}$ of $\mathcal{O}_K[[G]]$ is the kernel of this homomorphism.

Let $H \lhd_o G$ be an open normal subgroup of G, with a respective projection map $\pi_H : \mathcal{O}_K[[G]] \to \mathcal{O}_K[G/H]$. Let $\tau_H : \mathcal{O}_K[G/H] \to \mathcal{O}_K[G/G]$ be the projection $\sum a_i(g_i + H) \mapsto \sum a_i(g_i + G)$. Then $\overline{I_G} = \pi_H^{-1}(\ker \tau_H)$, since both π_H and τ_H are surjective, and $\pi_G = \tau_H \circ \pi_H$. Since $\ker \tau_H$ is just the augmentation ideal $I_{G/H} \lhd \mathcal{O}_K[G/H]$, we have that $\overline{I_G} = \varprojlim_{H \lhd_o G} I_{G/H}$.

Lemma 2.3.18. *Let G be a pro-p group. Then $(\overline{I_G}, \varpi) = J(\mathcal{O}_K[[G]])$.*

Proof. We have seen (2.3.16) that $(I_{G/H}, \varpi)$ is a maximal left ideal of $\mathcal{O}_K[G/H]$, hence its preimage is a maximal left ideal of $\mathcal{O}_K[[G]]$ (the projection is surjective). Hence the Jacobson radical is a subset of $(\overline{I_G}, \varpi)$. It is now enough to show that $(\overline{I_G}, \varpi)$ is contained in the Jacobson radical. The Jacobson radical of $\mathcal{O}_K[[G]]$ is the intersection of all maximal left ideals. In the morphism $\pi_H : \mathcal{O}_K[[G]] \to \mathcal{O}_K[G/H]$, the preimage of a maximal left ideal is itself a maximal left ideal. If $x \in \overline{I_G}$, then $\pi_H(x)$ is in $J(\mathcal{O}_K[G/H]) = (I_{G/H}, \varpi)$ (again, proposition 2.3.16 can be used since G is pro-p). Now any maximal ideal of $\mathcal{O}_K[G/H]$ is of the form $\pi_H^{-1}(M)$, since if Q is a maximal ideal, then either $\pi_H(Q) = \mathcal{O}_K[G/H]$ for all H, or $\pi_H(Q)$ is a maximal ideal of $\mathcal{O}_K[G/H]$ for some H. In the first case, $Q = \mathcal{O}_K[[G]]$ shows that Q would actually not be maximal. This shows the proposition. □

We also obtain the following corollary:

Corollary 2.3.19. *Let G be an Abelian pro-p group. Then $\mathcal{O}_K[[G]]$ is a local domain with unique maximal ideal $(\overline{I_G}, \varpi)$.*

2.4 Admissible Representations

Definition 2.4.1. Let G be a locally profinite group. We say that a smooth representation $(\pi, M) \in \mathrm{SRep}_R(G)$ is admissible, if for any open subgroup $H \leq_o G$ the H-invariant submodule $M^H = \{m \in M \mid \forall h \in H \ : \ \pi(h)m = m\}$ is finitely generated. The full subcategory of SRep_R spanned by the admissible representations is denoted $\mathrm{ARep}_R(G)$

Proposition 2.4.2. *Let R be a left-Noetherian ring, G locally profinite, and $(\pi, M) \in \mathrm{SRep}_R(G)$. Then the following are equivalent:*
a) (π, M) is admissible.
b) $\forall K \leq_{o,c} G$ compact open subgroup: M^K is finitely generated.

Proof. a) $\implies$ b) trivially.
b) $\implies$ a): e has a neighbourhood basis consisting of open compact groups in G. If H is any open subgroup, then $\exists K$ compact open subgroup of G with $K \subseteq H$. But then $M^H \leq M^K$ is a submodule. Since R is left-Noetherian and M^K is finitely generated by our assumption, M^H is finitely generated as well. □

Note that we have our representations over left-modules. If we instead work with right-modules, we obviously get an analogous statement with R being right-Noetherian. Unlike for smooth representations, the category of admissible representations is not Abelian in general. We will give sufficient conditions for this to be the case.

Lemma 2.4.3. *Let G be a locally profinite group, $R \in \mathrm{Ring}$ semisimple, such that there exists an $L \leq_{o,c} G$ open compact subgroup of G for which $\forall U \leq_o L$ open subgroup satisfies that $|L : U| \cdot 1_R \in R^\times$. Then $1 \in G$ has a neighbourhood basis consisting of compact open subgroups K, such that for all smooth representations $(\pi, M) \in \mathrm{SRep}_R(G)$, the restriction $(\pi\big|_K, M) \in \mathrm{SRep}_R(K)$ is semisimple.*

Proof. Fix some open set $V \subseteq G$. Since G is locally profinite, $\exists K \leq_{o,c} G$ open compact subgroup such that $K \subseteq L \cap V$. If $W \leq_o K$ is an open subgroup of K, then W is an open subgroup of L, and K is an open subgroup of L. But then $|L : W| = |L : K| \cdot |K : W|$, which shows that $|K : W|$ is invertible in R. We know that K is profinite, hence $1 \in K$ has a neighbourhood basis consisting of normal open subgroup. Since the index of an arbitrary open subgroup in K is invertible by the previous argument, the index of these normal subgroups (w.r.t K) is invertible as well. We can then apply theorem 2.2.10 for K, to any smooth representation $(\pi, M) \in \mathrm{SRep}_R(G)$. The semisimplicity of $(\pi\big|_K, M)$ follows. □

Lemma 2.4.4. *Let G be a locally profinite group, $R \in \mathrm{Ring}$ any ring, $K \leq G$ any subgroup satisfying that for all $(\pi, M) \in \mathrm{SRep}_R(G)$: the restriction $(\pi\big|_K, M) \in \mathrm{SRep}_R(K)$ is semisimple. Then the functor $\mathrm{SRep}_R(G) \to \mathrm{Ab}$, $M \mapsto M^K$, is exact.*

Proof. This follows trivially from the fact that M^K is a direct summand of M as an $R[K]$-module. □

Theorem 2.4.5. *Suppose that G is a locally profinite group, $R \in \text{Ring}$ a semisimple ring. Furthermore, assume that there is an open compact $K \leq_{o,c} G$ that satisfies the following:*

$$\forall U \leq_o K \text{ open subgroup: } |K : U| \cdot 1_R \in R^\times.$$

Then $\text{ARep}_R(G)$ is an Abelian category.

Proof. We consider $\text{ARep}_R(G)$ as a full subcategory of $\text{SRep}_R(G)$, which is Abelian (proposition 2.2.8). We will use lemma 1.0.3. 0 is trivially an admissible representation. In a direct sum of representations, the submodule $(M \oplus N)^K$ is just $M^K \oplus N^K$, which is clearly finitely generated, if both M^K and N^K are. Hence the direct sum of two admissible representations is admissible. It remains to show that if f is a morphism of admissible representations, then $\ker f$ and $\operatorname{coker} f$ are both admissible. By lemma 2.4.3, there is a neighbourhood basis of $1 \in G$ consisting of compact open subgroups, which satisfy that the restriction of any smooth representation of G to these subgroups is semisimple. By lemma 2.4.4, taking the K-invariants w.r.t such groups is an exact functor, hence if $0 \to V \to M \to W \to 0$ is exact in $\text{SRep}_R(G)$, we have that $0 \to V^K \to M^K \to W^K \to 0$ is exact as well. If M is admissible, then M^K is finitely generated; since R is semisimple, R is Noetherian, hence both V^K and W^K are finitely generated. But then for any open subgroup $U \leq_o G$, V^U is finitely generated as well, since $\exists K$ from the above neighbourhood basis with $K \subseteq U$. Similarly for W^U. This shows that both W and V are admissible. □

Corollary 2.4.6. *If G is a locally profinite group that contains a compact open pro-p subgroup, and $\mathbb{F}$ is a field with $\operatorname{char} F \neq p$, then $\text{ARep}_{\mathbb{F}}(G)$ is Abelian.*

The above corollary shows that the category of admissible representations over some $L/\mathbb{Q}_\ell$ where $\ell \neq p$ is Abelian, which is important for the local Langlands conjectures. For characteristics p, however, different approaches are needed, and the theorem is not true in general. For the base ring $\mathcal{O}_K$, an alternative description of admissibility is needed. To obtain an Abelian category, one must impose additional conditions on the representations.

2.4.1 Admissibility in Characteristic p

We shall now focus on the question of admissibility in the case when the coefficient ring is $\mathcal{O}_K$, where K is a p-adic number field. Whenever U is a profinite group, $\mathcal{O}_K[[U]]$ denotes the Iwasawa algebra of G with coefficients in $\mathcal{O}_K$, as defined in section 2.3.

We introduced the augmentation ideal of a group ring in section 2.3, for any group G. If M is a G-module (i.e. a $\mathbb{Z}[G]$- module), then M is a $\mathbb{Z}[H]$-module as well, whenever $H \leq G$.

Definition 2.4.7. The module of H-coinvariants of M is $M_H = M/I_H M$.

Lemma 2.4.8. *Let M be a locally compact topological G-module. Then for any subgroup $H \leq G$, we have that $(M^H)^\vee \simeq M_H$ and $(M_H)^\vee \simeq (M^\vee)^H$.*

Note: here $\vee$ is the Pontryagin duality functor.

Proof. We first show the second proposition: Both $(M_H)^\vee$ and $(M^\vee)^H$ are submodules of $M^\vee$. Now $\mu \in M^\vee$ satisfies $\mu \in (M_H)^\vee$ if and only if μ factors as $\mu = \nu \circ \pi$, where $\pi : M \to M_H$. This is further equivalent to the assertion that $\mu((h-1)m) = 0$ for all $m \in M, h \in H$. Equivalently: $h^{-1}\mu(m) = \mu(m)$ for all $m \in M, h \in H$. I.e: $h^{-1}\mu = \mu$, $\mu \in (M^\vee)^H$.

For the second statement, we will use the first one. For any G-module N: $(N_H)^\vee = (N^\vee)^H$. Dualizing gives $(N_H)^{\vee\vee} = ((N^\vee)^H)^\vee$. But then $N_H \simeq ((N^\vee)^H)^\vee$. Applying this for $N = M^\vee$, we get that $(M^\vee)_H \simeq ((M^{\vee\vee})^H)^\vee \simeq (M^H)^\vee$. □

Proposition 2.4.9. *Let G be a locally profinite group, R a ring, $(\pi, M) \in \text{SRep}_R(G)$ a smooth G-representation (equipped with the discrete topology). Then the Pontryagin dual group $M^\vee$ has a natural structure of an $R[[K]]$-module, for any compact open subgroup K of G.*

Proof. (π, M) is a smooth representation. By 2.2.6,

$$M = \bigcup_{U \leq_o G} M^U = \varinjlim_{U \leq_o G} M^U.$$

Applying the Pontryagin dual (M is given the discrete topology), by lemma 2.4.8 and the fact that the dual of a direct limit is the inverse limit of the dual groups, we obtain

$$M^\vee = \varprojlim_{U \leq_o G} M_U.$$

For any fixed open subgroup $U \leq_o G$, we have an $R[G/U]$-action on M^U. In general, for any topological left S-module N, $N^\vee$ is a topological right S-module, where the induced S-action is given by $\phi \cdot s = [n \mapsto \phi(s^{-1}n)]$. This shows that M_U is an $R[G/U]$-module. Furthermore, the action of $R[G/U]$ on M_U and the action of $R[G/V]$ on M_V are compatible for two open subgroups U, V. But then $R[[G]]$ acts on $M^\vee$. □

We shall now prove the alternative description of admissibility in characteristic p in several steps.

Lemma 2.4.10. *Let G be a locally pro-p group, $U \leq_{c,o} G$ a compact open subgroup of G, which is pro-p, and (π, M) a **smooth and $\mathcal{O}_K$-torsion** representation of G over $\mathcal{O}_K$, equipped with the discrete topology. Consider the following two statements:*

1.) *M^U is finitely generated over $\mathcal{O}_K$.*

2.) *$M^\vee$ is finitely generated over $\mathcal{O}_K[[U]]$.*

Here 2.) $\Longrightarrow$ 1.) holds. If M is admissible, then 1.) $\Longrightarrow$ 2.) holds as well.

Proof. 2.) $\Longrightarrow$ 1.): suppose that $M^\vee$ is finitely generated over $\mathcal{O}_K[[U]]$; then $(M^\vee)_U$ is finitely generated as well (since it is a quotient $\mathcal{O}_K[[U]]$-module of M). The action of $\mathcal{O}_K[[U]]$ on $(M^\vee)_U$ factors through the projection map $\mathcal{O}_K[[U]] \to \mathcal{O}_K[U/U]$ by definition; which is isomorphic to $\mathcal{O}_K$. This shows that $(M^\vee)_U$ is finitely generated over $\mathcal{O}_K$. But finitely generated torsion $\mathcal{O}_K$-modules are finite as sets (by the structure theorem of finitely generated modules over a principal ideal domain). Hence its dual is also finite as a set; in particular, it is finitely generated over $\mathcal{O}_K$.

1.) $\Longrightarrow$ 2.): We now assume that M is admissible. Suppose that M^U is finitely generated over $\mathcal{O}_K$ (again, it is then finite as a set), hence, $(M^\vee)_U$ is finitely generated over $\mathcal{O}_K[[G]]$, by lemma 2.4.8. We wish to apply Nakayama's lemma for compact modules (theorem 2.3.12), for $I = \overline{I_U}$ (the augmentation ideal of the Iwasawa algebra $\mathcal{O}_K[[U]]$). We need that $M^\vee$ is profinite, and that $\overline{I_U}^n \to 0$. $M = \bigcup M^U$ where U runs on all compact open subgroups of G, as M is smooth. Since M is admissible, each of the M^U are finitely generated. But finitely generated torsion $\mathcal{O}_L$ modules are finite as sets. Dualizing gives $M = \varprojlim M_U$, which shows that M is profinite We also need that $I^n \to 0$ in the sense of definition 2.3.11, for the augmentation ideal $I = \overline{I_G}$. I is nilpotent in each of the quotients $\mathcal{O}_K/\varpi^n\mathcal{O}_K[G/H]$ (H is an open normal subgroup), as it is nil in each of $\mathcal{O}_K[G/H]$, and $\mathcal{O}_K/\varpi^n\mathcal{O}_K[G/H]$ is finite as a set. But since $\mathcal{O}_K[[G]] = \varprojlim_{n,U} \mathcal{O}_K/\varpi^n\mathcal{O}_K[G/H]$, we have that for an open neighbourhood basis $\{U_\alpha\}$ of 0, I^n is contained in U. This shows that $I^n \to 0$.

$(M^\vee)_U$ is finitely generated over $\mathcal{O}_K \simeq \mathcal{O}_K[[G]]/I$, and $(M^\vee)_U$ is just $M^\vee/IM^\vee$ by definition; hence, from the second statement of the Nakayama lemma, we have that $M^\vee$ is finitely generated over $\mathcal{O}_K[[U]]$. □

Theorem 2.4.11. *Let G be a locally pro-p group. Let $(\pi, M) \in \mathrm{SRep}_{\mathcal{O}_K}(G)$, which is $\mathcal{O}_K$-torsion. Then (π, M) is an admissible representation over $\mathcal{O}_K$ if and only if $M^\vee$ is finitely generated as a $\mathcal{O}_K[[U]]$-module, for **any** open compact pro-p subgroup U.*

To be precise: if (π, M) is admissible, then for all compact open pro-p U, $M^\vee$ is finitely generated over $\mathcal{O}_K[[U]]$. But if there exists just one compact open pro-p U for which $M^\vee$ is finitely generated over $\mathcal{O}_K[[U]]$, then M is admissible.

Proof. Fix an open compact subgroup $U \leq_{c,o} G$: then U is a pro-p group.

" $\Longrightarrow$ ": if M is admissible, then M^U is finitely generated over $\mathcal{O}_K$. By lemma 2.4.10, $M^\vee$ is then finitely generated over $\mathcal{O}_K[[U]]$.

" $\Longleftarrow$ ": Let V be any compact open subgroup of G. We wish to show that M^V is finitely generated over $\mathcal{O}_K$. By lemma 2.3.10, if $V \leq U$, or $U \leq V$, then $M^\vee$ is finitely generated over $\mathcal{O}_K[[V]]$. Now using lemma 2.4.10, we have that in this case, M^V is finitely generated over $\mathcal{O}_K$. For a general V, look at $W = V \cap U$. W is again a compact open subgroup of G, and since $W \leq U$, M^W is finitely generated over $\mathcal{O}_K$, by the previous argument. But $\mathcal{O}_K$ is Noetherian, and $M^V \leq M^W$ is a sub-$\mathcal{O}_K$-module. □

Proposition 2.4.12. *Let G be a locally pro-p group, with $U \leq_{c,o} G$ a pro-p subgroup, and suppose that M is a finitely generated topological module over $\mathcal{O}_K[[U]]$, which is equipped with a compatible continuous (w.r.t the canonical topology on M) G-action. Then $M^\vee$ is a smooth, admissible $\mathcal{O}_K$-torsion representation.*

Proof. Let M be a finitely generated $\mathcal{O}_K[[U]]$- module. It is then compact, as it is the image of $\mathcal{O}_K[[U]]^n$, which is compact; hence its dual is discrete. The G-action is continuous on $M^\vee$, which means precisely that M is a smooth G-representation. $M^\vee$ is O_K-torsion: the image of any μ in $M^\vee$ (as a subset of $\mathbb{T} \subseteq \mathbb{C}$) is contained in $K/\mathcal{O}_K$. Now $K/\mathcal{O}_K$ is discrete, since $\mathcal{O}_K$ is an open subgroup; and $\mu : M \to K/\mathcal{O}_K$ is continuous. But then the image of μ is a compact subset of a discrete space; hence finite. It is then of the form $\varpi^{-r}\mathcal{O}_K/\mathcal{O}_K$, which is ϖ^r-torsion. Finally, $M^\vee$ is admissible, since by the Pontryagin duality theorem, $(M^\vee)^\vee \simeq M$, and $M^\vee$ is smooth and $\mathcal{O}_K$-torsion. But then by theorem 2.4.11, $M^\vee$ is admissible. □

Corollary 2.4.13. *Pontryagin duality gives a contravariant equivalence between the category of smooth, admissible, $\mathcal{O}_K$-torsion representations, and the category of finitely generated $\mathcal{O}_K[[U]]$-modules equipped with a compatible continuous (w.r.t the canonical topology on M) G-action.*

Proof. It is clear that the morphisms correspond to each other in the categories. □

Corollary 2.4.14. *If G is a locally pro-p group, which is also a p-adic Lie group, then the category of admissible and $\mathcal{O}_K$-torsion representatations of G over $\mathcal{O}_K$ is Abelian.*

Proof. Fix a compact open pro-p subgroup of G. By 2.3.4, $\mathcal{O}_K[[U]]$ is a Noetherian ring; hence the category of finitely generated modules over $\mathcal{O}_L[[U]]$ is Abelian. The category of continuous G-representations is Abelian. The intersection of these two full subcategories of $\mathcal{O}_L[[U]]$-Mod is also Abelian. By 2.4.13, category of admissible and $\mathcal{O}_K$-torsion representations of G over $\mathcal{O}_K$ is equivalent to the opposite of this category. Opposite categories of Abelian categories are Abelian. □

Proposition 2.4.15. *Let G be a locally pro-p group, M a smooth and $\mathcal{O}_K$-torsion representation of G, which furthermore satisfies that for some $n \in \mathbb{N}$, ϖ^n annihilates M. Then M is admissible if and only if there exists an $U \leq_{c,o} G$ compact open pro-p subgroup of G, which satisfies that M^U is a finitely generated $\mathcal{O}_K$-module.*

Proof. One direction is clear; admissibility implies that any compact open subgroup fixes only a finitely generated submodule. In particular, this is true for U.

Let K be any compact open subgroup of G. We wish to show that M^K is finitely generated as an $\mathcal{O}_K$-module. Now $U \cap K$ is again a compact open subgroup, which satisfies $M^K \subseteq M^{K\cap U}$, hence we can assume that $K \subseteq U$. In particular, K is pro-p as well.

K is an open subgroup of U, hence of finite index. U/K is then a finite p-group; in particular it has a normal subgroup of index p; which corresponds to a subgroup K' of U with index p. It is actually enough to consider this special case, since we have a sequence of subgroups $K < K_1 < \ldots < K_c < U$, such that each quotient is cyclic of order p. But then we iterate the special case for each K_i.

Now it remains to show the statement for this special case: U/K is cyclic of order p. Let $M[a]$ be the set of elements annihilated by a. Since M is $\mathcal{O}_K$-torsion, we have that $M = \bigcup_{r\in\mathbb{N}} M[\varpi^r]$. In particular, $M^K = \bigcup_r M^K[\varpi^r]$. The residue field $k = \mathcal{O}_K/(\varpi)$ acts on $M^K[\varpi]$. Let $g \in U/K$, and let us consider g as an element of the group ring $k[U/K]$. If $v \in (g-1)^{p-1}M^K[\varpi]$, then $(g-1)v = (g-1)^p = g^p v - v = v - v = 0$, which shows that $v \in M^U[\varpi]$. Hence $(g-1)^{n-1}M^K[\varpi]$ is finitely generated over $\mathcal{O}_K$. Now suppose that for some $k \in \mathbb{N}$, $(g-1)^{n-s}M^K[\varpi]$ is finitely generated over $\mathcal{O}_K$ (we have just shown this for $s = 1$). Then multiplication with $(g-1)$ on $(g-1)^{n-s-1}M^K[\varpi]$, has an image in $(g-1)^{n-s}M^K[\varpi]$, which is finitely generated over $\mathcal{O}_K$, and the kernel of this map is in $M^U[\varpi]$, by definition. $M^U[\varpi]$ is finitely generated over $\mathcal{O}_K$ as well; but then so is $(g-1)^{n-s-1}M^K[\varpi]$ ($\mathcal{O}_K$ is Noetherian). Iterating this argument gives that $M^K[\varpi]$ is finitely generated over $\mathcal{O}_K$.

Now $M^K[\varpi^n] \hookrightarrow M^K[\varpi^{n+t}]$ for any $t \in \mathbb{N}$. We can repreat the previous argument for $M^K[\varpi^n]/M^K[\varpi^{n-1}]$ to obtain that $M^K[\varpi^n]$ is finitely generated. Since some power of ϖ annihilates M (by assumption), we have that $M^K[\varpi^n] = M^K$ for some n. The claim follows. □

2.5 p-adic Banach Space Representations

In the p-adic Langlands correspondence, the representations on the "GL_n" side are actually p-adic Banach space representations. As we will see, these spaces can be obtained as a limit of smooth representations over finite rings. p-adic Banach were introduced in the the p-adic Langlands program by Schneider and Teitelbaum in [11]. In this section, we fix the following notation: $L/\mathbb{Q}_p$ is a finite extension, $\mathcal{O}_L$ is the ring of integers of L, $\mathfrak{p}_L$ is the unique maximal ideal of $\mathcal{O}_L$, and ϖ is the uniformizer of $\mathcal{O}_L$.

Definition 2.5.1. Let L be as described above. An L-Banach space is a locally convex complete topological vector space over L, such that the topology can be defined by a norm.

Note that we do not fix the norm in the definition, following the custom of the literature. Many of the theorems that exist for $\mathbb{R}$ or $\mathbb{C}$ vector spaces exist in the p-adic setting as well. In particular, bounded continuous linear operators are still continuous, and the open mapping theorem holds. We denote the space of continuous automorphisms of an L-Banach space V by $\mathrm{Aut}^c(V)$.

Definition 2.5.2. Let G be a locally profinite group, V an L-Banach space, $\pi : G \to \mathrm{Aut}^c(V)$ a function. We say that (π, V) is a continuous L-Banach space representation of G, if the map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous. The category of continuous L-Banach space representations is denoted by $\mathrm{Ban}_L(G)$; the morphisms are defined as G-equivariant continuous linear maps of L-Banach spaces.

Note that unlike in the previous sections, we require the map $G \times V \to V$ to be continuous w.r.t. the predefined topology of V, and not the discrete topology on V (i.e. the representation is continuous, but not "smooth"). This is because "most" of the L-Banach space representations that are smooth are finite-dimensional, hence uninteresting in the p-adic Langlands setting.

Lemma 2.5.3. *Let G be a profinite group. Then $\mathrm{Ban}_L(G)$ is an Abelian category.*

Proof. We will use lemma 1.0.3. Clearly, kernels, cokernels and finite direct sums of Banach spaces are again Banach spaces. Hence the category of G-representations on L-Banach spaces form an Abelian category, and this category contains $\mathrm{Ban}_L(G)$ as a full subcategory. Hence we can apply lemma 1.0.3 once more, this time for the subcategoty $\mathrm{Ban}_L(G)$ to obtain that $\mathrm{Ban}_L(G)$ is Abelian. The conditions of the lemma are trivially satisfied. □

Definition 2.5.4. Let V be an L-Banach space, equipped with a *continuous* L-linear G-action $\pi : G \times V \to V$. Suppose that we fix a norm $\|.\|$ on V. If G acts through norm preserving automorphisms of V, i.e. $\|gv\| = \|v\|$ for all $g \in G, v \in V$, then we say that (π, V) is a *unitary* L-Banach space representation.

Similarly to the $\mathbb{R}$ case, if $\phi : V \to V$ is a linear operator such that for all $v \in V$: $\|\phi v\| \leq C_\phi \|v\|$, then ϕ is continuous. In particular, for any fixed norm $\|.\|$, in any unitary L-Banach space representation, G acts through continuous linear operators. Note, however, that this does not imply a priori that the representation is continuous, and this is why the restriction of the continuity of $G \times V \to V$ is imposed. If (π, V) is a unitary L-Banach space representation of G w.r.t. a fixed norm $\|.\|$, we will say that the triple $(\pi, V, \|.\|)$ is a unitary representation.

Let $\mathrm{Ban}_L^U(G)$ be the full subcategory of $\mathrm{Ban}_L(G)$ defined by objects (π, V), which satisfy that there exists a norm $\|.\|$ on V which turns $(\pi, V, \|.\|)$ into a unitary representation.

Lemma 2.5.5. *The full subcategory $\mathrm{Ban}_L^U(G)$ of $\mathrm{Ban}_L(G)$ is Abelian.*

Proof. We can apply lemma 1.0.3. The direct sum of unitary representations is still unitary (the norm of the direct sum is the sum of norms), the kernels of unitary representations are trivially unitary. For cokernels, the norm on V/U is defined as $\|x + U\| \stackrel{\text{def}}{=} \inf_{u \in U} \|x + u\|$. $\|\pi(g)(x + U)\| = \|\pi(g)x + U\| = \inf_{u \in U} \|\pi(g)x + u\| = \inf_{u \in U} \|\pi(g)x + \pi(g)u\|$, since $\pi(g)$ is bijective on U. But then $\|\pi(g)(x + U)\| = \inf_{u \in U} \|\pi(g)(x + u)\| = \inf_{u \in U} \|x + u\| = \|x + U\|$ □

Lemma 2.5.6. *Let G be a locally profinite group, and let $(\pi, V, \|.\|)$ be a unitary L-Banach space representation of G. Let $V_0 \stackrel{\text{def}}{=} \{v \in V \ : \ \|v\| \leq 1\}$ be the closed unit ball of V.*

1.) *For any $n \in \mathbb{N}$: $\varpi^n V_0$ is mapped to itself via the G-action.*

2.) *There is a natural $\mathcal{O}_L/\varpi^n\mathcal{O}_L$-action on $V_0/\varpi^n V_0$.*

3.) *$V_0/\varpi^n V_0$ is a smooth G-representation over $\mathcal{O}_L/\varpi^n\mathcal{O}_L$.*

Proof. 1.): $|\varpi| = q^{-k}$ where q is the cardinality of the residue field of L, hence $\varpi^k V_0 = \{v \in V_0 : \|v\| \leq q^{-k}\}$. Since G acts through norm-preserving linear maps, $G\varpi^k V_0 = \varpi^k V_0$.

2.): We define $(x + \varpi^n\mathcal{O}_L)(v + \varpi^n V_0) \stackrel{\text{def}}{=} (xv + \varpi^n V_0)$. This is clearly a well-defined action of $\mathcal{O}_L/\varpi^n\mathcal{O}_L$.

3.): By 1.): $V_0/\varpi^n V_0$ is a quotient of two G-representations, hence itself a representation of G. By 2.), it is also a module over $\mathcal{O}_L/\varpi^n\mathcal{O}_L$. We need that it is smooth. We assumed that (π, V) is a continuous representation. Hence $G \times V \to V$ is continuous, which implies that $G \times V_0 \to V_0$ and $G \times V_0/\varpi^n V_0 \to V_0/\varpi^n V_0$ are both continuous. The quotient topology on $V_0/\varpi^n V_0$ is discrete, hence $G \times V_0/\varpi^n V_0 \to V_0/\varpi^n V_0$ is continuous w.r.t the discrete topology on $V_0/\varpi^n V_0$, showing that the representation is smooth. □

Lemma 2.5.7. *Let $(V, \|.\|)$ be an L-Banach space. Then*

$$V \simeq \left(\varprojlim_{n\in\mathbb{N}} V_0/\varpi^n V_0 \right) \otimes_{\mathcal{O}_L} L \simeq V_0 \otimes_{\mathcal{O}_L} L \simeq V_0[\varpi^{-1}] \tag{2.3}$$

as L-vector spaces.

Proof. Clearly, $V_0 \simeq \varprojlim_{n\in\mathbb{N}} V_0/\varpi^n V_0$ as an $\mathcal{O}_L$-module, since V_0 is complete. $V = V_0[\varpi^{-1}]$, since for any $v \in V$, there is a $v_0 \in V_0$ and $c \in K$ such that $v = cv_0$, and c is of the form $\varpi^{-m}c_0$, where $c_0 \in \mathcal{O}_L$. The last isomorphism follows from the fact that L is the quotient field of $\mathcal{O}_L$, and $\mathcal{O}_L$ is a DVR. □

As a consequence of this lemma, if V_0 is already equipped with an action of G, then this action extends (linearly) to V.

Definition 2.5.8. Let G be a locally pro-p group, $(\pi, V, \|.\|)$ a unitary L-Banach space representation of G. We say $(\pi, V, \|.\|)$ is an admissible representation, if for all $n \in \mathbb{N}$, $V_0/\varpi^n V_0$ is an admissible representation.

By lemma 2.5.6, the reductions $V_0/\varpi^n V_0$ are smooth. Each of the $V_0/\varpi^n V_0$ are *torsion* $\mathcal{O}_L$-representations, and we restricted our interest to locally pro-p groups; hence the theorems of section 2.4.1 apply. In particular, for any compact open subgroup $U \leq_{c,o} G$, we have an action of the Iwasawa algebra $\mathcal{O}_L[[U]]$ on $(V_0/\varpi^n V_0)^\vee$. The usual definition of admissibility for L-Banach spaces uses this dual Iwasawa-action.

2.6 Representations in the p-adic Langlands Setting

We now describe the three main categories of representations on which the functor of Colmez is defined. We try to follow their notations, but will eventually deviate to avoid assigning multiple meanings to the symbol $\mathrm{Rep}_{\mathcal{O}_L} G$. Let G be a p-adic Lie group, L a finite extension of $\mathbb{Q}_p$. Theorem 2.4.14 implies that in this setting, the category $\mathrm{ARep}_{\mathcal{O}_L} G$ of (smooth) admissible representations over $\mathcal{O}_L$ is Abelian.

Definition 2.6.1. $\mathrm{Rep}_{tors}\, G$ is the full subcategory of G-representations (π, M) over $\mathcal{O}_L$, satisfying the following conditions:

1. (π, M) is smooth and satisfies that for each open compact subgroup K of G, M^K is of *finite length* over $\mathcal{O}_L$ (in particular, M is admissible).

2. M is of finite length over $\mathcal{O}_L[G]$;

3. π acts through a central character.

Note that finite length submodules over $\mathcal{O}_L$ are finite *as sets*, since the only simple module over $\mathcal{O}_L$ (up to isomorphism) is $\mathcal{O}_L/\varpi\mathcal{O}_L$.

Proposition 2.6.2. *Let $(\pi, M) \in \mathrm{Rep}_{tors}\, G$. Then M is a torsion $\mathcal{O}_L$-module. In particular, M is a torsion $\mathcal{O}_L[G]$-module.*

Proof. Let $m \in M$ be any nonzero element. Then, since M is smooth, we have that $\mathrm{Stab}(m)$ is an open subgroup of G (proposition 2.2.6). Since G is profinite, there is an open compact subgroup K of G such that $K \leq \mathrm{Stab}(m)$, but then $m \in M^K$. By the assumption on M, M^K is of finite length over $\mathcal{O}_L$. But then M^K is a finite p-group, since simple modules over $\mathcal{O}_L$ are isomorphic to $\mathcal{O}_L/\varpi$, which is a finite p-group, and extending a finite p-group with a finite p-group gives again a finite p-group. The finite composition series of M^K gives that M^K is a finite p-group. In particular, there is a power of p that annihilates M^K, hence m. □

Proposition 2.6.3. *Suppose that M is a torsion module over $\mathcal{O}_L$. Then M equipped with the discrete topology is a topological $\mathcal{O}_L$-module.*

Proof. Suppose that ϖ^n annihilates M for some n. Then clearly we have an $\mathcal{O}_L/\varpi^n\mathcal{O}_L$-module structure on M. Let $f_n : \mathcal{O}_L \to \mathcal{O}_L/\varpi^n\mathcal{O}_L$ be the modulo ϖ^n map. f_n is continuous, since $\mathcal{O}_L = \varprojlim \mathcal{O}_L/\varpi^n$. Then in the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_L \times M & \xrightarrow{f_n \times id} & \mathcal{O}_L/p^n\mathcal{O}_L \times M \\ \downarrow \Phi & \swarrow \overline{\Phi} & \\ M & & \end{array}$$

the map $\overline{\Phi}$ is continuous, since its domain is discrete; and $f_n \times id$ is continuous, since it is a product of continuous maps.

Now lets consider a general torsion $\mathcal{O}_L$-module M; in this case, $M = \bigcup_n M[\varpi^n]$ (since $\mathcal{O}_L$ is a DVR, separated in the ϖ-adic topology). We need that $\Phi : \mathcal{O}_L \times M \to M$ is continuous. We have that $M[\varpi^n] = \{m \in M \mid \varpi^n m = 0\}$ is a topological module over $\mathcal{O}_L$, since ϖ^n annihilates $M[\varpi^n]$. This means that $\Phi_n : \mathcal{O}_L \times M[\varpi^n] \to M[\varpi^n]$ is continuous.

Let $m \in M$. We need that $\Phi^{-1}(m)$ is an open set. $m \in M[\varpi^n]$ for some large enough n. Now fix an $x \in \Phi^{-1}(m)$. For N large enough, $m \in M[\varpi^n]$ and $x \in \Phi_N^{-1}(m)$. Then $\Phi_N^{-1}(m)$ is an open set of $\mathcal{O}_L \times M[\varpi^N]$, which contains x. $\mathcal{O}_L \times M[p^N] \to \mathcal{O}_L \times M$ is a product of open maps ($M[p^N] \to M$ is trivially an open map, since M is discrete), hence itself an open map. This shows that $\Phi_N^{-1}(m)$ is an open neighbourhood of x in $\Phi^{-1}(m)$. In particular, $\Phi^{-1}(m)$ is an open set of $\mathcal{O}_L \times M$. □

Proposition 2.6.4. $\mathrm{Rep}_{tors}\, G$ *is equal (as a subcategory of* $\mathrm{ARep}_{\mathcal{O}_L}\, G$*) to the subcategory $\mathcal{C}$ defined by the following properties of any object $(M, \pi) \in \mathcal{C}$:*

1. *(M, π) is smooth and admissible as a G-representation;*
2. *(M, π) is of finite length over $\mathcal{O}_L[G]$;*
3. *M is $\mathcal{O}_L$-torsion;*
4. *π acts through a central character.*

Compared to the definition of $\mathrm{Rep}_{tors}\, G$*, the admissibility condition is weaker (imposing that M^K is finitely generated instead of finite length), but the additional condition that M is $\mathcal{O}_L$ torsion is necessary.*

Proof. It is enough to show that any finitely generated torsion $\mathcal{O}_L$-module is of finite length. This is however trivial, since finitely generated torsion modules over $\mathcal{O}_L$ are (by the fundamental theorem of finitely generated modules over a PID) finite direct sums of $\mathcal{O}_L/\varpi^n$; these are finite as sets, hence in particular of finite length as $\mathcal{O}_L$-modules. □

Corollary 2.6.5. $\mathrm{Rep}_{tors}\, G$ *is an Abelian category.*

Proof. We use the characterization of proposition 2.6.4. $\mathrm{Rep}_{tors}\, G$ is a full subcategory of the category of admissible and $\mathcal{O}_L$-torsion representations, which is Abelian by corollary 2.4.14. We will use proposition 1.0.3. 0 is in $\mathrm{Rep}_{tors}\, G$, and it is closed under taking finite direct sums. Kernels and cokernels of finite length modules are of finite length, and the G action is through central characters on each of them. □

Proposition 2.6.6. *If (π, M) is in* $\mathrm{Rep}_{tors}\, G$*, then any $\mathcal{O}_L$-submodule of M that is finitely generated is of finite length over $\mathcal{O}_L$. In particular, for any open subgroup K of G, M^K is of finite length over $\mathcal{O}_L$.*

Proof. We proved this statement in the proof of proposition 2.6.4. Since (π, M) is admissible, M^K is a finitely generated $\mathcal{O}_L$-module, hence the previous lemma can be applied. □

We have seen that any $M \in \mathrm{Rep}_{tors}\, G$ is $\mathcal{O}_L$-torsion. In fact, the following stronger proposition holds as well:

Proposition 2.6.7. *Let $(\pi, M) \in \mathrm{Rep}_{tors}\, G$. Then there exists an n such that ϖ^n annihilates M.*

Proof. Since M is of finite length over $\mathcal{O}_L[G]$, it is enough to show the proposition for irreducible elements of $\mathrm{Rep}_{tors}\, G$ (we can iterate the composition series to get a finite bound on n). Clearly, the elements annihilated by ϖ form a sub-G-representation as well. Then by the irreducibility of M, $M[\varpi] = 0$ or $M[\varpi] = M$. But in the latter case, M could not be a torsion module over $\mathcal{O}_L$. Hence $M[\varpi] = 0$, and the claim follows. □

To avoid using the same notation for two different categories, we will add an upper index C to the notation in the following definitions.

Definition 2.6.8. $\mathrm{Rep}^C_{\mathcal{O}_L}\, G$ is the full subcategory of $\mathcal{O}_L$ representations, consisting of objects (π, M), such that

1. M is separated and complete in its p-adic topology;
2. M is torsion-free;
3. For all $n \in \mathbb{N}$, $M/p^n M$ is in $\mathrm{Rep}_{tors}\, G$.

Definition 2.6.9. $\mathrm{Rep}^C_L\, G$ is the category of L-representations that are equipped with an $\mathcal{O}_L$-lattice, belonging to $\mathrm{Rep}_{\mathcal{O}_L}\, G$.

Such representations are complete, since their $\mathcal{O}_L$-net is already complete. We then obained an L-Banach space, equipped with a complete G-stable $\mathcal{O}_L$-net. It can be proven that any element of $\mathrm{Rep}^C_L\, G$ is a unitary Banach-space representation. But then it is clear that any such representation is admissible as an L-Banach space representation, in the sense of definition 2.5.8.

2.7 Fontaine's Equivalence and (φ, Γ)-modules

We now consider one of the main steps in establishing the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, the category of étale (φ, Γ)-modules. This section is based on Fontaine's and Ouyang's book [12] and Colmez's article [2].

2.7.1 Important Power Series Rings

Consider a p-adic number field $L/\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$. Then $\mathcal{O}_L$ is a local ring with residue field k_L, (unique) maximal ideal $\mathfrak{m}_L$ and uniformizer ϖ.

Definition 2.7.1. $\mathcal{O}_\mathcal{E} = \{f(T) = \sum_{k\in\mathbb{Z}} a_k T^k \mid a_k \in \mathcal{O}_L \wedge \lim_{k\to-\infty} a_k = 0\}$. $\mathcal{O}_\mathcal{E}$ will turn out to be a local ring. We denote its residue field by $k_\mathcal{E}$, its maximal ideal by $\mathfrak{m}_\mathcal{E}$, and its field of fractions by $\mathcal{E}$. Furthermore, $\mathcal{O}^+_\mathcal{E} = \mathcal{O}_L[[T]] \leq \mathcal{O}_\mathcal{E}$, $\mathcal{E}^+ = \mathcal{O}^+_\mathcal{E}[\frac{1}{p}] \leq \mathcal{E}$, and $k^+_\mathcal{E} = k_L[[T]] \leq k_\mathcal{E}$. The fact that $X^+ \leq X$ follows from the next proposition (for $X = \mathcal{O}_\mathcal{E}, \mathcal{E}, k_\mathcal{E}$).

Proposition 2.7.2. *The following are satisfied.*

a) *$\mathcal{O}_\mathcal{E}$ is indeed a local ring.*

b) *In fact, $\mathcal{O}_\mathcal{E}$ is a DVR with uniformizer ϖ.*

c) $k_\mathcal{E} = k_L((T))$.

d) $\mathcal{E} = \mathcal{O}_\mathcal{E}[\frac{1}{p}]$.

Proof. We first need that $\mathcal{O}_{\mathcal{E}}$ is indeed a ring. It is clearly an Abelian group. If $p(T) = \sum_{n\in\mathbb{Z}} a_n T^n$ and $q(T) = \sum_{n\in\mathbb{Z}} b_n T^n$, then $p \cdot q$ is well-defined: the coefficient of T^k is an infinite sum with terms tending to 0; and by basic p-adic analysis, such sums converge.

We show that the unique maximal ideal of $\mathcal{O}_{\mathcal{E}}$ is $\mathfrak{m}_L \mathcal{O}_{\mathcal{E}}$. It is enough to show that every element $a \notin \mathfrak{m}_L \mathcal{O}_{\mathcal{E}}$ is a unit. By definition, a has a coefficient a_k not in $\mathfrak{m}_L$. By the condition that the coefficients tend to 0 at $-\infty$, we have a minimal such a_k. Since T has an inverse, we can assume without loss of generality that a_0 is this minimal unit coefficient. If ϖ denotes the uniformizer of $\mathcal{O}_L$, we have that a is of the form $a = p(T) + \varpi h(T)$ where $p(T)$ is a formal power series with coefficients in $\mathcal{O}_L$, and with a constant coefficient in $\mathcal{O}_L^{\times} \cong \mathcal{O}_L \setminus \mathfrak{m}_L$; and h is some element of $\mathcal{O}_{\mathcal{E}}$. We know that $p(T)$ has an inverse (it even has an inverse in $\mathcal{O}_L[[T]]$). Multiplying with $p(T)^{-1}$, we have that a is a unit if and only if $1+\varpi hp^{-1}$ is. But since $(1+\varpi hp^{-1})(1-\varpi hp^{-1}+(\varpi hp^{-1})^2-\ldots) = 1$, hence a is a unit.

Since $\mathfrak{m}_L = \varpi \mathcal{O}_L$, by the previous reasoning we have $\mathfrak{m}_{\mathcal{E}} = \mathfrak{m}_L \mathcal{O}_{\mathcal{E}} = \varpi \mathcal{O}_L \mathcal{O}_{\mathcal{E}} = \varpi \mathcal{O}_{\mathcal{E}}$. In particular, $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with uniformizer ϖ. Taking the quotient, we obtain $k_{\mathcal{E}} = k_L((T))$, because sufficiently small coefficients are always multiples of ϖ, by the definition.

Finally, we need that ϖ has an inverse if p has. Since ϖ divides p, if p is invertible, then so is ϖ. □

For "mod p" representations there are two more families of rings we need to consider: the ring of formal Laurent series $\mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$ (k_ε is just the special case $n = 1$), and the ring of formal power series $\mathcal{O}_L/\varpi^n\mathcal{O}_L[[T]]$. Notice that these are the rings we obtain when reducing mod ϖ^n either $\mathcal{O}_{\mathcal{E}}$ or $\mathcal{O}_{\mathcal{E}}^+$. The above constructed rings ($\mathcal{O}_{\mathcal{E}}, \mathcal{O}_{\mathcal{E}}^+, k_{\mathcal{E}}, k_{\mathcal{E}}^+, \mathcal{E}, \mathcal{E}^+$, $\mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$) are all $\mathcal{O}_L$- modules, hence each of them has a natural p-adic topology, inherited from $\mathcal{O}_L^{\mathbb{Z}}$ (we call this topology the strong topology). However, the topology we consider on these rings is not always the strong topology:

1. We give $\mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$ (in particular, $k_{\mathcal{E}}$), the strong topology.
2. On $\mathcal{O}_{\mathcal{E}}$, the topology is the weak topology defined by the reduction mod ϖ map $\mathcal{O}_{\mathcal{E}} \to k_{\mathcal{E}}$. A neighbourhood basis of 0 is given by $p^k\mathcal{O}_{\mathcal{E}} + T^n\mathcal{O}_{\mathcal{E}}$. The reason for this choice of topology is basically that we want proposition 2.7.3 to hold.
3. $\mathcal{E}$ is given the topology via $\mathcal{E} = \varinjlim_{n\in\mathbb{N}} p^{-n}\mathcal{O}_{\mathcal{E}}$.
4. The +-versions of the rings are given the subspace topology.

Proposition 2.7.3. *$\mathcal{O}_{\mathcal{E}} \simeq \varprojlim_{n\in\mathbb{N}} \mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$ as topological rings.*

Proof. Let $P \in \varprojlim_{n\in\mathbb{N}} \mathcal{O}_L/\varpi^n\mathcal{O}_L((T)) \simeq \varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathcal{E}}/\varpi^n\mathcal{O}_{\mathcal{E}}$. We denote the image of P by the reduction mod ϖ^n map in $\mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$ by P_n. Clearly, P_n is a formal Laurent series, with minimal nonzero coefficient c_n. The kth coefficient of P is the inverse limit of the kth coefficient of P_n, its valuation is the first n where it appears. But then P is of the form $\sum_{i=-\infty}^{\infty} a_i T^i$ where $a_i \in \mathcal{O}_L$. Since P has a nonzero coefficient of minimal index mod ϖ^n for all n, $\lim_{i\to-\infty} a_i = 0$. Any element of $\mathcal{O}_{\mathcal{E}}$ is the inverse limit of itself mod ϖ^n. Furhtermore, the collection $(P \bmod \varpi^n)_{n\in\mathbb{Z}}$ is uniquely defined by P. This shows that the two sides are isomorphic as rings. By definition, if we endow $\mathcal{O}_{\mathcal{E}}$ with the projective limit topology (via the ring isomorphism we just proved), then the map $\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}/\varpi O_{\mathcal{E}} = k_{\mathcal{E}}$ is continuous. For the weak topology on $\mathcal{O}_{\mathcal{E}}$ to coincide with the projective limit topology, it is enough to show that the reduction mod ϖ^n maps are all continuous, when $\mathcal{O}_{\mathcal{E}}$ is given the weak topology. The preimage of a basis element of $\mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$ is clearly a basis element of $\mathcal{O}_{\mathcal{E}}$. The proposition follows. □

We introduce a so-called (φ, Γ)-module structure on these rings. Consider the map $\varphi : T \mapsto (1+T)^p - 1$, extended linearly to any of the rings above. We call φ the *Frobenius endomorphism.*

Proposition 2.7.4. *Let A be one of $\mathcal{O}_{\mathcal{E}}, \mathcal{O}_{\mathcal{E}}^+, k_{\mathcal{E}}, k_{\mathcal{E}}^+, \mathcal{E}, \mathcal{E}^+$, $\mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$, φ the Frobenius endomorphism of A. Then*

1.) *φ is $\mathcal{O}_L$-linear*

2.) *φ is continuous*

Proof. 1.): $\{T^n\}$ ($n \in \mathbb{Z}$ or $\mathbb{N}$, depending on the choice of A) is a free generating set of A over $\mathcal{O}_L$ (or $\mathcal{O}_L/\varpi^n\mathcal{O}_L$ for some n), and φ is defined on each element of this set. By the adjointness of the free and forgetful functors, we get that φ is a linear map over $\mathcal{O}_L$ (resp. $\mathcal{O}_L/\varpi^n\mathcal{O}_L$). The claim follows.

2.): Continuity is clear on $\mathcal{O}_L/\varpi^n\mathcal{O}_L((T))$; since linear maps on finite rank free modules are continuous; and here the image of $\bigcup_{n=1}^K \mathcal{O}_L T^n$ has an image in a finite rank module. The continuity on $\mathcal{O}_\mathcal{E}$ follows by 2.7.3. The continuity on the rest of the rings are immediate from the constructions. □

The following proposition shows why φ is called Frobenius.

Proposition 2.7.5. *By 2.3.6 and 2.3.7, we have isomorphisms*

$$\mathcal{O}_K[[\Delta]] \simeq \mathcal{O}_\mathcal{E}^+, \;\; \mathcal{O}_K/\varpi_K^n\mathcal{O}_K[[\Delta]] \simeq \mathcal{O}_K/\varpi_K^n\mathcal{O}_K[[T]],$$

where Δ is a topological group (with multiplicative notation), that is isomorphic to the additive group of $\mathbb{Z}_p$. Then (via these isomorphisms) the φ of each power series ring corresponds to the endomorphism $v \mapsto v^p$ of the respective completed group algebra.

Proof. Let $\delta \in \Delta$ be a topological generator of Δ (which, in additive notation can be chosen to be 1). The above isomorphisms are given by $\delta \mapsto (1+T)$. But then $\varphi(1+T) \longleftrightarrow \delta^p \longleftrightarrow (1+T)^p$ implies the proposition, by the continuity of φ (the set generated by δ is dense in Δ, and by 2.3.9, $\mathcal{O}_L[\Delta]$ is dense in the completed group algebra. □

Corollary 2.7.6. *Let A be either $\mathcal{O}_\mathcal{E}^+$ or $k_\mathcal{E}^+$.*

1. *The Frobenius $\varphi : A \to A$ is actually a ring endomorphism.*
2. *The Frobenius endomorphism $\varphi : A \to A$ is flat.*

Proof. From proposition 2.7.5, we obtain that ϕ is multiplicative: since $\delta \mapsto \delta^p$ is just the multiplication on $\mathbb{Z}_p$ with p; which is clearly a group homomorphism. By $\mathcal{O}_K$-linearity, we have that ϕ is multiplicative on $\mathcal{O}_K[\Delta]$, which is dense in $\mathcal{O}_K[[\Delta]$. The product of limits is the limit of the products; hence ϕ is multiplicative on $\mathcal{O}_K[[\Delta]]$ as well.

For the flatness of φ: $p\mathbb{Z}_p$ is a a compact open subgroup of $\mathbb{Z}_p$ (which is isomorphic to Δ). This shows that $\varphi(\Delta)$ is compact and open in Δ. Then by proposition 2.3.8, $\mathcal{O}_K[[\Delta]]$ is free and finitely generated over $\mathcal{O}_K[[\varphi(\Delta)]] = \varphi(\mathcal{O}_K[[\Delta]])$. In particular, it is a flat module over A. □

Proposition 2.7.7. *$\varphi : k_\mathcal{E}^+ \to k_\mathcal{E}^+$ is a local ring endomorphism (i.e. it maps the unique maximal ideal of $k_\mathcal{E}^+$ to itself).*

Proof. $T \mapsto (1+T)^p - 1 = 1 + Tq(T) - 1 = Tq(T)$ for some polynomial $q(T)$, by the binomial expansion. □

Consider now the following $\mathcal{O}_L$-linear $\mathbb{Z}_p^\times$-action σ on any of the above rings: $\sigma(a) : T \mapsto (1+T)^a - 1$, extended $\mathcal{O}_L$-linearly.

Proposition 2.7.8. *We have the following properties of σ for any of the rings defined above:*

a) *σ is indeed an $\mathcal{O}_L$-linear $\mathbb{Z}_p^\times$-action on the rings.*

b) *For each $a \in \mathbb{Z}_p^\times$, $\sigma(a)$ is a continuous map.*

c) *$\sigma(a)$ and φ commute.*

d) *$\sigma(a)$ is a ring endomorphism for each a.*

Proof. The linearity holds by definition. If $a, b \in \mathbb{Z}_p^\times$, then $\sigma(a)\sigma(b)(T-1) = (1+T)^{ab}$, which shows that σ is indeed a group action. It is enough to check that the φ and $\sigma(a)$ commute for the $\mathcal{O}_L$-basis $(1+T)^k$. $(1+T)^k \overset{\varphi}{\mapsto} (1+T)^{kp} \mapsto \sigma(a)(1+T)^{kpa}$, and p-adic exponentiation satisfies $x^{uv} = x^{vu}$. □

Proposition 2.7.9. *By 2.3.6 and 2.3.7, we have isomorphisms*

$$\mathcal{O}_K[[\Delta]] \simeq \mathcal{O}_\mathcal{E}^+, \;\; \mathcal{O}_K/\varpi_K^n\mathcal{O}_K[[\Delta]] \simeq \mathcal{O}_K/\varpi_K^n\mathcal{O}_K[[T]],$$

where Δ is a topological group (with multiplicative notation), that is isomorphic to the additive group of $\mathbb{Z}_p$. Then (via these isomorphisms) the endomorphism $\sigma(a)$ for some $a \in \mathbb{Z}_p^\times$ of each power series ring corresponds to the endomorphism $v \mapsto v^a$ of the respective completed group algebra.

Proof. The proof is basically the same as of proposition 2.7.5. □

2.7.2 (φ, Γ)-modules in General

We now define the general notion of (φ, Γ)- modules. (φ, Γ)- modules over the rings defined in the previous section act as the intermediate objects of the p-adic Langlands correspondence: the **D** functor of Colmez maps certain GL_2 representations to étale (φ, Γ)-modules, and the functor of Fontaine sends such modules to n-dimensional Galois-representations.

Definition 2.7.10. Let $R \in \mathrm{Ring}$ be a topological ring, equipped with a a continuous ring endomorphism $\varphi : R \to R$. Suppose that Γ is a topological group, and that we have a fixed Γ-action σ on R, satisfying that $\Gamma \times R \to R$, $(\gamma, r) \mapsto \sigma(\gamma)r$ is continuous. A (φ, Γ)-module over R is a topological R-module M, *finitely generated as a module*, equipped with a continuous module endomorphism $\varphi_M : M \to M$, and an R-linear action σ_M of Γ, which satisfies that $\Gamma \times M \to M$; $(\gamma, m) \mapsto \sigma_M(\gamma)m$ is continuous, satisfying the following conditions:
  i) φ_M and Γ commute: i.e. for any $\gamma \in \Gamma$, $\sigma_M(\gamma) \circ \varphi_M = \varphi_M \circ \sigma_M(\gamma)$.
  ii) φ_M is φ-semilinear.
  iii) $\sigma_M(\gamma)$ is $\sigma(\gamma)$-semilinear for each $\gamma \in \Gamma$.
If we do not require M to be finitely generated over R, we call M a *possibly non- finitely generated* (φ, Γ)-module over R. We fully acknowledge the ridiculousness of this terminology (but at least it avoids causing any confusion).

In the previous section, we proved that $\mathcal{O}_\mathcal{E}, \mathcal{O}_\mathcal{E}^+, \mathcal{E}, \mathcal{E}^+, k_\mathcal{E}$, and $k_\mathcal{E}$ are (φ, Γ)-modules over themselves. We now concentrate on one of these rings. In particular, we will describe how one can obtain a (φ, Γ)-module over one of the rings from a (φ, Γ)-module over another. A natural way to obtain a module over a different ring is to "change the scalars", i.e. take the tensor product $B \otimes_A M$. In the case of (φ, Γ)-modules, a similar construction works; which we shall describe now.

We need the following well-known lemma (it can be found in most commutative algebra textbooks):

Lemma 2.7.11. *Let A be a commutative Noetherian ring, $I \lhd$ an ideal of A. Then the I-adic completion of A is flat over A.*

Lemma 2.7.12. *$\mathcal{O}_\mathcal{E}$ is flat over ${\mathcal{O}_\mathcal{E}}^+$.*

Proof. We have that $\mathcal{O}_K((T))$ is the localization of $\mathcal{O}_\mathcal{E}+$ at at the ideal (T). $\mathcal{O}_\mathcal{E}$ is isomorphic to the ϖ-adic completion of $\mathcal{O}_K((T))$ as rings, hence it is flat over $\mathcal{O}_K((T))$ by lemma 2.7.11. But then $\mathcal{O}_\mathcal{E}$ is trivially flat over ${\mathcal{O}_\mathcal{E}}^+$. □

Proposition 2.7.13. *Suppose that M is a module over ${\mathcal{O}_\mathcal{E}}^+$.*

1.) *If M is finite as a set, then $\mathcal{O}_\mathcal{E} \otimes_{{\mathcal{O}_\mathcal{E}}^+} M = 0$.*

2.) *If $M, N \in \mathcal{O}_\mathcal{E}^+$-Mod, and M surjects to N with finite kernel, then*

$$\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} M \simeq \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} N.$$

3.) *If $N \leq M$ is a submodule of finite index, then*

$$\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} M \simeq \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} N.$$

Proof. Suppose that M is a module that is a finite set. We claim that $T^n M = 0$ for some n. Clearly, for some n, $T^n M = T^{n+1} M$ (otherwise $T^n M$ would be an infinite descending chain). Then T acts bijectively on $N \stackrel{\text{def}}{=} T^n M$. If $m \in N$, we have that for some k_m, $T^{k_m} m = 0$. Take $\ell = \prod_{m \in N} k_m$. We have that $(T^\ell - 1)N = 0$. But $T^\ell - 1$ is invertible in ${\mathcal{O}_\mathcal{E}}^+$, hence $N = 0$; i.e. $T^n M = 0$.

Now T^n is invertible in $\mathcal{O}_\mathcal{E}$. $T^m(a \otimes m) = a \otimes 0 = 0$. Hence T^m is an invertible element that annihilates $\mathcal{O}_\mathcal{E} \otimes_{{\mathcal{O}_\mathcal{E}}^+} M$. The first claim follows.

The tensor product functor with $\mathcal{O}_\mathcal{E}$ is right exact. But then an exact sequence of the form $0 \to F \to M \to N \to 0$ where F is finite as a set, gives an exact sequence $0 \to \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} M \to \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} N \to 0$

For the third proposition, we have the sequence $0 \to N \to M \to F \to 0$ with F being finite. By lemma 2.7.12, $\mathcal{O}_\mathcal{E}$ is a flat module over $\mathcal{O}_\mathcal{E}^+$. But then, applying the tensor product with $\mathcal{O}_\mathcal{E}$ on this sequence shows the desired isomorphism. □

Proposition 2.7.14. *Suppose that M is a possibly non-finitely generated (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}^{+}$. Then*

a) $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^{+}} M$ *is a (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}$.*

b) In fact, $M \mapsto \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}{}^{+}} M$ is an exact functor between the categories of possibly non-finitely generated (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}}{}^{+}$ and possibly non-finitely generated (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}}$ This functor clearly restricts to an exact functor from (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}}{}^{+}$ to (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}}$.

Proof. The φ-action on the tensor product $T = \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}{}^{+}} M$ is given by $\varphi_T(a \otimes m) = \varphi(a) \otimes \varphi_M(m)$. Similarly for the Γ-action. All the properties of a (φ,Γ)-module are trivially satisfied. □

The most important property of (φ,Γ)-modules is them being *étale*. If M is a (φ,Γ)-module over R, we use the notation $M_\varphi = R \otimes_\varphi M$. Here R has an R-R-bimodule structure, where multiplication from the left is just the usual multiplication of R, but multiplication from the right is defined by φ; $r' \circ r \stackrel{\text{def}}{=} r' \cdot \varphi(r)$.

Definition 2.7.15. Let R, φ, Γ as in definition 2.7.10, but we require that R is commutative. We say that a possibly non-finitely generated (φ,Γ)-module M is étale, if the endomorphism φ_M induces a surjective map $1 \otimes \varphi_M : M_\varphi \to M$; $r \otimes m \mapsto r\varphi_M(m)$.

We say that a (φ,Γ)-module M (i.e. M is now finitely generated over R) is étale, if the map $1 \otimes \varphi_M$ is a linear isomorphism.

Notice that M is étale if and only if $\varphi(M)$ generates M as an R-module.

Fact 2.7.16. *A (φ,Γ)-module M that is finitely generated over R is étale w.r.t. one of the definitions above, if and only if it is étale w.r.t. the other.*

Proposition 2.7.17. *Let M be a possibly non-finitely generated étale (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}{}^{+}$. Then $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}{}^{+}} M$ is étale over $\mathcal{O}_{\mathcal{E}}$.*

Proof. It is enough to show that the image of the module endomorphism $\varphi_\otimes$ generates $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}{}^{+}} M$, where $\varphi_\otimes :\ r \otimes m \mapsto \varphi(r) \otimes \varphi_M(m)$. Here $\phi : R \to R$ is a ring homomorphism, hence 1 is in its image. $\varphi_M : M \to M$ satisfies that $\varphi_M(M)$ generates M as an $\mathcal{O}_{\mathcal{E}}{}^{+}$-module ($M$ is étale). But then $1 \otimes \varphi_M(m)$ is in the image of $\varphi_\otimes$; and elements of this form clearly generate $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}{}^{+}} M$ as an $\mathcal{O}_{\mathcal{E}}$-module. □

2.7.3 The Equivalence of Fontaine

There are several versions of the equivalence of Fontaine. The versions we need, and state here, are the same as the ones from the article [2] of Colmez.

Definition 2.7.18. The following categories of (φ,Γ)-modules are considered.

1. The category of étale (φ,Γ)-modules over $\mathcal{O}_{\mathcal{E}}$, which are of finite length over $\mathcal{O}_{\mathcal{E}}$ is denoted by $\Phi\Gamma^{et}_{tors}$.

2. The category of étale (φ,Γ)- modules over $\mathcal{O}_{\mathcal{E}}$, are free (and of finite rank, by definition) over $\mathcal{O}_{\mathcal{E}}$, is denoted by $\Phi\Gamma^{et}(\mathcal{O}_{\mathcal{E}})$.

3. The category of finite-dimensional étale (φ,Γ)-modules over $\mathcal{E}$ is denoted by $\Phi\Gamma^{et}(\mathcal{E})$.

It is immediate, by proposition 2.7.14, that tensoring with $\mathcal{E}$ associates to any (φ,Γ)-module in $\Phi\Gamma^{et}(\mathcal{O}_{\mathcal{E}})$ a (φ,Γ)-module in $\Phi\Gamma^{et}(\mathcal{E})$. In the following statements we will use the notation $\mathcal{G}_p$ for the absolute Galois group of $\mathbb{Q}_p$.

Definition 2.7.19. On the Galois-side of the p-adic Langlands program, we have the following categories of representations.

1. $\mathrm{Rep}_{tors}\,\mathcal{G}_p$ is the category of continuous representations of $\mathcal{G}_p$ over $\mathcal{O}_L$, which are of finite length over $\mathcal{O}_L$.

2. $\mathrm{Rep}_{\mathcal{O}_L}\,\mathcal{G}_p$ is the category of continuous representations V of $\mathcal{G}_p$ over $\mathcal{O}_L$, which are free, and for each n, $V/\varpi^n V$ is in $\mathrm{Rep}_{tors}\,\mathcal{G}_p$.

3. $\mathrm{Rep}_{\mathcal{O}_L} \mathcal{G}_p$ is the category of continuous representations of $\mathcal{G}_p$ over L, which are finite-dimensional vector spaces over L.

Theorem 2.7.20 (Fontaine)**.** *The categories 1., 2. and 3. of definition 2.7.18 and the categories of definition 2.7.19 are eqivalent respectively.*

Chapter 3

The Montréal Functor

In this chapter, we give an overview of the functor of Colmez Then we switch to an algebraic point of view, and use it to deduce a certain finitary property of the functor. We try to be as thorough as possible, however we will omit proofs that depend on the characterization of irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and the properties of standard presentations of Rep_{tors}.

3.1 The Functor of Colmez

We now give a description of the functor of Colmez, for $\mathrm{GL}_2(\mathbb{Q}_p)$). We will then use the discussion of the previous section, to derive some properties using "purely algebraic" methods. The main idea of Colmez was to establish a functor from smooth, admissible mod p representations to étale (φ, Γ)-modules. Then, one can use the equivalence of Fontaine (section 2.7), to connect to the Galois side of the Langlands philosophy.

3.1.1 (φ, Γ)-module structure on $\mathcal{O}_L$-modules

Let $L/\mathbb{Q}_p$ be a finite extension, and suppose that $G = \mathrm{GL}_2(\mathbb{Q}_p)$). Whenever Π is a smooth, admissible representation over $\mathcal{O}_L$, and of finite length over $\mathcal{O}_L[G]$, we will give an étale (φ, Γ)-module structure on certain subrepresentations of Π. Then, by taking an inverse limit on all such representations, we obtain an étale (φ, Γ)-module $\mathbf{D}(\Pi)$. The map $\Pi \mapsto \mathbf{D}(\Pi)$ will turn out to be a functor with good properties. Throughout this section, we will always work with $\mathcal{O}_L$-modules that are also objects of Rep_{tors} as wel. As we proved it in the previous sections, such modules are actually modules over $\mathcal{O}_L/\varpi^n\mathcal{O}_L$ as well (for some n).

An étale (φ, Γ)-module is necessarily a module over $\mathcal{O}_{\mathcal{E}}$. The first idea in establishing the functor $\mathbf{D}$ of Colmez is to find an $\mathcal{O}_{\mathcal{E}}^+$-module structure, and then take the tensor product with $\mathcal{O}_{\mathcal{E}}$, to obtain an $\mathcal{O}_{\mathcal{E}}$-module. The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ will give the $\mathcal{O}_{\mathcal{E}}^+$-structure via the following isomorphism:

Proposition 3.1.1. *Using the notation of 2.7, we have that:*

$$\mathcal{O}_{\mathcal{E}}^+ \simeq \mathcal{O}_L \left[\left[\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}\right]\right] \tag{3.1}$$

where the ring on the right hand side is the Iwasawa algebra of $\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ *with coefficients in* $\mathcal{O}_L$.

Proof. Notice that

$$(\mathbb{Z}_p, +) \simeq \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

as topological groups. Then, by 2.3.6, $\mathcal{O}_L[[T]] \simeq \mathcal{O}_L\left[\left[\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)\right]\right]$. But $\mathcal{O}_{\mathcal{E}}^+$ is just the formal power series ring $\mathcal{O}_L[[T]]$. □

From now on, let $\Pi \in \mathrm{Rep}_{tors}\, G$ (defined in section 2.6), and let M be a subset of Π which is stable under the action of $N_0 = \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. Π is always given the discrete topology, and M is equipped with the discrete topology as well.

Lemma 3.1.2. *M has a natural structure of a topological* $\mathcal{O}_L\left[\left[\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)\right]\right]$*-module.*

Proof. Π is smooth, hence $\Pi = \bigcup_n \Pi^{H_n}$, where H_n is the nth standard neighbourhood basis element of 0: $H_n = \{g \in G : g \equiv 1_G \pmod{\varpi^n}\}$. Now N_0/p^n acts on $M \cap \Pi^{H_n}$. These actions are compatible, hence define an action of $\mathcal{O}_L[[N_0]]$. This action is continuous, since the action of $\mathcal{O}_L$ and N_0 are both continuous, and together they form a dense set in $\mathcal{O}_L[[N_0]]$ (proposition 2.3.9). □

The lemma implies that we have a continuous action of $\mathcal{O}_\mathcal{E}^+$ on M. To be precise (following the notation of Colmez), we can write $D_{Ped}(M)$ (*Ped* as in pedantic) for

$$D_{Ped}(M) = \mathcal{O}_\mathcal{E}^+ \otimes_{\mathcal{O}_\mathcal{E}^+} M.$$

Fact 3.1.3. *$M \simeq D_{Ped}(M)$ as $\mathcal{O}_L$-modules.*

Corollary 3.1.4. *The Pontryagin dual $M^\vee$ is also equipped with a continuous action of ${\mathcal{O}_\mathcal{E}}^+$.*

In fact, this corollary will be used to define the functor of Colmez. From now on, we only require M to be a topological ${\mathcal{O}_\mathcal{E}}^+$-module; in particular M can be the original M or its dual $M^\vee$ as well. We shall give a (φ, Γ)- module structure on this new M over $\mathcal{O}_\mathcal{E}^+$. In order to do that, we need the stronger assumption on M; that it is stable under the (continuous) P^+-action (again following the notations of Colmez's original article), where

$$P^+ = \begin{pmatrix} \mathbb{Z}_P \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}.$$

We will denote the isomorphism of 3.1.3 by $\iota : M \simeq D_{Ped}(M)$. We define $\varphi_M : D_{Ped}(M) \to D_{Ped}(M)$ by

$$\varphi_M(\iota(v)) \stackrel{\text{def}}{=} \iota\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot v\right),$$

and for each $a \in \mathbb{Z}_p^\times$, we define $\sigma_a : D_{Ped}(M) \to D_{Ped}(M)$ by

$$\sigma_a(\iota(v)) \stackrel{\text{def}}{=} \iota\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Proposition 3.1.5. *The above φ_M, and the $\mathbb{Z}_p^\times$-action defined by σ_a turn $D_{Ped}(M)$ into a possibly non-finitely generated (φ, Γ)-module over $\mathcal{O}_\mathcal{E}^+$.*

Proof. Clearly, the actions of φ_M and σ_a are $\mathcal{O}_L$-linear. Since the action of P^+ is continuous on M, the endomorphisms φ_M and σ_a are continuous as well. Furthermore, these actions commute for all $a \in \mathbb{Z}_p^\times$.

It remains to prove the semi-linearity conditions. To deduce that φ_M is φ-semilinear, consider the embedding

$$\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \hookrightarrow \mathcal{O}_L\left[\left[\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}\right]\right].$$

By theorem 2.3.9, the $\mathcal{O}_L$-translates of the image of this embedding give a dense set, hence by the continuity of φ_M, it is enough to show the φ-semi- linearity for $\lambda = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where $a \in \mathbb{Z}_p$.

$$\varphi_M(\lambda\iota(v)) = \iota\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \lambda v\right) = \iota\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} v\right) =$$
$$= \iota\left(\begin{pmatrix} p & pa \\ 0 & 1 \end{pmatrix} v\right) = \iota\left(\begin{pmatrix} 1 & pa \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} v\right) =$$
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^p \cdot \varphi_M(\iota(v))$$

By proposition 2.7.5, $v \mapsto v^p$ corresponds exactly to φ via the isomorphism of 2.3.6. The φ-semi-linearity of φ_M follows. For the action σ of $\mathbb{Z}_p^\times$, the exact same argument can be repreated, simply by replacing "p" with "x" in the above calculation, where $x \in \mathbb{Z}_p^\times$. At the end, one uses proposition 2.7.9. □

By now, we can associate to any continuous P^+-representation (or its dual) a (φ, Γ)- module over ${\mathcal{O}_\mathcal{E}}^+$. The difficulty is to associate a (φ, Γ)-module, that is étale, and finite-dimensional.

Definition 3.1.6. We use the notation $\mathcal{W}(\Pi)$ for the set of sub-$\mathcal{O}_L$-modules W of Π, that satisfy the following properties:

1. W is stable under the action on $\mathrm{GL}_2(\mathbb{Z}_p)$ and $Z(G)$,

2. W is finitely generated over $\mathcal{O}_L$.

3. W generates Π as an $\mathcal{O}_L[G]$-module.

The reason for these conditions will be explained in the next section. Note, that by lemma 2.6.6, W is actually of finite length over $\mathcal{O}_L$. Condition 2. and 3. forces any $W \in \mathcal{W}(\Pi)$ to be both sufficiently small, and sufficiently large, in some sense. The reason for the conditions is that these ensure that the compact induction $I(W)$ is well-defined and surjects to Π (see proposition 3.1.10).

Proposition 3.1.7. *If $\Pi \in \mathrm{Rep}_{tors}\, G$, then $\mathcal{W}(\Pi) \neq \emptyset$. More precisely: there exists a compact open subgroup $K \leq_{c,o} G$ such that $\Pi^K \in \mathcal{W}(\Pi)$.*

Proof. The construction is explicit. By our assumption, Π is of finite length over $\mathcal{O}_L[G]$. We claim that $\exists n \in \mathbb{N}$ such that the "standard" open pro-p subgroup $K_n = \{g \in \mathrm{GL}_2(\mathbb{Q}_p) \mid g \equiv 1 \mod p^n\}$ satisfies that Π^{K_n} generates Π. Let $0 = \Pi_0 < \ldots < \Pi_{n-1} < \Pi_m = \Pi$ be a composition series of Π, and $v_i \in \Pi_i \setminus \Pi_{i-1}$. Then clearly $\{v_i\}_{i=1}^m$ is a generating set of Π. By 2.2.6 part c), for each $i = 1, \ldots m$ there exists an open subgroup H_i such that $v_i \in \Pi^{H_i}$. But then $H = \bigcap_{i=1}^m H_i$ satisfies that for each i: $v_i \in \Pi^H$. H is a finite intersection of open sets, hence itself open. Since K_n forms a neighbourhood basis of 1, $\exists n$ s.t. $K_n \subseteq H$. But then $\Pi^H \subseteq \Pi^K$, hence Π^K generates Π. Since $K_n \leq \mathrm{GL}_2(\mathbb{Z}_p)$, Π^{K_n} is clearly stable under the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ and the center of G. Since Π is admissible, the subspace Π_n^K is finitely generated over $\mathcal{O}_L$. □

Proposition 3.1.8. *Let $W \in \mathcal{W}(\Pi)$. Then Π is generated as an $\mathcal{O}_L$-module by elements of the form $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w$, where $n \in \mathbb{Z}$, $a \in \mathbb{Q}_p$ and $w \in W$.*

Proof. We know that W generates Π as a left $\mathcal{O}_L[G]$-module. It is enough to show $gW \subseteq \left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)W$, whenever $g \in \mathrm{GL}_2(\mathbb{Q}_p)$.

1. $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p)$ if $|x|_p = 1$. But then by the $\mathrm{GL}_2(\mathbb{Z}_p)$-stability of W, we have $\left(\begin{smallmatrix} p^n & a \\ 0 & x \end{smallmatrix}\right)W = \left(\begin{smallmatrix} p^n & ax^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)W \subseteq \left(\begin{smallmatrix} p^n & x^{-1}a \\ 0 & 1 \end{smallmatrix}\right)W$. Hence it is enough to show that for any $g \in G$, $gW \subseteq \left(\begin{smallmatrix} p^n & a \\ 0 & x \end{smallmatrix}\right)W$, where $|x|_p = 1$.

2. Let $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$ be arbitrary. We have that $z \neq 0$, and $x/z = p^n u$ where $|u|_p = 1$. Then by the $Z(G)$-stability of W,

$$\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)W = \left(\begin{smallmatrix} p^n & yu^{-1}z^{-1} \\ 0 & u^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} uz & 0 \\ 0 & uz \end{smallmatrix}\right)W \subseteq \left(\begin{smallmatrix} p^n & yu^{-1}z^{-1} \\ 0 & u^{-1} \end{smallmatrix}\right)W.$$

   We can apply point 1. From now on, it is enough to show that for each $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, $gW \subseteq BW$ where B is the subgroup of upper triangular matrices of $\mathrm{GL}_2(\mathbb{Q}_p)$.

3. From the Iwasawa-decomposition of $\mathrm{GL}_2(\mathbb{Q}_p)$, $g = bk$, where b is upper-triangular, and $k \in \mathrm{GL}_2(\mathbb{Z}_p)$. But then $gW = bkW \subseteq bW$ by the $\mathrm{GL}_2(\mathbb{Q}_p)$-stability of W. The proof is complete, using point 2.

□

Let us denote $K = \mathrm{GL}_2(\mathbb{Z}_p)$, $Z = Z(G)$ Let $\Pi \in \mathrm{Rep}_{tors}\, G$, W in $\mathcal{W}(\Pi)$. Then $I(W)$ is the compact induction

$$C\text{-}\mathrm{Ind}_{K\cdot Z}^G W = \{\phi : G \to W \text{ such that } \phi \text{ is of finite support modulo } KZ,\ k\phi(h) = \phi(kh) \text{ if } k \in KZ\}.$$

We give a G-action on $I(W)$ via $g\phi(h) = \phi(hg)$. If $\phi \in I(W)$, then we can consider it as a function on G/KZ (which is the Bruhat-Tits tree of G): if $gh^{-1} \in KZ$, then $\phi(h) = gh^{-1}\phi(h) = \phi(g)$ (as W is KZ-stable). The condition that ϕ should have finite support modulo KZ can then be understood as "ϕ has finite support in G/KZ".

Lemma 3.1.9. *$I(W) \simeq_{\mathcal{O}_L} \bigoplus_{g\in G/KZ} W$.*

Proof. We map ϕ to the collection of its values on the inverses: the gth coordinate of ϕ is $\phi(g^{-1})$ (we could map ϕ to $\phi(g)$, but we follow the notation of the article of Colmez). Since ϕ has finite support in G/KZ, this collection is in the direct sum. Conversely, any element of the direct sum defines a ϕ in $I(W)$. □

Proposition 3.1.10. *Consider the map $\Phi : \bigoplus_{g\in G/KZ} w_g \mapsto \sum_{g\in G/KZ} g \cdot w_g$.*

i.) *Φ is a well-defined $I(W) \to \Pi$ map;*

ii.) *Φ is G-equivariant;*

iii.) *Φ is surjective.*

Proof. i.): holds by the previous lemma: explicitly, $\phi \mapsto \sum_{g\in G/KZ} g\phi(g^{-1})$. The image is a well-defined element of Π, since W is assumed to be KZ-invariant. ii.): $\Phi(h\phi) = \sum g\phi(g^{-1}h) = \sum g\phi((h^{-1}g)^{-1})$. By a change of variables $r = h^{-1}g$, this is equal to $\sum_r hr\phi(r) = h\Phi(\phi)$. All of these summations are performed on on G/KZ. iii.): W generates Π as an $\mathcal{O}_L[G]$-module. This means that any element can be written as $\sum g_i w_i$. The proposition follows. □

We denote the kernel of Φ with $R(W,\Pi)$; we have the short exact sequence

$$0 \longrightarrow R(W,\Pi) \longrightarrow I(W) \xrightarrow{\Phi} \Pi \longrightarrow 0. \tag{3.2}$$

We give an alternative descrption of some elements of $I(W)$. Let $g \in G$, $v \in W$.

$$[g,v](h) = \begin{cases} hg \cdot v, & \text{if } hg \in KZ \\ 0, & \text{if } hg \notin KZ. \end{cases} \tag{3.3}$$

Here we consider $[g,v]$ as a $G \to W$ function. We shall see that it is actually an element of $I(W)$. We use the notation $[g,W] = \bigcup_{v\in W}\{[g,v]\}$

Proposition 3.1.11. *$[g,v]$ satisfies*

1.) *$[g,v]$ is indeed in $I(W)$;*

2.) *If we identify $I(W)$ with $\bigoplus_{g\in G/KZ} W_g$, where $W_g \simeq W$, then $[g,v]$ corresponds to $v \in W_g$;*

3.) *$\Phi([g,v]) = gv$;*

4.) *$[g,v] = g[1,v]$.*

Proof. If $k \in KZ$, then for any $g,h \in G$, we have that $hg \in KZ \iff khg \in KZ$. But then

$$k[g,v](h) = \begin{Bmatrix} khg \cdot v & \text{if } hg \in KZ \\ 0 & \text{if } hg \notin KZ \end{Bmatrix} = \begin{Bmatrix} khg \cdot v & \text{if } khg \in KZ \\ 0 & \text{if } khg \notin KZ \end{Bmatrix} = [g,v](kh).$$

We need that $[g,v]$ has constant support modulo KZ. $hg \in KZ$ if and only if $h^{-1}KZ = gKZ$. Hence the only class on which $[g,v](h)$ is nonzero, is $W_{h^{-1}} = W_g$ (since the value of $\phi(h)$ of any ϕ is in $W_{h^{-1}}$, see the proof of 3.1.9). I.e., the support of $[g,v]$ module KZ is not only finite, but in fact a single point. The proof of 1.) is finished. The value of $[g,v]$ at this single point is hgv, for any h such that $h^{-1}KZ = gKZ$. But then we can choose h to be g^{-1}, hence $hgv = v$. This shows 2.). 3.) follows from 2.) immediately. 4.) is trivial: $[1,gv](h) = h1gv = hgv = [g,v](h)$. □

To put it simply, $[g,v]$ is the element of $I(W)$ that has a single non-zero coordinate, located in the gth direct summand, with value v: $I(W) = \bigoplus_{g\in G}[g,W]$.

3.1.2 Standard Presentations

The use of standard presentations is essential to prove properties of the functor of Colmez. We will only list the most important properties of standard presentations, and omit most of the proofs.

A standard presentation, to put it simply, is a choice of $W \in \mathcal{W}(\Pi)$ such that the kernel $R(W,\Pi)$ has a "nice" generating set. Fix an element $g \in G$. Suppose that $y \in W \cap g^{-1}W$. Then we have that for the map Φ of proposition 3.1.10, $\Phi([g,y]) = gy = \Phi([1,gy])$. Both y and gy are elements of W, hence Φ is well defined on $[g,y]$ and $[1,gy]$. This shows that $r_g(y) := [g,y] - [1,gv] \in R(W,\Pi)$. If $g \in KZ$, $[g,y] = g[1,y] = [1,gy]$; in this case $r_g = 0$. If $g \notin KZ$, then $W_g \neq W_1$ shows that $[g,y] \neq [1,gv]$ (they are nonzero in different coordinates); hence $r_g(y)$ is a nonzero element of $R(W,\Pi)$.

Definition 3.1.12. We say that the sequence of diagram 3.2 is a *standard presentation* of Π, if $R(W,\Pi)$ is generated by elements of the form $r_P(y)$, where

$$P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

We use the notation $\mathcal{W}^{(0)}(\Pi) \stackrel{\text{def}}{=} \{W \in \mathcal{W}(\Pi)$, such that W gives a standard presentation of $\Pi\}$.

The quotient set G/KZ is the Bruhat-Tits tree of G. It has naturally a root element, which we denote by σ_0; it corresponds to 1. The tree is indeed a tree graph, which has a well-defined notion of distance on the set of its vertices, which we shall denote by d. Now for some $n \in \mathbb{N}$, consider the following sub-$\mathcal{O}_L$-module of Π: $W^{[n]} = \sum_{d(s,\sigma_0)\leq n} sW$. Then we have that $W^{[0]} = W$, and $W \subseteq W^{[n]}$, since $W^{[i]} \subseteq W^{[j]}$ whenever $i \leq j$. By the triangle inequality, we have $(W^{[n]})^{[m]} = W^{[n+m]}$. Moreover, the action of K (and hence KZ) fixes σ_0, and the action of G is distance-preserving on the tree. Hence $W^{[n]}$ is preserved by the action of KZ. From this, the following fact is trivial:

Fact 3.1.13. *If $W \in \mathcal{W}(\Pi)$, then $W^{[n]} \in \mathcal{W}(\Pi)$.*

Lemma 3.1.14. *If $V, W \in \mathcal{W}(\Pi)$, then for sufficiently large N, $V \leq W^{[N]}$.*

Proof. Since W is in $\mathcal{W}(\Pi)$, V is finitely generated over $\mathcal{O}_L$. Since $W \in \mathcal{W}(\Pi)$, W generates Π as an $\mathcal{O}_L[G]$-module, hence $\Pi = \bigcup_n W^{[n]}$. But then each of the finitely many generators of V are in some $W^{[N]}$ for N sufficiently large. □

We state the following lemma without proof.

Lemma 3.1.15. *If $W \in \mathcal{W}^{(0)}(\Pi)$, then $W^{[n]} \in \mathcal{W}^{(0)}(\Pi)$.*

The lemma implies the following propositions immediately.

Proposition 3.1.16. *If Π admits a standard presentation (i.e. $\exists W \in \mathcal{W}^{(0)}(\Pi)$), then for any $W'' \in \mathcal{W}(\Pi)$, there exists a $W' \in \mathcal{W}^{(0)}(\Pi)$ that contains W''. In other words, the subset $\mathcal{W}^{(0)}(\Pi)$ is cofinal in $\mathcal{W}(\Pi)$.*

Proof. There exists a $W \in \mathcal{W}^{(0)}(\Pi)$ by our assumption. By lemma 3.1.15, $W^{[n]}$ is also an element of $\mathcal{W}^{(0)}(\Pi)$. By lemma 3.1.14, since both W and W' are elements of $\mathcal{W}(\Pi)$, for large enough N, $W' \subseteq W^{[N]}$. But then we can simply set $W'' = W^{[N]}$. □

Proposition 3.1.17. *If $W_1, W_2 \in \mathcal{W}^{(0)}(\Pi)$, then there exists a $W \in \mathcal{W}^{(0)}(\Pi)$, such that $W_i \subseteq W$.*

Proof. By lemma 3.1.14, there exists an N, such that $W_2 \subseteq W_1^{[N]}$. Also, by definition, $W_1 \subseteq W_1^{[N]}$. By lemma 3.1.15, $W_1^{[N]} \in \mathcal{W}^{(0)}(\Pi)$. □

Theorem 3.1.18. *Any $\Pi \in \mathrm{Rep}_{tors}\, G$ admits a standard presentation.*

We shall not prove this theorem here, due to the nature of the proof: it strongly depends on the classification of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Describing this classification is out of the scope of this text. Most of the proofs and theorems we discussed until now would work for $\mathrm{GL}_2(K)$ for some p-adic number field K as well. However, theorem 3.1.18 may fail. Analogous statements may also fail for generalizations defined for GL_n, instead of GL_2.

We denote the Bruhat-Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)/Z\cdot\mathrm{GL}_2(\mathbb{Z}_p$ by $\mathcal{T}$. Note that two vertices $a,b\in\mathcal{T}$ are connected by an edge if and only if there exist representatives $g,h\in G$ such that $[g]=a,[h]=b$ and $g^{-1}h=\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$. We set $I_{\mathcal{A}}(W,\Pi)=\sum_{a\in\mathcal{A}}aW\subseteq\Pi$.

One of the most important properties of standard presentations is that the following lemma holds.

Lemma 3.1.19. *Suppose that $W\in\mathcal{W}^{(0)}(\Pi)$. Let $\mathcal{A}_1\subseteq\mathcal{A}_2$ be connected subtrees of $\mathcal{T}$. Let $\mu\in I_{\mathcal{A}_1}(W,\Pi)^\vee$ such that μ is 0 on sW, whenever s represents an external vertex of the tree. Then μ can be uniquely extended by zeros to an element $\overline{\mu}\in I_{\mathcal{A}_2}(W,\Pi)$.*

3.1.3 The definition of D

Let Π be an element of $\mathrm{Rep}_{tors}\,G$, and $W\in\mathcal{W}(\Pi)$. Recall that $[g,W]=\{[g,v]\ :\ v\in W\}$ is a subset of $I(W)$.

Definition 3.1.20. Let $I_{\mathbb{Z}_p}(W)$ be the sub-$\mathcal{O}_L$- module of $I(W)$, generated by $[\left(\begin{smallmatrix}p^n&a\\0&1\end{smallmatrix}\right),W]$, where a is an element of $\mathbb{Q}_p$, satisfying that $a+p^n\mathbb{Z}_p\subseteq\mathbb{Z}_p$.

We use the notation $\mathrm{D}^\natural{}_W(\Pi)\stackrel{\mathrm{def}}{=}\Phi(I_{\mathbb{Z}_p}(W))^\vee$, where $\vee$ is the Pontryagin duality functor, and Φ is the surjection $I(W)\to\Pi$, defined in proposition 3.1.10.

If one thinks of $I(W)$ as the direct sum $\bigoplus_{g\in G/KZ}W_g$, then $I_{\mathbb{Z}_p}(W)$ is generated by the elements with coordinates only in W_x for $x=\left(\begin{smallmatrix}p^n&a\\0&1\end{smallmatrix}\right)^{-1}$. But then the image by Φ is simply the $\mathcal{O}_L$-submodule generated by $\{\left(\begin{smallmatrix}p^n&a\\0&1\end{smallmatrix}\right)\cdot w\}$. It turns out that $\Phi(I_{\mathbb{Z}_p}(W))$ is more or less independent of the choice of W, and is instead something that describes the representation Π itself.

Proposition 3.1.21. *Let $W_2\leq W_1$ be two elements of $\mathcal{W}(\Pi)$. Then the induced map $\Phi(I_{\mathbb{Z}_p}(W_2))\hookrightarrow\Phi(I_{\mathbb{Z}_p}(W_1))$ turns $\Phi(I_{\mathbb{Z}_p}(W_2))$ into a finite index sub-$\mathcal{O}_L$-module of $\Phi(I_{\mathbb{Z}_p}(W_1))$.*

Proof. We have that $W_1\leq W_2^{[n]}$ for n sufficiently large; and that $(W_2^{[n]})^{[1]}=W_2^{[n+1]}$. Hence it is enough to show the proposition for $W_2\leq W_2^{[1]}$. This means that $W_1\subseteq\Phi(I_{\mathbb{Z}_p}(W_2))+\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}W_2$. Now if $n\geq1$, then $\left(\begin{smallmatrix}p^n&i\\0&1\end{smallmatrix}\right)\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}\in P^+$. Hence $\Phi(I_{\mathbb{Z}_p}(W_1)/\Phi(I_{\mathbb{Z}_p}(W_2)$ is a quotient of $\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}W_2$, which is of finite length over $\mathcal{O}_L$, hence finite as a set. The proposition follows. □

Proposition 3.1.22.

$$\mathcal{O}_{\mathcal{E}}\otimes_{\mathcal{O}_{\mathcal{E}^+}}\mathrm{D}^\natural{}_{W_1}(\Pi)\simeq\mathcal{O}_{\mathcal{E}}\otimes_{\mathcal{O}_{\mathcal{E}^+}}\mathrm{D}^\natural{}_{W_2}(\Pi)$$

Proof. By the previous proposition 3.1.21, and the exactness of the Pontryagin duality functor, we have that

$$0\to K\to\mathrm{D}^\natural{}_{W_1}(\Pi)\to\mathrm{D}^\natural{}_{W_2}(\Pi)\to0$$

is an exact sequence with K a finite module. The claim follows from proposition 2.7.13. □

Definition 3.1.23. For $\Pi\in\mathrm{Rep}_{tors}\,G$, we define

$$\mathbf{D}(\Pi)=\varprojlim_{W\in\mathcal{W}(\Pi)}\mathcal{O}_{\mathcal{E}}\otimes_{\mathcal{O}_{\mathcal{E}^+}}\mathrm{D}^\natural{}_W(\Pi)$$

via the isomorphisms $\mathcal{O}_{\mathcal{E}}\otimes_{\mathcal{O}_{\mathcal{E}^+}}\mathrm{D}^\natural{}_{W_1}(\Pi)\simeq\mathcal{O}_{\mathcal{E}}\otimes_{\mathcal{O}_{\mathcal{E}^+}}\mathrm{D}^\natural{}_{W_2}(\Pi)$.

By theorem 3.1.18, the set $\mathcal{W}^{(0)}$ and hence by 3.1.16, we have that $\mathbf{D}$ can be defined as an inverse limit on $\mathcal{W}^{(0)}(\Pi)$, instead of $\mathcal{W}(\Pi)$. In fact, Colmez defines $\mathbf{D}$ as a limit on $\mathcal{W}^{(0)}(\Pi)$. We use the above definition instead, since it makes the analogue to other Montréal functors (where standard presentations are not present) more apparent. The main point of standard presentations is that they have nice properties, and due to the cofinality of $\mathcal{W}^{(0)}(\Pi)$ in $\mathcal{W}(\Pi)$, these properties translate to properties of $\mathbf{D}(\Pi)$. We shall see (but it is intuitively clear, since it is a limit via isomorphisms) that $\mathbf{D}$ is essentially $\mathrm{D}^\natural{}_W$, but it does not depend on the choice of $W\in\mathcal{W}(\Pi)$. Π is a smooth G-representation, hence it is a continuous G-representation when endowed with the discrete topology. Note that Π with the discrete topology is also a topological module over $\mathcal{O}_L$, by proposition 2.6.3 4.). But then $\mathrm{D}^\natural{}_W(\Pi)=\Phi(I_{\mathbb{Z}_p}(W))^\vee$ is the Pontryagin-dual of a discrete topological $\mathcal{O}_L$-module, hence compact. The dual action of $\mathcal{O}_L$ is still continuous, hence turning $\mathrm{D}^\natural{}_W(\Pi)$ into a topological $\mathcal{O}_L$-module.

Proposition 3.1.24. $\mathbf{D}(\Pi) \simeq \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathrm{D}^{\natural}{}_W(\Pi)$ *as topological* $\mathcal{O}_{\mathcal{E}}$*-modules, for any* $W \in \mathcal{W}^{(}\Pi)$*, via the natural projection map of the inverse limit.*

Proof. By proposition 3.1.22, the projective limit in the definition of $\mathbf{D}(\Pi)$ is the limit of isomorphisms. Furhtermore, if $W_1, W_2 \in \mathcal{W}^{(}\Pi)$, there is a common upper bound W of W_i, by proposition 3.1.14 in $\mathcal{W}(\Pi)$. Hence an element of $\mathbf{D}(\Pi)$ is uniquely determined by its image in $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathrm{D}^{\natural}{}_W(\Pi)$. □

We wish to apply the discussion at the beginning of this section, in particular lemma 3.1.2, to define a (φ, Γ)-module structure on M. However, $\Phi(I_{\mathbb{Z}_p}(W))$, and hence its dual are not closed under the action of P^+. The P^+-action is instead given on a finite-index (and complete) submodule $\mathrm{D}_W^+(\Pi)$ of $\mathrm{D}^{\natural}{}_W(\Pi)$; by proposition 2.7.13,

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathrm{D}_W^+(\Pi) = \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathrm{D}^{\natural}{}_W(\Pi),$$

and from the (φ, Γ)-module structure of $\mathrm{D}_W^+(\Pi)$ over ${\mathcal{O}_{\mathcal{E}}}^+$, we obtain a (φ, Γ)-module structure on $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathrm{D}_W^+(\Pi)$, over $\mathcal{O}_{\mathcal{E}}$.

Definition 3.1.25. Let $W \in \mathcal{W}(\Pi)$. $\mathrm{D}_W^+(\Pi)$ is the set of $\mu \in \Pi^\vee$, which are 0 on $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right) \cdot W$ whenever $a + p^n\mathbb{Z}_p$ is **not** fully contained in $\mathbb{Z}_p$.

An element μ of $\mathrm{D}_W^+(\Pi)$ is a function on Π, hence we can consider the restriction $\mu\big|_{\Phi(I_{\mathbb{Z}_p}(W))}$, which is just an element of $\mathrm{D}^{\natural}{}_W(\Pi)$. We claim that $r : \mu \mapsto \mu\big|_{\Phi(I_{\mathbb{Z}_p}(W))}$ is an injective map $\mathrm{D}_W^+(\Pi) \to \mathrm{D}^{\natural}{}_W(\Pi)$.

Proposition 3.1.26. *We have that*

1.) $\mathrm{D}_W^+(\Pi)$ *is a finite index topological submodule of* $\mathrm{D}^{\natural}{}_W(\Pi)$ *via* r*.*

2.) $\mathrm{D}_W^+(\Pi)$ *is stable under the* P^+*-action inherited from* $\Pi^\vee$*.*

3.) $\mathrm{D}_W^+(\Pi)$ *is a complete* $\mathcal{O}_L$*-module.*

Proof. 1.): By proposition 3.1.8, we know that elements of the form $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w$ generate Π as an $\mathcal{O}_L$-submodule, where $w \in W, a \in \mathbb{Q}_p, n \in \mathbb{Z}$. Notice that since W is an $\mathcal{O}_L$ submodule and $\mathcal{O}_L$-elements commute with elements of G, we have that these elements generate Π as a $\mathbb{Z}$-submodule as well.

Let $\mu \in \mathrm{D}_W^+(\Pi)$. The restriction of μ to $\Phi(I_{\mathbb{Z}_p}(W))$ is a linear map from $\mathrm{D}_W^+(\Pi)$ to $\mathrm{D}^{\natural}{}_W(\Pi)$. Its kernel consists of the μ which are zero on $\Phi(I_{\mathbb{Z}_p}(W)) = \langle \left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w \ : \ a+p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p \rangle$. But then such a μ is zero on all elements of the form $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w$, where $w \in W, a \in \mathbb{Q}_p, n \in \mathbb{Z}$, hence $\mu = 0$ in $\Pi^\vee$ (since these elements generate Π as an Abelian group, and μ is a group homomorphism). We obtained that $\mathrm{D}_W^+(\Pi) \leq \mathrm{D}^{\natural}{}_W(\Pi)$ but still need that this is a topological embedding (i.e. the restriction $r : \mu \mapsto \mu\big|_{\Phi(I_{\mathbb{Z}_p}(W))}$ gives a homeomorphism to its image. The embedding $i : \Phi(I_{\mathbb{Z}_p}(W)) \hookrightarrow \Pi$ is a topological embedding, hence $i^\vee$ (which is just the precomposition with the embedding i) is a quotient map $\Pi^\vee \to \mathrm{D}^{\natural}{}_W(\Pi)$; in particular, it is a closed map.

Closed subgroups of $\Pi^\vee$ are precisely the annihilators of closed subgroups of Π. W is a closed subgroup of Π, hence $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)W$ is a closed subgroup as well. The annihilator of any such subgroup is closed in $\Pi^\vee$. Now take $J = \bigcap \mathrm{Ann}(\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)W)$ where the intersection is over matrices where $a + p^n\mathbb{Z}_p \not\subseteq \mathbb{Z}_p$. But J is precisely $\mathrm{D}_W^+(\Pi)$. This shows that $\mathrm{D}_W^+(\Pi)$ is a closed subgroup in $\Pi^\vee$.

The restriction of a continuous closed map to a closed subset is still a closed map. In particular, $i^\vee\big|_{\mathrm{D}_W^+(\Pi)}$ is a closed map. Now clearly $r = i^\vee\big|_{\mathrm{D}_W^+(\Pi)}$; which gives that r is bijective (as we have proved at the beginning of the proof), and closed and continuous; hence a homeomorphism.

It remains to show that $\mathrm{D}_W^+(\Pi)$ is of finite index. Elements of $\mathrm{D}^{\natural}{}_W(\Pi)$ which are also elements of $\mathrm{D}_W^+(\Pi)$ are precisely the continuous group homomorphisms from $\Phi(I_{\mathbb{Z}_p}(W))$ to $\mathbb{T}$ that can be extended as 0 to Π continuously. Let $\nu \in \mathrm{D}^{\natural}{}_W(\Pi)$ We claim that if $\nu\big|_W \equiv 0$, then ν extends to Π via zeros. This is an immediate consequence of 3.1.19. We set $N = \{\mu \in \mathrm{D}^{\natural}{}_W(\Pi) \ : \ \mu\big|_W \equiv 0\}$. We have just seen hat $N \subseteq \mathrm{D}_W^+(\Pi)$. Now W is a finite set, since it is torsion, and finite length over $\mathcal{O}_L$. N is the annihilator of W, hence taking the quotient by N corresponds to taking the embedding

$W \hookrightarrow \Phi(I_{\mathbb{Z}_p}(W))$; hence the quotient $\mathrm{D}^\natural{}_W(\Pi)/N$ is a finite set. But then clearly $\mathrm{D}^\natural{}_W(\Pi)/\mathrm{D}^+_W(\Pi)$ is finite as well.

2.): It is enough to show that whenever $\mu \in \mathrm{D}^+_W(\Pi)$, then $g\mu \in \mathrm{D}^+_W(\Pi)$, for any $g \in P^+$. P^+ is clearly generated by $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$, hence it is enough to show the proposition for g of this form. Let $v \in \Pi$ be of the form $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right) w$ where $w \in W$ and $a + p^n\mathbb{Z}_p \not\subseteq \mathbb{Z}_p$. By definition, $g\mu \in \mathrm{D}^+_W(\Pi)$ if and only if $g\mu(v) = 0$ for any such v. As a reminder: $g\mu(v) \stackrel{\text{def}}{=} \mu(g^{-1}v)$. Now for $g = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$:

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mu(v) \stackrel{\text{def}}{=} \mu\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \cdot w\right) = \mu\left(\begin{pmatrix} p^{n-1} & p^{-1}a \\ 0 & 1 \end{pmatrix} w\right)$$

Here $p^{-1}a + p^{n-1}\mathbb{Z}_p$ is a circle with a larger radius and origin of larger norm than $a + p^n\mathbb{Z}_p$, which was not contained in $\mathbb{Z}_p$. Hence clearly $p^{-1}a + p^{n-1}\mathbb{Z}_p \not\subseteq \mathbb{Z}_p$. The assumption on μ implies then that the above expression is 0.

Now for $g = \left(\begin{smallmatrix} u & z \\ 0 & 1 \end{smallmatrix}\right)$ with $u \in \mathbb{Z}_p^\times$, $z \in \mathbb{Z}_p$:

$$\begin{pmatrix} u & z \\ 0 & 1 \end{pmatrix} \mu(v) \stackrel{\text{def}}{=} \mu\left(\begin{pmatrix} u & z \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \cdot w\right) =$$

$$= \mu\left(\begin{pmatrix} u^{-1} & -u^{-1}z \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p^n & a \\ 0 & 1 \end{pmatrix} \cdot w\right) = \mu\left(\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p^n & a - z \\ 0 & 1 \end{pmatrix} \cdot w\right)$$

Let $\rho \in \mathbb{Z}_p$ be such that $a + p^n\rho \notin \mathbb{Z}_p$. Such a ρ exists by the assumption on v. We claim that $|a - z + p^n\rho|_p > 1$. Here $-z \in \mathbb{Z}_p$ and $a + p^n\rho \notin \mathbb{Z}_p$ shows that $|a + p^n\rho|_p > |-z|_p$. But if $|s|_p \neq |t|_p$, then $|s + t|_p = \max(|s|_p, |t|_p)$ for any $s, t \in \mathbb{Q}_p$. In particular, $|a - z + p^n\rho|_p = \max(|a + p^n\rho|_p, |-z|_p) = |a + p^n\rho)|_p > 1$. This shows that (by the assumption $\mu \in \mathrm{D}^+_W$), that $\mu\left(\left(\begin{smallmatrix} p^n & a-z \\ 0 & 1 \end{smallmatrix}\right) \cdot w\right) = 0$. But then $\mu\left(\left(\begin{smallmatrix} u^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & a-z \\ 0 & 1 \end{smallmatrix}\right) \cdot w\right) = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \mu\left(\left(\begin{smallmatrix} p^n & a-z \\ 0 & 1 \end{smallmatrix}\right) \cdot w\right) = 0$, as desired.

3.): It is enough to show that $r(\mathrm{D}^+_W(\Pi))$ is closed in $\mathrm{D}^\natural{}_W(\Pi)$; since $\mathrm{D}^\natural{}_W(\Pi)$ is a profinite group, and closed subgroups of profinite groups are again profinite, via the "same" projective limit. $\mathrm{D}^\natural{}_W(\Pi)$ is profinite via a ϖ-adic limit, hence $r(\mathrm{D}^+_W(\Pi))$ is too; i.e. $r(\mathrm{D}^+_W(\Pi))$ is complete.

As observed in the proof of 1.), $r = i^\vee\big|_{\mathrm{D}^+_W(\Pi)}$, and $i^\vee$ is a closed map, $\mathrm{D}^+_W(\Pi)$ is a closed set; hence $r(\mathrm{D}^+_W(\Pi))$ is a closed set. □

We obtain that, by proposition 2.7.13, $\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} \mathrm{D}^\natural{}_W(\Pi) \simeq \mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} \mathrm{D}^+_W(\Pi)$, and by proposition 2.7.14, $\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} \mathrm{D}^+_W(\Pi)$ is a (φ, Γ)-module over $\mathcal{O}_\mathcal{E}$. To apply this proposition, we needed that $\mathrm{D}^+_W(\Pi)$ has a *continuous* $P+$-action; but this is clearly satisfied, since the action of P^+ is the restriction of the action of P^+ on $\Pi^\vee$, which is the dual of a continuous P^+-representation (by the smoothness of Π), hence itself a continuous P^+-representation.

Corollary 3.1.27. *The P^+-action on each of the $\mathcal{O}_\mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}^+} \mathrm{D}^\natural{}_W(\Pi)$ turns $\mathbf{D}(\Pi)$ into a possibly non-finitely generated (φ, Γ)-module over $\mathcal{O}_E$.*

Proof. If $W_1 \subseteq W_2$ are elements of $\mathcal{W}(\Pi)$, then $\mathrm{D}^+_{W_2}(\Pi) \to \mathrm{D}^+_{W_1}(\Pi)$ is a surjective map of possibly non-finitely generated (φ, Γ)-modules; since the P^+-action on both is the restriction of the P^+-action on $\Pi^\vee$. When the tensor product with $\mathcal{O}_\mathcal{E}$ is applied to compatible (ϕ, Γ)-modules, the resulting modules will be compatible (φ, Γ)-modules by 2.7.14. Hence the (φ, Γ)-action is compatible with the inverse limit. □

3.1.4 The Étale Property

So far, we have established that $\mathbf{D}(\Pi)$ is a possibly non-finitely generated (φ, Γ)-module over $\mathcal{O}_\mathcal{E}$. Now we wish to prove the following:

1. $\mathbf{D}(\Pi)$ is a (φ, Γ)-module over $\mathcal{O}_\mathcal{E}$, i.e. it is finitely generated over $\mathcal{O}_E$.

2. $\mathbf{D}(\Pi)$ is étale as a (φ, Γ)-module over $\mathcal{O}_\mathcal{E}$.

We prove the first point only with the algebraic approach of Emerton ([5]), in section 3.3. For now, we prove that $\mathbf{D}(\Pi)$ is an étale, possibly non-finitely generated (φ, Γ)-module over $\mathcal{O}_{\mathcal{E}}$.

We will use the notation

$$I_U^{\Pi}(W)_0 = \{\mu \in \Pi^\vee \ : \ \mu\big|_{\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)W} = 0 \text{ if } a + p^n\mathbb{Z}_p \not\subseteq U\}.$$

Note that with this notation, $\mathrm{D}_W^+(\Pi) = I_{\mathbb{Z}_p}^{\Pi}(W)_0$.

Lemma 3.1.28. *Consider the map $\psi_\Pi : \bigoplus_{i=1}^{p-1} \varpi^\vee = (\varpi^\vee)^p \to \varpi^\vee$, defined by*

$$(\mu_0, \dots, \mu_{p-1}) \mapsto \sum_{i=0}^{p-1} \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\mu_i.$$

The restriction of ψ_Π to $(\mathrm{D}_W^+(\Pi))^p$ is denoted by ψ. Then ψ is an injective map, with its image contained in $\mathrm{D}_W^+(\Pi)$; and its cokernel in $\mathrm{D}_W^+(\Pi)$ is finite as a set.

Proof. $\mathrm{D}_W^+(\Pi)$ is closed under the action of P^+, by proposition 3.1.26. As $\mu_i \in \mathrm{D}_W^+(\Pi)$, and $\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) \in P^+$, the image of ψ is in $\mathrm{D}_W^+(\Pi)$.

For the injectivity of ψ, we claim that $\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\mu$ is an element of $I_{i+p\mathbb{Z}_p}^{\Pi}(W)_0$, whenever $\mu \in \mathrm{D}_W^+(\Pi)$. We have to show that if $v \in \left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)W$ for some a, n with $a + p^n\mathbb{Z}_p \not\subseteq i + p\mathbb{Z}_p$, then $\left(\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\mu\right)(v) = 0$. We set $v = \left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w$ for some $w \in W$.

$$\left(\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\mu\right)(v) = 0 \iff \mu\left(\left(\begin{smallmatrix} p^{-1} & -p^{-1}i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w\right) = 0 \iff \mu\left(\left(\begin{smallmatrix} p^{n-1} & p^{-1}(a-i) \\ 0 & 1 \end{smallmatrix}\right)w\right) = 0.$$

We know that $a + p^n\mathbb{Z}_p \not\subseteq i + p\mathbb{Z}_p$. Multiplying both sides with p^{-1}, and subtracting $p^{-1}i$ gives $p^{-1}a - p^{-1}i + p^{n-1} \not\subseteq \mathbb{Z}_p$. But then since $\mu \in \mathrm{D}_W^+(\Pi) = I_{\mathbb{Z}_p}^{\Pi}(W)_0$, we have that the above expression is indeed 0, and the claim is true. Now, $\mathbb{Z}_p = \coprod_{i=0}^{p-1} i + p\mathbb{Z}_p$. By proposition 3.1.8, Π is generated by $\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)W$, where $a \in \mathbb{Q}_p$, $n \in \mathbb{Z}$. But on such elements, $\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\mu_i$ is nonzero if and only if $a + p^n\mathbb{Z}_p \subseteq i + p\mathbb{Z}_p$. Hence the supports of $\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\mu_i$ are pairwise disjoint. The injectivity of ψ follows.

Let $R_i \subseteq \Phi(I(W))$ be $\sum \left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)W$, where $a + p^n\mathbb{Z}_p \subseteq i + p\mathbb{Z}_p$. By lemma 3.1.19, if $\mu \in \Phi(I_{i+p\mathbb{Z}_p}(W))^\vee$, which also happens to be 0 on $\sum \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)W$, then μ can be extended with zeros to some $\lambda_i \in \Pi^\vee$. We claim that then $\lambda_i = \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)\mu_i$ for some $\mu_i\, \mathrm{D}_W^+(\Pi)$. Indeed, consider $\mu_i \stackrel{\text{def}}{=} \left(\begin{smallmatrix} p^{-1} & -p^{-1}i \\ 0 & 1 \end{smallmatrix}\right)\lambda_i$. Let a, n such that $a + p^n\mathbb{Z}_p \not\subseteq \mathbb{Z}_p$. Then

$$\mu_i\left(\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w\right) = \lambda_i\left(\left(\begin{smallmatrix} p & i \\ 0 & i \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right)w\right) = \lambda_i\left(\left(\begin{smallmatrix} p^{n+1} & pa+i \\ 0 & 1 \end{smallmatrix}\right)w\right) = 0,$$

since $i + ap + p^{n+1}\mathbb{Z}_p \not\subseteq i + \mathbb{Z}_p$ and λ_i is supported elements generated with a, n, where this continment is fulfilled.

But then the image of ψ contains any μ which is 0 on W and on $\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)$ for each $i = 0, \dots p-1$. The set of these μ is denoted by N. Then for the cokernel: $\mathrm{D}_W^+(\Pi)/\operatorname{im}\psi \subseteq N$, and N is the annihilator of $Q \stackrel{\text{def}}{=} W + \sum \left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right)W$ in $\mathrm{D}_W^+(\Pi)$. But quotients with the annihilator are the duals of the objects they annihilate; in this case, N is the dual of Q. This shows that N is of finite length over $\mathcal{O}_L$, hence finite as a set. □

Theorem 3.1.29. *$\mathbf{D}(\Pi)$ is an étale possibly non-finitely generated (φ, Γ)-module over $\mathcal{O}_{\mathcal{E}}$.*

Proof. From lemma 3.1.28, we have that ψ is injective with finite cokernel. If we consider $\mathrm{D}_W^+(\Pi)$ as a module over $\mathcal{O}_{\mathcal{E}}{}^+$, the map ψ becomes $(\mathrm{D}_W^+(\Pi))^p \to \mathrm{D}_W^+(\Pi)$; $(\mu_0, \dots, \mu_{p-1}) \mapsto \sum_{i=0}^{p-1}(1 + T)^i\varphi(\mu_i)$. This shows, that the image of φ generates a finite cokernel submodule of $\mathrm{D}_W^+(\Pi)$. Taking the tensor product with $\mathcal{O}_{\mathcal{E}}$, we get that ψ induces an isomorphism $\mathbf{D}(\Pi)^p \to \mathbf{D}(\Pi)$, by proposition 2.7.13. But then $\psi_{\mathbf{D}}$ shows that the image of $\varphi(\mathbf{D}(\Pi))$ generates $\mathbf{D}(\Pi)$ as an $\mathcal{O}_{\mathcal{E}}$-module. Equivalently, the possibly non-finitely generated (φ, Γ)-module structure on $\mathbf{D}(\Pi)$ is étale. □

Our main goal is to more or less prove the following theorem.

Theorem 3.1.30 (Main Theorem)**.** $\mathbf{D}$ *is an exact functor from* $\mathrm{Rep}_{tors}\, G$ *to* $\Phi\Gamma^{et}_{tors}(\mathcal{O}_{\mathcal{E}})$.

The exactness of $\mathbf{D}$ follows from the properties of standard presentations. The proof is not particularly complicated, but we will not include it here.

Theorem 3.1.31. $\mathbf{D}$ *is an exact functor.*

It remains to show that $\mathbf{D}(\Pi)$ is finitely generated over $\mathcal{O}_{\mathcal{E}}$, whenever $\Pi \in \mathrm{Rep}_{tors}\, G$. We will prove this statement with a completely different, mostly algebraic machinery.

3.2 Skew Polynomial Rings

Reminder: in the introduction, we used the notation $F^*M = R \otimes_F M$, whenever $F : R \to R$ is a ring endomorphism.

Definition 3.2.1. Suppose that R is a ring, $F : R \to R$ is a flat ring endomorphism. We define the ring $R[F]$ to be the quotient of the free R-algebra generated by a single variable $\hat{F}$, which satisfies the relations $\hat{F}r = F(r)\hat{F}$ for any $r \in R$. We call $R[F]$ the skew polynomial ring in one indeterminate over R, twisted by F.

In non-commutative algebra, one usually considers a more general definition of skew polynomial rings: the ring endomorphism of the ring does not need to be flat; furthermore, the commutation relation is of the less restrictive form $a\hat{F} = F(a)\hat{F} + \delta(r)$ for some derivation δ. For our goals, however, the above definition is sufficiently general.

Example 3.2.2. If F is just the identity $R \to R$, we obtain the usual polynomial ring: $R[F] \simeq R[X]$.

Theorem 3.2.3 (Skew Hilbert basis theorem)**.** *If R is a left (right) Noetherian ring, then for any* automorphism $\alpha : R \to R$, *the skew polynomial ring $R[\alpha]$ is again left (right) Noetherian.*

For a proof (even when a derivation δ is present) see [13]. However, if the endomorphism F is not an automorphism, the theorem might fail, even if we require F to be flat. However, it is true that a skew polynomial ring (in the sense of 3.2.1) for a flat endomorphism F, over a (left) Noetherian ring R is still (left) coherent.

Since we have an injection of rings $R \hookrightarrow R[F]$, any left $R[F]$-module $M \in R[F]$-Mod has a left R-module structure. Additionally, the multiplication from the left by $\hat{F}$ gives a map $M \to M$, defined by $m \mapsto \hat{F}m$. This map is F-semilinear, since $\hat{F}(am) = (\hat{F}a)m = F(a)\hat{F}m$. It hence gives a linear map $F^*M \to M$; $a \otimes m \mapsto aFm$.

Proposition 3.2.4. *The map $\phi : F^*M \to M$; $r \otimes m \mapsto rFm$ is indeed R-linear.*

Proof. Additivity is clear. $r\phi(r' \otimes m) = r(r'Fm) = (rr')Fm = \phi(rr' \otimes m) = \phi(r(r' \otimes m))$. □

We shall now prove the following theorem. The proof is a transcription of the proof found in [5] (although we work with non- commutative rings, and sometimes give more details).

Theorem 3.2.5. *Suppose that $R \in Ring$ is left-Noetherian, $F : R \to R$ is a flat endomorphism. Then $R[F]$ is left-coherent.*

Suppose that the following is an exact sequence of left $R[F]$-modules.

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Then the following diagram (over R) has exact rows, since F is flat:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^*M' & \longrightarrow & F^*M & \longrightarrow & F^*M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_{M'}} & & \downarrow{\scriptstyle \phi_M} & & \downarrow{\scriptstyle \phi_{M''}} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array} \tag{3.4}$$

We can use the snake lemma (or the long exact sequence of the homology of chain complexes) to obtain an exact sequence:

$$\begin{aligned} 0 \to \ker\phi_{M'} \to \ker\phi_M \to \ker\phi_{M''} \to \\ \to \operatorname{coker}\phi_{M'} \to \operatorname{coker}\phi_M \to \operatorname{coker}\phi_{M''} \to 0 \end{aligned} \tag{3.5}$$

Lemma 3.2.6. *Let $M \in R[F]$-Mod, $M \leq R[F]$. Then M is finitely generated over $R[F]$ $\iff$ $\operatorname{coker} \phi_M$ is finitely generated over R.*

Proof. " $\implies$ ": Suppose that we have surjection $R[F]^m \to M$, then we have an exact sequence of $R[F]$- modules, given by $0 \to Z \to R[F]^m \to M \to 0$. Considering the diagram 3.4, we get the kernel-cokernel sequence 3.5; which ends with $\operatorname{coker} \phi_{R[F]^m} \to \operatorname{coker} \phi_M \to 0$. We claim that $R^m \simeq \operatorname{coker} \phi_{R[F]^m}$. This finishes the proof of this direction of the lemma.

$\operatorname{coker} \phi_{R[F]^m} \simeq R[F]^m / \operatorname{im} \phi_{R[F]^m} \simeq (R[F]/ \operatorname{im} \phi_{R[F]})^m$. Consider $\phi_{R[F]} : F^*R[F] \to R[F]$, it is given by $r \otimes p(F) \mapsto rFp(F)$, where p is some polynomial of F. But then clearly, the image is just $F \cdot R[F]$, hence $\operatorname{coker} \phi_{R[F]} = R$.

" $\impliedby$ ": suppose that $\operatorname{coker} \phi_M$ is finitely generated. Since $M \leq R[F]$, the following is a well-defined subset of M:

$$M^{\leq d} \stackrel{\text{def}}{=} M \cap \bigoplus_{i=0}^{D} RF^i.$$

Clearly, $M^{\leq d}$ is a left R-submodule of M. Since R is Noetherian and $\bigoplus_{i=0}^{D} RF^i$ is a finitely generated left R-module, $M^{\leq d}$ is finitely generated over R for each d. Furthermore, $M = \bigcup_{d \geq 0} M^{\leq d}$. We claim that there is an index D such that the map $M^{\leq D} \to \operatorname{coker} \phi_M$ is surjective. R is Noetherian, hence $\operatorname{coker} \phi_M$ is a Noetherian R-module. If there was no such D, then the images of the modules $\phi_M(M^{\leq d})$ would form an infinite ascending chain in $\operatorname{coker} \phi_M$.

Since $M^{\leq D} \to \operatorname{coker} \phi_M$ is surjective, for any $q \geq D$, we have that $M^{\leq q} \leq M^{\leq D} + FM$. Furthermore, $FM \cap M^{\leq q} \subseteq FM^{\leq q-1}$; hence $M^{\leq q} \subseteq M^{\leq D} + FM^{\leq q-1}$. Recursively, we obtain that $M^{\leq q} \subseteq A[F]M^{\leq D}$. q was arbitrary; thus we have $M = A[F]M^{\leq D}$. But $M^{\leq D}$ is finitely generated over A; its finite generating set finitely generates M over $A[F]$. □

Lemma 3.2.7. *Let M be a finitely generated left $R[F]$-module. Then M is finitely presented $\iff$ $\ker \phi_M$ is finitely generated over R.*

Proof. M is finitely generated over $R[F]$, hence we have an exact sequence $0 \to M' \to R[F]^m \to M \to 0$ for some $R[F]$-module M'. Then we have a diagram of the form 3.4, from which we obtain a short exact sequence

$$\ker \phi_{R[F]^m} \to \ker \phi_M \to \operatorname{coker} \phi_{M'} \to R^m \to \operatorname{coker} \phi_M \to 0,$$

since we have shown that $R^m \simeq \operatorname{coker} \phi_{R[F]^m}$ in the proof of lemma 3.2.6. $\ker \phi_{R[F]^m} = 0$, since $1 \otimes p(F) \mapsto Fp(F)$ is injective. Hence the above sequence is changed to

$$0 \to \ker \phi_M \to \operatorname{coker} \phi_{M'} \to R^m \to \operatorname{coker} \phi_M \to 0.$$

But then, since R is Noetherian, $\ker \phi_M$ is finitely generated over R if and only if $\operatorname{coker} \phi_{M'}$ is. By 3.2.6, $\operatorname{coker} \phi_{M'}$ is finitely generated over R if and only if M' is finitely generated over $R[F]$; but this is furthermore equivalent to M being finitely presented via the sequence $M' \to R[F]^m \to M \to 0$. □

Proof of Theorem 3.2.5. Suppose that $M \leq R[F]$ is a finitely generated left submodule. We need that M is finitely presented. We have an exact sequence $0 \to M \to R[F] \to M'' \to 0$ for some $R[F]$- module M''. The associated kernel cokernel sequence begins with $0 \to \ker \phi_M \to \ker \phi_{R[F]}$. We have seen in the proof of 3.2.7, that $\ker \phi_{R[F]} = 0$; it follows that $\ker \phi_M = 0$. In particular, $\ker \phi_M$ is finitely generated over R, hence, by 3.2.7, M is finitely presented over $R[F]$. □

3.3 The Algebraic Point of View

Now, we shall consider a general theory of DVRs, that will allow us to deduce finitary properties of the functor of Colmez. This section is closely following the article [5].

Suppose that A is a discrete valuation ring, with uniformizer t, residue field k, and equipped with a flat *local* endomorphism $F : A \to A$. Clearly A is Noetherian, hence, by 3.2.5, $A[F]$ is left coherent. For us, this ring A will be one of the "+"-version power series rings, defined in section 2.7.1. If M is a module over A, then we will use the notation $M[t]$ to denote the submodule of M annihilated by t.

Definition 3.3.1. We say that an A-module M is admissible as an A-module, if M is A-torsion, and $M[t]$ is finite dimensional over k.

Here if $m \in M[t]$ and if $a - b \in tA$, then $(a-b)m \in Atm = 0$; which shows that $M[t]$ is indeed a k-vector space. If A is a DVR, we have an embedding $A \hookrightarrow \hat{A}$, where $\hat{A}$ is the t-adic completion of A.

Lemma 3.3.2. *Let $M \in A$-Mod be admissible as an A-module. Then $M[t^n]$ is of finite length over A. As a consequence, since $M[t^n]$ is torsion, and finitely generated over a PID, $M[t^n] \simeq \bigoplus_{i=1}^{k} (A/t^iA)^{r_i}$.*

Proof. $M[t]$ is of finite length, since it is finite-dimensional over k. We proceed by induction. In the exact sequence

$$0 \to M[t] \to M[t^{n+1}] \xrightarrow{\cdot t} tM[t^{n+1}] \to 0$$

the last term $tM[t^{n+1}]$ is contained in $M[t^n]$, which is of finite length by the induction hypothesis. Hence $tM[t^{n+1}]$ is of finite length as well, and $M[t^{n+1}]$ is then a finite length extension of a finite length module, i.e. itself of finite length. □

Proposition 3.3.3. *Let $M \in A$-Mod be admissible as an A-module. Then*

$$M \simeq (K/A)^r \oplus \bigoplus_{j=1}^{N} (A/t^jA)^{r_j}.$$

Proof. We have that $M[t] = \bigoplus_{i=1}^{r_1^{(1)}} A/t$ Consider the exact sequence Clearly, $M[t] \subseteq M[t^n]$. We have that $M[t^n] \simeq \bigoplus_{i=1}^{N_n} (A/t^iA)^{r_i^{(n)}}$. But each of the A/t^iA submodules contain precisely one copy of A/tA. This shows that $r_1^{(1)} = \sum_{i=1}^{N_n} r_i^{(n)}$. In particular, when n is increased, the total number of direct summands stays the same. Some of the direct summands "terminate" in some A/t^kA, while others do not. Since M is torsion, $M = \varinjlim_n M[t^n]$. The non-terminating direct summands give $\varinjlim A/t^nA \simeq K/A \simeq \hat{K}/\hat{A}$. □

Corollary 3.3.4. *Let $M \in A$-Mod be admissible as an A-module. Then*

$$M^\vee \simeq \hat{A}^r \oplus \text{ torsion part.}$$

Definition 3.3.5. We call the integer r the *corank* of the module M; it is simply the free rank of the dual module $M^\vee$.

Proposition 3.3.6. *If M is an admissible A-module, then M is Artinian.*

Proof. By the structure theorem 3.3.3, it is enough to show that K/A is an Artinian A-module. $K/A \simeq A[t^{-1}]/A \simeq \bigcup t^{-n}A/A$ as A-modules. We claim that the only submodules of K/A are of the form A/t^nA. This follows from the fact that the submodule generated by any element is of this form. But submodules of this form can not form an infinite strictly decreasing chain. □

Now that the basic properties of admissible A-modules are established, we investigate the interaction of such modules with the endomorphism F.

Let Add_A^F be the full subcategory of $A[F]$-modules that are finitely generated over $A[F]$ and admissible as an A-module.

Lemma 3.3.7. *If $M \in \mathrm{Add}_A^F$, then F^*M and M have the same corank.*

Proof. F is local, hence $A \otimes_F (K/A)^r = A \otimes_F (\varinjlim A/t^nA)^r = \varinjlim (A \otimes_F A/t^nA) \simeq (K/A)^r$. □

Theorem 3.3.8. Add_A^F *satisfies the following:*

1.) *If $M \in \mathrm{Add}_A^F$, then M is finitely presented over $A[F]$.*

2.) Add_A^F *is an Abelian category.*

3.) *If $M \in \mathrm{Add}_A^F$, then M is of finite length over $A[F]$.*

Proof. The map $\phi_M : F^*M \to M$, which is a module homomorphism. M is finitely generated over $A[F]$, hence $\operatorname{coker}\phi_M$ is finitely generated over A, by lemma 3.2.6. By lemma 3.2.7, it is enough to show that $\ker\phi_M$ is finitely generated over A. Now F^*M and M have the same corank. Consider $\operatorname{coker}\phi_M$. It is finitely generated over A by lemma 3.2.6 as M is finitely generated over $A[F]$. But then $\ker\phi_M$ is finitely generated over A as well. 3.2.7 implies that M is finitely presented over $A[F]$.

2.): We will use proposition 1.0.3. Clearly, 0 is in Add_A^F, and if two modules are in Add_A^F, then so is their direct sum. We need that quotients and submodules of $M \in \mathrm{Add}_A^F$ are again in Add_A^F. If N is quotient of some $M \in \mathrm{Add}_A^F$, then N is again trivially finitely generated over $A[F]$, $M[t]$ surjects to $N[t]$, and N is A-torsion. Now suppose that N is an $A[F]$-submodule of $M \in \mathrm{Add}_A^F$. Then $N[t] \le M[t]$, hence it is finite-dimensional over k. N is A-torsion, since M is A-torsion. By 1.), M is finitely presented over $A[F]$, and M/N is in Add_A^F by the previous reasoning, so M/N is finitely presented as well. Hence $M \to M/N$ is a morphism of finitely presented $A[F]$-modules. By 3.2.5, $A[F]$ is left coherent, implying that the category of finitely presented modules over $A[F]$ is Abelian. In particular, it contains all kernels, hence N is also finitely presented. This shows that N is finitely generated as an $A[F]$-module.

3.): By proposition 3.3.6, M is Artinian as an A-module, hence it is Artinian as an $A[F]$-module. By 2.), any submodule of M is finitely generated over $A[F]$, hence M is Noetherian. Any module that is both Noetherian and Artinian is of finite length. □

The following holds in general:

Lemma 3.3.9. *For any regular local ring A, the global dimension of A is equal to its Krull dimension. In particular, if A is a DVR, its global dimension is 1. Furthermore, if A is a DVR with uniformizer t and residue field $k = A/t$, $M \in A$-Mod, then $\mathrm{Tor}_0(M,k) = M/tM$, and $\mathrm{Tor}_1(M,k) = M[t]$.*

In our setting, each of the Tor-modules is equipped with a $k[F]$-module action: on M/tM, $F : m + tM \mapsto F(m) + tM$ is well defined. On $M[t]$, the action $F : m \mapsto \frac{F(t)}{t}F(M)$. By the locality of F, $F(t) \in tA$, and hence $F(t)/t$ is well-defined (A is a UFD).

Proposition 3.3.10. *Suppose that M is a finitely generated left $A[F]$-module, which is torsion over A. Then if M/tM is torsion over $k[F]$, then M is admissible as an A-module.*

Proof. It is enough to show that $M[t]$ is finite-dimensional over k. First, let $M = \langle m\rangle$ be a cyclic $A[F]$-module, and $M_0 = Am$ be the A-module generated by m. A is a DVR, M is A-torsion, hence $M_0 \simeq A/t^r$ for some $r \in \mathbb{N}$. Taking the tensor product with $A[F]$, the injection $M_0 \to M$ gives an exact sequence:

$$0 \to N \to A[F] \otimes_A M_0 \to M \to 0. \tag{3.6}$$

Modding out by t, we get a short exact sequence

$$N/tN \to A[F] \otimes_A M_0/t(A[F] \otimes_A M_0) \to M/tM.$$

Since $M_0 \simeq A/t^r$, we have that

$$A[F] \otimes_A M_0/t(A[F] \otimes_A M_0) \simeq k[F]$$
$$(A[F] \otimes_A M_0)[t] \simeq k[F]$$

as $A[F]$-modules.

Using the first isomorphism, we obtain

$$N/tN \to k[F] \to M/tM,$$

where M/tM is $k[F]$-torsion. But then there exists an element $n \in N$ such that its image in $A[F] \otimes_A M_0/t(A[F] \otimes_A M_0)$ is non-zero (the kernel of $k[F] \to M/tM$ can not be zero). Let us then denote the $A[F]$-submodule of $A[F] \otimes_A M_0$, generated by the image of this n (denoted by n'). Then since the $n' + t(A[F] \otimes_A M_0) \neq 0$, clearly $n' \neq 0$, hence $M' \neq 0$. Let $M'' = (A[F] \otimes_A M_0)/M'$, i.e. we have the exact sequence

$$0 \to M' \to A[F] \otimes_A M_0 \to M'' \to 0.$$

The associated long exact sequence of Tor-modules, by lemma 3.3.9, and the two isomorphisms above, gives

$$0 \to M'[t] \to k[F] \to M''[t] \to M'/tM' \to k[F] \to M''/tM'' \to 0.$$

M' is a cyclic $A[F]$-module, since it is generated by n': but then M'/tM is cyclic over $k[F]$. By the construction of n', its image in $k[F] = A[F] \otimes_A M_0/t(A[F] \otimes_A M_0)$ is non-zero. But then the image of any $m' \in M'$ is nonzero (its generator is not in the kernel); i.e. M' injects to $k[F]$. But then we have the short exact sequence:

$$0 \to M'[t] \to k[F] \to M''[t] \to 0.$$

M' is a non-zero submodule of a torsion A-module, hence $M'[t] \neq 0$. Now $M''[t]$ is a quotient of $k[F]$, by the non-zero $M'[t]$; and any such quotient is finite- dimensional ($k[F]$ is just the standard polynomial ring $k[X]$). But then M'' is admissible, and in fact M'' is an object of Add_A^F. By theorem 3.3.8, Add_A^F is Abelian, hence any quotient of M'' is again in Add_A^F. By the exact sequence of the diagram 3.6, $0 \xrightarrow{\phi} N \to A[F] \otimes_A M_0 \to M \to 0$. As $M'' = (A[F] \otimes_A M_0)/M'$, with $M' \subseteq \mathrm{im}(\phi)$, M'' surjects to M. The claim for cyclic M follows.

Suppose now that M is generated by the elements $m_1, \dots m_n$, as an $A[F]$-module. Let M' be the $A[F]$-module generated by m_1, and $M'' = M/M'$, i.e. $0 \to M' \to M \to M'' \to 0$ is exact. Now M'' is generated by $n-1$ elements, and it is torsion over A; furthermore, $M''/tM'' \simeq (M/M')/t(M/M')$ is a quotient of M/tM, hence $k[F]$-torsion. By induction, M'' is admissible over A. Now the long exact sequence of Tor-modules, associated to $0 \to M' \to M \to M'' \to 0$ gives, by lemma 3.3.9

$$0 \to M'[t] \to M[t] \to M''[t] \to M'/tM' \to M/tM \to M''/tM'' \to 0.$$

M'' is admissible, hence $M''[t]$ is finite-dimensional over k, hence torsion over $k[F]$. By assumption, M/tM is $k[F]$-torsion. But then M'/tM' is an extension of $k[F]$-torsion modules, i.e. itself $k[F]$-torsion. By induction, M' is admissible as well. Actually, both M' and M'' are admissible as A-modules and finitely generated over $A[F]$, hence elements of Add_A^F, which is an Abelian category by theorem 3.3.8. Hence M is also admissible. □

We shall now focus on the application of the theory established in this section to the functor of Colmez. Throughout section 3.1, we considered representations from the category $\mathrm{Rep}_{tors}\, G$, with $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and coefficients in $\mathcal{O}_L$, where L is a p-adic number field. Whenever Π is an object of $\mathrm{Rep}_{tors}\, G$, the module $\mathrm{D}_W^+(\Pi)$ is a (φ, Γ)-module over $\mathcal{O}_{\mathcal{E}}{}^+$. $\mathcal{O}_{\mathcal{E}}{}^+ = \mathcal{O}_L[[T]]$ is a local ring with maximal ideal (T, ϖ), hence we can not choose it as the ring "A" of this section. Instead, we work with $A := k_{\mathcal{E}}^+ \simeq k_L[[T]]$, which is indeed a DVR. Also, $\varphi : k_L[[T]] \to k_L[[T]]$ is a flat endomorphism (by corollary 2.7.6), and it is local (by proposition 2.7.7).

Suppose that $\Pi \in \mathrm{Rep}_{tors}\, G$. Then there is an A-action on $\Pi[\varpi]$, since Π is smooth: indeed, $\Pi = \bigcup \Pi^U$ where U runs on compact open normal subgroups of G; hence $k[G/U]$ acts on $\Pi^U[\varpi]$, and these actions are compatible. For now, we will simply assume that Π is ϖ-torsion. The general case will follow easily.

Proposition 3.3.11. *Let M be an A-module which is a topological A-module with the discrete topology. Then M is admissible as an A-module (in the sense of definition 3.3.1). if and only if M is admissible as a $\mathbb{Z}_p$-representation over $\mathcal{O}_K$ (in the sense of 2.4.1).*

Proof. $A = k[[t]]$ is isomorphic to the Iwasawa-algebra $k[[\mathbb{Z}_p]]$, by proposition 2.3.6. We can consider M as a $\mathcal{O}_L[[\mathbb{Z}_p]]$-module, via the surjection $\mathcal{O}_L[[\mathbb{Z}_p]] \to k[[\mathbb{Z}_p]]$.

First suppose that M is admissible as an A-module. Then M is smooth by assumption, and trivially $\mathcal{O}_K$-torsion. We also have that $M[t]$ is finite-dimensional over k, hence finitely generated over $\mathcal{O}_L$. Since ϖ annihilates M, we can apply proposition 2.4.15, hence it is enough to show that there is one pro-p subgroup U of $\mathbb{Z}_p$ which satisfies M^U is finitely generated over $\mathcal{O}_L$. Equivalently: M^U is finite-dimensional over k. Let δ be a topological generator of $\mathbb{Z}_p$, and we write addition in $\mathbb{Z}_p$ multiplicatively. We have that δ corresponds to $1 + t$ in $k[[t]]$, hence t corresponds to $\delta - 1$. If $v \in M[t]$, then $tv = 0$, i.e. $(\delta - 1)v = 0$. By the assumption, $M[t]$ is finite dimensional over k; but this means precisely that $M^{\mathbb{Z}_p}$ is finitely generated over $\mathcal{O}_K$.

Conversely: if M is admissible as a $\mathbb{Z}_p$- representation over $\mathcal{O}_L$, then any pro-p subgroup of $\mathbb{Z}_p$ fixes only a module that is finitely generated over $\mathcal{O}_L$ (hence finite- dimensional over k). But then $M[t]$ is finite- dimensional over k. □

Note that since $\mathrm{Rep}_{tors}\, G$ is Abelian (by theorem 2.6.5), any subrepresentation would be admissible as a G-representation. However, being admissible as an A-module (or, equivalently, by the above proposition, being admissible as a $\mathbb{Z}_p$-representation) is a much stronger statement, as it requires the fixed point submodule of smaller groups to be finitely generated.

Proposition 3.3.12. *If M is an admissible A-module, which is a topological A-module with the discrete topology, then the Pontryagin dual of M, $M^\vee$ is a finitely generated topological A-module.*

Proof. This is immediate from the previous proposition (3.3.11), and using proposition 2.4.11. □

Our main goal is to show the following theorem:

Theorem 3.3.13. *Let $\Pi \in \mathrm{Rep}_{tors}\, G$, which satisfies $\Pi = \Pi[\varpi]$. Let $W \in \mathcal{W}(\Pi)$. Then $\Phi(I_{\mathbb{Z}_p}(W))$ is admissible as a $k_{\mathcal{E}}^{+} = \mathcal{O}_L/\varpi\mathcal{O}_L[[t]]$-module.*

Note that by proposition 3.3.12, we immediately obtain that the Pontryagin-dual of $\Phi(I_{\mathbb{Z}_p}(W))$ is finitely generated. But $\Phi(I_{\mathbb{Z}_p}(W))^\vee$ is just $\mathrm{D}^\natural{}_W(\Pi)$. Taking the tensor product with $\mathcal{O}_{\mathcal{E}}$ gives:

Proposition 3.3.14. *Let $\Pi \in \mathrm{Rep}_{tors}\, G$ be such that $\Pi = \Pi[\varpi]$. Then $\mathbf{D}(\Pi)$ is finitely generated over $\mathcal{O}_{\mathcal{E}}$.*

Theorem 3.3.15. *Let $\Pi \in \mathrm{Rep}_{tors}\, G$. Then $\mathbf{D}(\Pi)$ is finitely generated over $\mathcal{O}_{\mathcal{E}}$.*

Proof. $\mathrm{Rep}_{tors}\, G$ is Abelian. Hence $\Pi[\varpi^k]$ is also in $\mathrm{Rep}_{tors}\, G$. By proposition 2.6.7, there exists some n such that $\Pi[\varpi^n] = \Pi$. We can apply the previous statement for $\Pi[\varpi^k]/\Pi[\varpi^{k-1}]$. By theorem 3.1.31, $\mathbf{D}$ is an exact functor. We obtain the proof with induction on k, and proving the claim for $\Pi[\varpi^k]$ (which terminates in finitely many steps, as $\Pi[\varpi^n] = \Pi$ for some n). □

In the rest of this section, we prove theorem 3.3.13. Now $\Phi(I_{\mathbb{Z}_p}(W))$ is just $\sum \left(\begin{smallmatrix} p^n & a \\ 0 & 1 \end{smallmatrix}\right) W$, where $a + p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p$. Clearly, this is the case if and only if $a \in \mathbb{Z}_p$. Also, the Frobenius F of A is just the multiplication with $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. But then:

Proposition 3.3.16. *$\Phi(I_{\mathbb{Z}_p}(W))$ is the $A[F]$-submodule of Π, generated by W.*

We can reformulate the above theorem as:

Theorem 3.3.17. *Let $\Pi \in \mathrm{Rep}_{tors}\, G$ with $\Pi = \Pi[\varpi]$. Let $W \in \mathcal{W}(\Pi)$, and let $M(\Pi, W)$ the $A[F]$-submodule of V, generated by V_0. Then $M(\Pi, W)$ is admissible as an A-module.*

Note that since V_0 is finite-dimensional over a finite field, it is finite as a set. But then $M(\Pi, W)$ is finitely generated over $\mathcal{A}[F]$. We can hence use 3.3.10. It remains to show that $M(\Pi, W)/tM(\Pi, W)$ is torsion over $k[F]$.

Proposition 3.3.18. *In the setting of 3.3.17 $M(\Pi, W)/tM(\Pi, W)$ is $k[F]$-torsion.*

Proof. Since $W \in \mathcal{W}(\Pi)$, we have that $GW = \Pi$. If P is the Borel subgroup of G, then by the Iwasawa-decomposition $\Pi = k[P]W$ (as mentioned in the proof of 3.1.8). Furhtermore, by the same proof, $P \subseteq F^{-\mathbb{N}} \left(\begin{smallmatrix} 1 & \mathbb{Q}_p \\ 0 & 1 \end{smallmatrix}\right) F^{\mathbb{N}}\, \mathrm{GL}_2(\mathbb{Z}_p) Z(G)$. This shows (since W is $\mathrm{GL}_2(\mathbb{Z}_p)Z(G)$-invariant) that $\Pi = k[F^{-1}]M(\Pi, W)$. In particular, every element of $\Pi/M(\Pi, W)$ is annihilated by some power of F. It implies that the Tor^A-modules of $\Pi/M(\Pi, W)$ are $k[F]$-torsion. Now we have a short exact sequence of A-modules

$$0 \to M(\Pi, W) \to \Pi \to \Pi/M(\Pi, W) \to 0$$

From the long exact sequence of Tor-modules (taking the tensor product w.r.t $k = A/t$), it is enough to show that $\Pi/t\Pi$ is $k[F]$-torsion (since then $\Pi/M(\Pi, W)[t]$ and $\Pi/t\Pi$ are both torsion, hence $M(\Pi, W)/tM(\Pi, W)$ is torsion as well. This is precisely the statement of proposition 3.3.19. □

Proposition 3.3.19. *Let Π in $\mathrm{Rep}_{tors}\, G$. Then $\Pi/\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) - 1\right)\Pi$ is $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$-torsion.*

The proof we give is based on Emerton's proof of a much more general theorem in [14]. Throughout this proof, H^i denotes group cohomology.

Let $N_0 = \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ of G, $V_K = \left(\begin{smallmatrix} 1+p^k\mathbb{Z}_p & \mathbb{Z}_p \\ p^k\mathbb{Z}_p & 1+p^k\mathbb{Z}_p \end{smallmatrix}\right)$, and $U_k = \left(\begin{smallmatrix} 1+p^k\mathbb{Z}_p & p^k\mathbb{Z}_p \\ p^k\mathbb{Z}_p & 1+p^k\mathbb{Z}_p \end{smallmatrix}\right)$ Then clearly, $N_0 = \bigcap_k V_i = \varprojlim V_i$.

Lemma 3.3.20. $\Pi/t\Pi \simeq H^1(N_0, \Pi)$.

Proof. Clearly, $H^0(N_0, \Pi) = \Pi^{N_0}$. Furthermore, we have that $\Pi^{N_0} \simeq \mathrm{Hom}_{k[[N_0]]}(k, \Pi)$, via the homomorphism $\rho : v \mapsto [\mu_v : 1 \mapsto v]$. ρ is trivially $k[[N_0]]$-linear and injective. It is surjective, because any $\mu : k \to \Pi$ (as N_0 acts trivially on k) must map 1 to an element on which N_0 acts trivially, i.e. to some $v \in \Pi^{N_0}$. It is then enough to show that the Ext^1 functor is $\Pi/t\Pi$. It can be calculated via the projective resolution of k. One such resolution is $0 \to k[[t]] \xrightarrow{\cdot t} k[[t]] \to k \to 0$. Applying the Hom functor and removing the "-1th" term, we get the half-exact sequence

$$0 \to \mathrm{Hom}_{k[[t]]}(k[[t]], \Pi) \to \mathrm{Hom}_{k[[t]]}(k[[t]], \Pi) \to 0.$$

Then clearly $\mathrm{Ext}^1(k, \Pi) = \Pi/t\Pi$, and we have shown the claim. ☐

Lemma 3.3.21. $H^1(N_0, \Pi) \simeq \varinjlim_k H^1(V_i, \Pi)$.

Proof. We have $N_0 = \bigcap_k V_i$, hence $\bigcup_k \Pi^{V_i}$. Similarly for any injective resolution: we have a diagram of chain complexes of the form

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Pi^{V_1} & \longrightarrow & I_1^{V_1} & \longrightarrow & I_2^{V_1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Pi^{V_2} & \longrightarrow & I_1^{V_2} & \longrightarrow & I_2^{V_2} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \dots & & \dots & & \dots & &
\end{array}$$

This is a direct limit of cochain complexes, which commutes with taking cohomology of cochain complexes. The claim follows. ☐

Lemma 3.3.22. *Each of the $H^1(V_i, \Pi)$ are F-invariant.*

Proof. Let $f = \left(\begin{smallmatrix} p & 0 \\ 1 & 0 \end{smallmatrix}\right)$. The action of F is the multiplication with $\left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right)$. For upper-triangular matrices, this is precisely the conjugation with f. The action of F on the first cohomology groups $H^1(V_i, \Pi)$ is given by the Hecke action

$$(F\mu)(g) = \sum_{u \in N_0/F(N_0)} F(\mu(gu^{-1}))$$

(where μ is a crossed homomorphism). This action is induced by the map $F : \Pi^{V_i} \to \Pi^{fV_i f^{-1}}$ composed with the summation on the FN_0F^{-1}-cosets. A direct calculation shows the F-invariance of the H^1 spaces. ☐

Lemma 3.3.23. *Each of the $H^m(V_i, \Pi)$ are finitely generated over k; and hence F-torsion.*

Proof. We can choose an injective resolution of Π that is admissible. This is because by theorem 2.4.11, the dual of torsion admissible representations are finitely generated over the Noetherian Iwasawa-algebra $\mathcal{O}_L[[U]]$ for some pro-p compact open subgroup U of G that contains V_i. But the category of finitely generated modules over any Noetherian ring has enough projectives. Dualizing gives an injective resolution consisting of admissible representations. Now in such an injective resolution, each of the $(I^r)_i^V$ is finitely generated over k, as V_i is an open subgroup of G, and I^r is admissible. Clearly, the cohomology groups $H^m(V_i, \Pi)$ are then also finitely generated over k. In particular, $H^1(V_i, \Pi)$ is finite as a set, hence F-torsion. ☐

The proof of proposition 3.3.19 follows from the previous lemmas, since $\Pi/t\Pi$ is the direct limit of F-torsion modules.

3.3.1 Final Notes

We completely proved, in several steps, the following theorem:

Theorem 3.3.24. *Let $\mathcal{C}$ be the full subcategory of* $\operatorname{Rep}_{tors} G$*, which consists of objects annihilated by ϖ. Then* $\mathbf{D}$ *is a functor from* C *to the category of finitely generated* (ϕ, Γ)*-modules over* $\mathcal{O}_{\mathcal{E}}$*.*

We did not prove any statements about standard presentations, but we used the fact that all object of $\operatorname{Rep}_{tors} G$ admit them. Using a property of standard presentations, we showed that, in fact, the image of $\mathbf{D}$ consists of étale (φ, Γ)-modules. Without proof (the proof again depends on the properties of standard presentations) we claimed that $\mathbf{D}$ is exact. Using this statement, we proved (3.3.15) that $\mathbf{D}(\Pi)$ is finitely generated for all $\Pi \in \operatorname{Rep}_{tors} G$.

Top sum things up, we obtained:

Theorem 3.3.25. $\mathbf{D}$ *is an exact functor from* $\operatorname{Rep}_{tors} G$ *to* $\Phi\Gamma^{et}_{tors}(\mathcal{O}_{\mathcal{E}})$*.*

Now we define the functor $\mathbf{D}$ for the other two categories defined in section 2.6.

For $\Pi \in \operatorname{Rep}^{C}_{\mathcal{O}_L} G$, we define $\mathbf{D}(\Pi)$ as the inverse limit of $\varprojlim_{n \in \mathbb{N}} \mathbf{D}(\Pi/\varpi^n)$. For $\Pi \in \operatorname{Rep}^{C}_{L} G$ with $\mathcal{O}_L$-lattice Π_0, we define $\mathbf{D}(\Pi)$ to be $L \otimes \mathbf{D}(\Pi_0)$.

We shall not prove the following theorem (but we did prove many parts of it).

Theorem 3.3.26. *Using the above definitions*

1. $\mathbf{D}$ *is an exact functor from* $\Pi \in \operatorname{Rep}^{C}_{\mathcal{O}_L} G$ *to* $\Phi\Gamma^{et}(\mathcal{O}_{\mathcal{E}})$*.*
2. $\mathbf{D}$ *is an exact functor from* $\operatorname{Rep}^{C}_{L} G$ *to* $\Phi\Gamma^{et}(\mathcal{E})$

There are actually several generalizations of the functor of Colmez, even for any reductive algebraic groups over $\mathbb{Q}_p$, or even K, where K is a p-adic number field (which is distinct, in general, from L). For details, see [4], [3]. Surprisingly, many of the properties of $\mathbf{D}$ exist in this much more general setting. The difference comes from two main points: on the one hand, one can not guarantee the existence of standard presentations. Hence these generalizations are only half-exact. On the other hand, the algebraic point of view of Emerton (and section 3.3) relies heavily on the fact that $k[[\mathbb{Z}_p]]$ is a DVR. This fails for $k[[\mathcal{O}_K]]$ where K is a p-adic number field, and the analogues in higher-dimensional algebraic groups do not seem to allow to deduce finitary properties. In fact, one does not know whether or not the image of these generalized Montréal functors are finitely generated as (φ, Γ)-modules.

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