ON CONIC SECTIONS AND SOME OF THEIR APPLICATIONS

MSC THESIS IN MATHEMATICS

BY ILLÉS FAZEKAS

SUPERVISOR: GYÖRGY KISS

DEPARTMENT OF GEOMETRY

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A **diplomamunka** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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a hallgató aláírása

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Acknowledgments

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I would also like to say special thanks to my former English teacher Barbara Ávár in recognition of her efforts into advancing my knowledge of the English language.

1 Introduction

My BSc thesis also revolved around conics, this is a seamless continuation of the work laid out there. Conic sections proved essential throughout the history of mathematics ever since the Ancient Greeks. In this piece of work, we aspire to give an insight into miscellaneous, often intriguing and beautiful properties of conic sections and their utilization in applications, following classical, finite and projective approaches.

We first present an alternative characterization of conics and some of their properties. We then proceed to finite projective planes and some of their applications, before moving back to the real plane to investigate pencils of circles. The last instance we unveil is a less-known yet even more eye-catching projective theorem regarding conics, Frégier's Theorem.

As the reader may see themself, throughout the thesis, visualization and illustration are of prime importance. Where possible and appropriate, especially in the last chapter, we strive to give synthetic geometric proofs to theorems, many of which I rigged up myself, providing my own ideas and interpretation in addition to or in place of the sources.

2 A Useful Property of Conic Sections

2.1 An alternative characterization of conic sections

Conic sections have a less-known definition that is equivalent to the original one, which we explore in the following section, specifically with respect to the parabola and the ellipse. Note that there is an equivalent definition for the hyperbola as well, but we will not discuss it due to space constraints. The following characterizations and properties of conics have greatly been sourced from [4]. Let's consider the following lemmas:

Lemma 2.1 *Let v be a circle with center G and radius r, and F (as the focus) be a point inside the circle but not coinciding with G. Then, the centers of circles that pass through F and are tangent to v lie on an ellipse with foci at F and G whose major axis is of length r.*

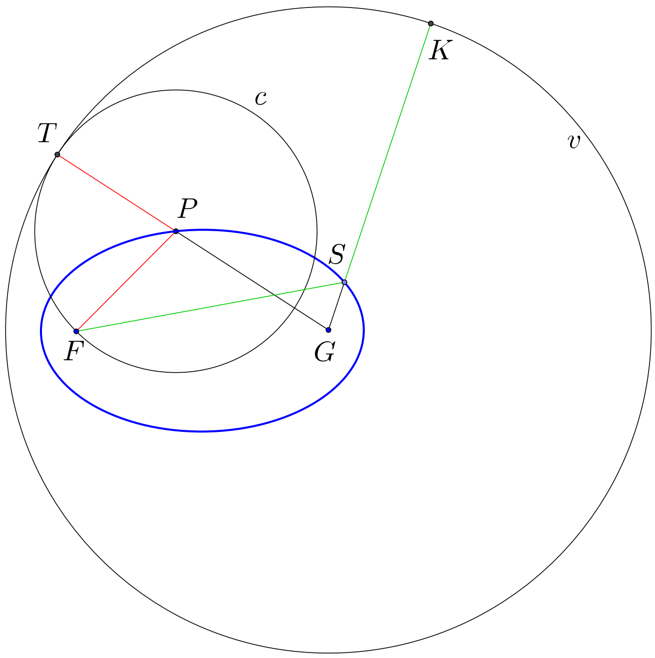

Figure 1: Ellipse and its directing circle

Proof Let c be a circle with center P that passes through F and is tangent to v in T (Figure 1). Then, due to the tangency properties of circles, points T, P and G collinear, yielding:

$$r = TG = TP + PG = FP + PG.$$

Taking this into account, one obtains that the sum of the distances of such P points from both F and G is constant. This means that the set of such points lies on an ellipse, although we remark that it would need further discussion to prove that such points make up the whole ellipse.

Lemma 2.2 *Let v be a line, and F a point that does not coincide with it. Then, the centers of circles that pass through F and are tangent to the line form a parabola with focus at F and directrix v.*

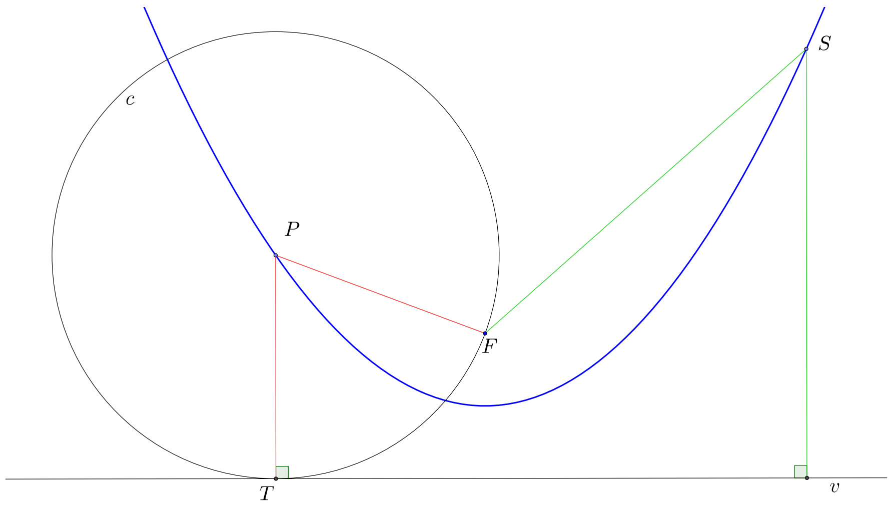

Figure 2: Parabola and its directrix

Proof Again, consider a circle with center P that passes through F and is tangent to the line v (Figure 2). Using the tangency property, we can deduce that $PT \perp v$. Thus, the distance from P to v is exactly PT, and since $PT = PF$ (as T and F lie on circle c), these points satisfy the definition of a parabola. In this case also, albeit left out, it would require further reasoning to prove that these points make up the whole parabola.

2.2 The reflective property of conics

We will now focus on a property of parabolas and ellipses that relates the focal points to tangent lines. Note that, as with the previous definitions, this can also be extended to hyperbolas.

Theorem 2.3 *For any point P on a parabola or an ellipse γ, the tangent drawn at P bisects the angle spanned by the radii connecting P with the foci. (In the case of the parabola, the other focus is the point at infinity lying on the parabola's axis, and also on all lines parallel to it, i.e. lines perpendicular to the directrix.)*

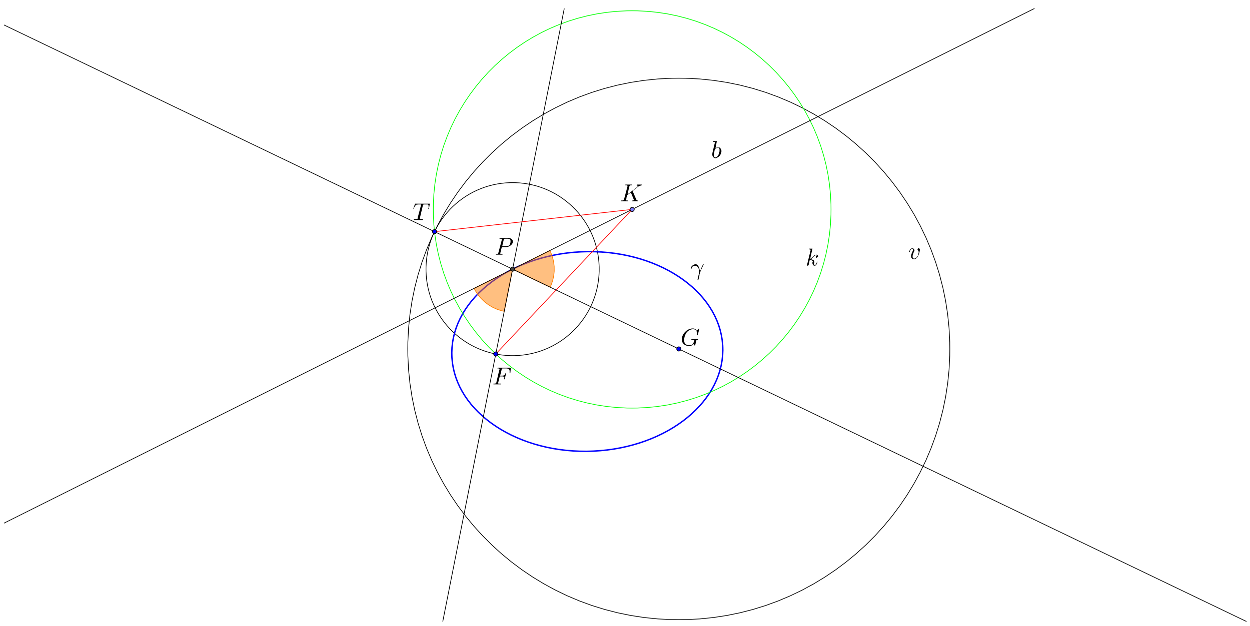

Figure 3: Ellipse and its tangent

Proof Let γ be an ellipse with focus F and directrix v (Figure 3). According to the previous lemma for ellipses, the circle c centered around P and with radius PF is tangent to v. Let T be the point of tangency. Let b be the angle bisector of $\angle TPF$, and K any point on it different from P. Since K lies on the bisector, $KT = KF$. Thus, the circle k centered around K with radius KF also passes through T. However, since T lies on v, and $P \neq K$, it follows that k does not touch v, implying that K cannot lie on γ.

We obtained that the intersection of the ellipse and the angle bisector consists of a single point, namely P, from which we can coclude that the tangent also acts as an angle bisector.

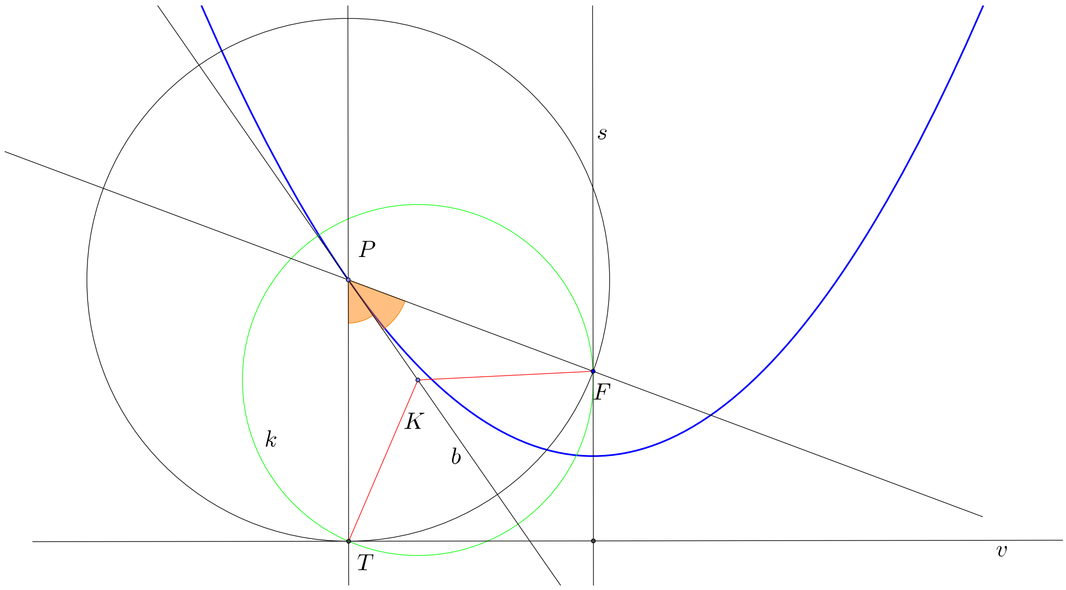

Figure 4: Parabola and its tangent

The reasoning in the previous proof can be analogously applied to the parabola as depicted on Figure 4, leading to the conclusion that this angle bisector has exactly one point in common with the curve. However, this per se does not necessarily imply tangency as b could still be parallel to the axis of the parabola. Nevertheless, this case can be excluded because if PT were parallel to the axis, PT and PF would coincide, which would imply that the focus lies on the directrix, contradicting that γ is a parabola. This proves that the tangent is again also the angle bisector.

2.3 Applications

The reflective property of parabolas is used in parabolic antennas whereby axially inbound satellite signals (signals whose path is parallel to the axis, i.e. they are travelling through the parabola's focus at infinity) are collected at the paraboloid-shaped antenna's focus following reflection, or to be precise, this holds on every cross-section through the paraboloid's axis. This minimizes the need for signal amplification and reduces the required surface area for signal reception. Similarly, this property is used in parabolic microphones for remote audio surveillance where sound waves are focused at the microphone's pickup point after reflection.

Conversely, a parabolic mirror can be used to disperse axially outgoing waves. This is exploited in car headlights where the light source is positioned at the focal point of a paraboloid-shaped reflector, making light rays bounce back off in the direction of the vehicle's motion.

An important application of the ellipse is the lithotripter, a non-invasive medical device used to break kidney stones with mechanical vibrations. The lithotripter is ellipsoid-shaped, with the targeted kidney stone placed at one of its foci. Mechanical vibrations are emitted in all directions from the ellipsoid's outer focus, ensuring that low-energy vibrations do not harm surrounding tissues. These low-energy vibrations then reflect off the ellipsoid's surface and converge on the other focus, where they take effect on the kidney stone, reducing the risks and discomfort associated with surgery.

3 Finite Projective Planes

As opposed to the classical approach, one can consider projective planes from a finite geometric point of view by endowing the incidence structure with a coordinate system over a finite field. Conic sections may still be defined, in analogy with the classical approach, as quadratic curves over finite fields. These, in addition to being interesting by themselves, offer elegant solutions to combinatorial and coding theoretic problems. In this chapter, we intend to give a short insight into this approach. The theoretical background on such planes is based on source [6].

3.1 Building projective planes with axioms of incidence

On the classical projective plane, the following two widely known axioms hold:

1. For any two arbitrary distinct points on the projective plane, there exist a unique line passing through them.

2. For any two arbitrary distinct lines on the projective plane, there exists a unique point at which these lines intersect.

These axioms establish a solid basis to proceed to define the abstract projective plane.

Definition 3.1 *Let us consider the triple* $\Pi = (\mathcal{P}, \mathcal{L}, I)$ *where* $\mathcal{P}$ *and* $\mathcal{L}$ *are disjoint sets and* I *is a relation defined on* $\mathcal{P} \times \mathcal{L}$*, called incidence relation, where* $\mathcal{P}$ *would form the set of points and* $\mathcal{L}$ *the set of lines.* Π *is called an abstract projective plane if it satisfies the following axioms:*

1. $\forall P_1, P_2\ (P_1 \neq P_2) \in \mathcal{P}\ \exists! l \in \mathcal{L}$ *such that* $(P_1, l), (P_2, l) \in I$*, the formalization of the statement that for any two arbitrary points on the projective plane, there exists a unique line joining them.*

2. $\forall l_1, l_2\ (l_1 \neq l_2) \in \mathcal{L}\ \exists! P \in \mathcal{P}$ *such that* $(P, l_1), (P, l_2) \in I$*, the formalization of the statement that any two arbitrary lines on the projective plane intersect at a single point.*

3. $\forall l \in \mathcal{L}\ |\{\ P \in \mathcal{P} \mid (P, l) \in I\}| \geq 3$*, i.e., every line on the plane contains at least three points.*

4. $\forall P \in \mathcal{P}$*-re* $|\{\ l \in \mathcal{L} \mid (P, l) \in I\}| \geq 3$*, i.e., every point on the plane is contained in at least three lines, otherwise stated, through every point pass at least three lines.*

Axioms 1. and 2., and axioms 3. and 4. are dual to each other, interchanging the roles of points and lines in one yields the other. Thus, a very robust tool is at disposal: if one formulates a theorem regarding points, lines and incidence, then dualizing it by interchanging the roles of lines and points, the corresponding theorem automatically holds too, sometimes saving us immense effort at proving a theorem.

Similarly, every projective plane admits a dual plane per se, whose points correspond to the lines of the original plane, and whose lines are associated with the points of the original plane, furthermore, these paired objects coincide on the dual plane if and only if they did on the original one.

One can easily show that the dual plane satisfies the above axioms, it is a projective plane itself, and even that dualization as an operation on projective planes is involutive: the dual of the dual of a projective plane is the plane itself.

Definition 3.2 *A projective plane* Π *is called finite if* $\mathcal{P}$ *and* $\mathcal{L}$ *are finite sets.*

The simplest example of a finite projective plane is the so called Fano plane (see figure below). One can easily verify that all axioms of a projective plane are satisfied by this configuration. Why it is the simplest example will turn out soon from the process of equipping it with a coordinate system.

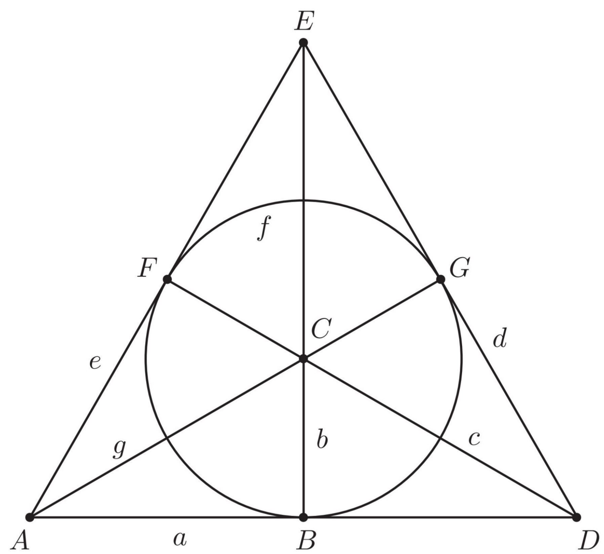

Figure 5: Fano plane

Formally, the structure of the plane appears as follows:

$$\mathcal{P} = \{A, B, C, D, E, F, G\}$$

$$\mathcal{L} = \{a, b, c, d, e, f, g\}$$

with incidence relation I as depicted.

Theorem 3.3 *Let Π be a finite projective plane. If there exists a line on Π consisting of $n+1$ points, then, the following are true:*

1. *Every line on Π consists of $n+1$ points.*
2. *Every point on Π is incident with $n+1$ lines.*
3. *Π contains n^2+n+1 points.*
4. *Π contains n^2+n+1 lines.*

Proof Let l be the line given in the assumption, and let P_1, P_2, ..., $P_{n+1} \in l$. Let H be a point not incident with l. According to Axiom 1, we can connect H with points P_i, and the resulting lines do not coincide as H is not on l. However, according to Axiom 2, all lines passing through H intersect l, but these intersection points can only be P_2, ..., P_{n+1}, thus, exactly $n+1$ lines pass through H. Duality warrants that if there exists a point S incident with $n+1$ lines, then, every line not passing through S contains $n+1$ points.

Let k be an arbitrary line different from l. Then, in line with Axiom 4, there exists at least one line passing through $k \cap l$ apart from k and l themselves. This line, according to Axiom 3, contains a point besides $k \cap l$. Let this point be denoted by T. As T is not incident with l, $n+1$ lines must pass through it. However, T is not incident with k either, so k must also contain $n+1$ points. This proves statement 1.

Let P be an arbitrary point on the finite projective plane Π. Then, upon invoking Axioms 3 and 4, one can deduce that the plane contains a line not crossing P. According to the above, there are $n+1$ points on this line, meaning $n+1$ lines cross P. This verifies statement 2.

As per Axiom 1, all points on the plane may line up if one lists all points connected by a line with an arbitrary, priorly fixed point P on the plane. We have already shown that exactly $n+1$ lines pass through P, all of which contain n points not counting in P, thereby, the total number of points on the plane is $n(n+1)+1 = n^2+n+1$.

Utilizing our immense tool, duality, we deduce that the total number of lines on the plane is also n^2+n+1, which completes the proof.

The previous theorem gives rise to the following definition:

Definition 3.4 *Let Π be a finite projective plane with each line containing $n+1$ points, or, dually, with each point crossed by $n+1$ lines. Then the integer n is called the order of the projective plane.*

We may remark that according to Axioms 3 and 4, $n+1 \geq 3$, i.e. $n \geq 2$, which, substituted into the above formula, yields that the order of a finite projective plane is at least of order $2^2 + 2 + 1 = 7$, this minimal property is attained by Fano's configuration, depicted above.

It is also worth mentioning that the problem of determining whether an integer qualifies as the order of a finite projective plane is still open. The orders of all known finite projective planes are prime powers, and conjecture is that all are. However, no proof has yet been provided apart from partial results excluding the existence of certain orders.

3.2 Analytic geometrical approach

As with the classical projective plane, one can still introduce homogenous coordinates over some finite projective planes in a similar way. In the fortunate case when Desargues's theorem, known from classical geometry, holds on the plane, we can use coordinate vectors over a finite field. Unsurprisingly, we will use vectors of length 3 which will be invariant under multiplication by a non-zero field element. The introduction of this coordinate system is illustrated below on the Fano plane over $\mathbb{F}_2$.

Let the triple $(a : b : c)$ be equivalent by definition with triple $(\lambda a : \lambda b : \lambda c)$ where a, b, c, $\lambda \in \mathbb{F}_2$, and $\lambda \neq 0$. Necessarily, as we are over $\mathbb{F}_2$, $\lambda = 1$, the coordinates are automatically normed.

a, b and c can attain 0 and 1 independently, yielding 8 possibilites, but we have to exclude the vector $(0 : 0 : 0)$ as done on the classical projective plane because the all-zero vector does not correspond to any point. This gives us 7 viable points, as depicted on the figure of the Fano plane below:

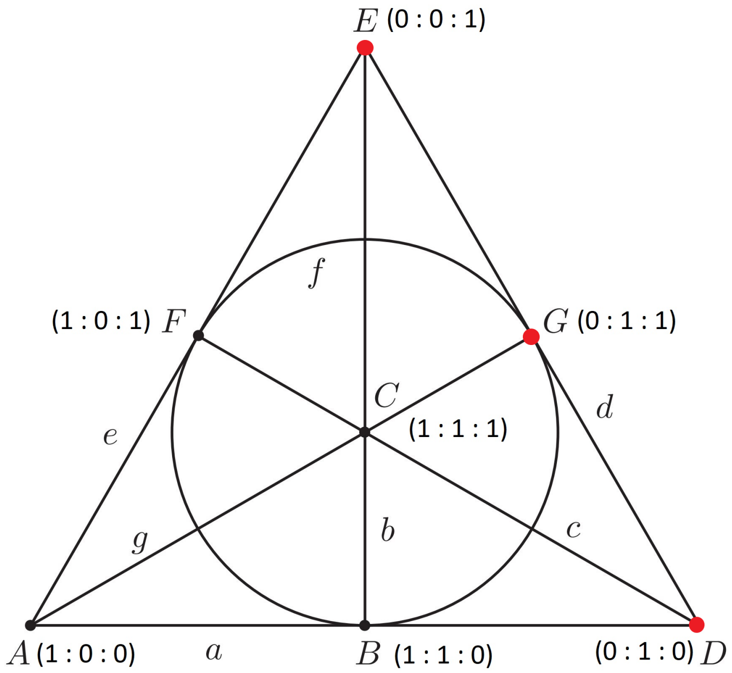

Figure 6: Coordinates over the Fano plane

A theorem we are accustomed to it on the classical projective plane, namely that three different points are collinear if and only if their coordinate vectors are linearly dependent, transitions smoothly to finite planes. Otherwise put, if two points lie on a line, so does their non-trivial linear combination.

However, non-zero coefficients over $\mathbb{F}_2$ can only be 1-s, hence, linear combination reduces to a simple sum. Thus, if we select two arbitrary points on the plane, the point represented by the sum of their corresponding vectors is automatically incident with the line determined by them and no other point is. (According to Axiom 3, such a point exists, according to the combinatorial restrictions imposed on the coordinates, no other does.) The above figure represents the formulation of coordinates on the Fano plane up to "rotation".

Analogously with the classical projective plane, points with coordinates beginning with 0 belong to the ideal line of the plane (points E, G, D, marked red on the figure). One can further establish specifically for the ideal line that the vectors of its points are closed under addition due to the first 0-s, the sum of any two produces the third.

3.3 Application of finite projective planes for secret sharing protocols

A frequently arising problem in information technology is the situation whereby we intend to limit the use of a virtual room, database, safe, document with shared access etc. used by multiple persons by setting the minimum number of members, greater than or equal to two, required to execute reading, writing or other transactions under the supervision of one another. This may be achieved utilizing finite projective planes.

In the case of two potential users, let the secret to be shared be an ideal point on a finite projective plane. The keys distributed to the users wishing to access the information will be two ordinary points on the plane such that the line crossing these two points also passes through the prefixed ideal point as shown below where the the red line denotes the ideal line of the plane, S is the secret point, and P_1 és P_2 are the keys the users have been granted.

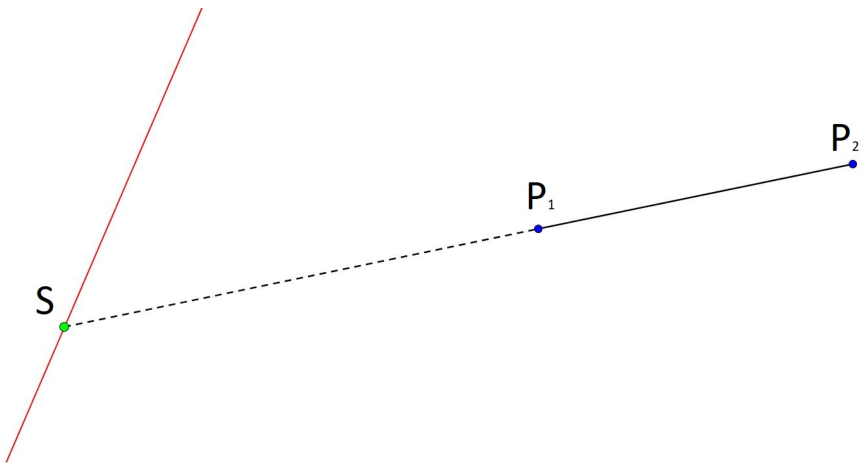

Figure 7: Secret sharing

In the case of three potential users, let us consider a 3-dimensional finite projective space and an ideal point in one of the ideal lines of this space. Let this point be the secret to be shared. Hand out the users three points of a conic section, of three points we clearly know that they are not collinear, therefore, they span a plane. The prefixed ideal line cuts out the secret point from this plane, however, out of the three, no two of them are sufficient for determining the secret, thereby offering protection.

A huge advantage of the above method is that it is a so-called perfect secret sharing scheme, namely, the distributed points uniquely determine the secret point, but missing any one of them, the unauthorized intruder, even in possession of their own key, has the same statistical chance of uncovering the secret point as when attempting to blind by trial and error due to the fact that the ideal line contains $q+1$ points - the probability of hitting the secret point is therefore $\frac{1}{q+1}$ - and the same number of lines pass through any pre-distributed point from which the intruder might try and cross the ideal line.

3.4 Finite affine planes

This section has heavily utilized source [8].

After removing the ideal line (points with coordinates starting with 0) from the finite projective plane of order q over $\mathbb{F}_q$, we obtain an affine plane with $q^2+q+1-(q+1)=q^2$ points. On this plane, all coordinates start with 1, so we can detach the first component from each, producing points of the form (a, b), where a and b are elements of $\mathbb{F}_q$, and q is a prime power.

On this affine plane, points, lines and their incidence turns out to be as follows:

- Points: (a, b) $a, b \in \mathbb{F}_q$
- Lines: $[c]$, $[m, k]$, c, m, $k \in \mathbb{F}_q$
- Incidence:
  1. $(a, b)\ I\ [c] \iff a = c$
  2. $(a, b)\ I\ [m, k] \iff b = ma + k$.

The lines of the affine plane can be divided into parallel classes: they split into lines spanned by a constant, of the form $[c]$, and lines with a different slope m, denoted by $[m, k]$, where the classes are disjoint from each other. This implies that at a fixed point in the affine plane, exactly one line passes through from a fixed parallel class.

Hence we obtained the affine plane by removing the ideal point from each line on the projective plane, each line of the affine plane contains q points. Intuitively, one would assume that every point on the affine plane is crossed by $q+1$ lines. This can be relatively easily shown, yet the proof of this statement is omitted due to the length of this work.

On this plane, similarly to the classical Euclidean or projective plane, we can analytically define conic sections, we will illustrate this on the parabola.

Definition 3.5 *Let q be an odd prime power, and consider the affine plane over $\mathbb{F}_q$. On this plane, we define the following object to be a parabola of order q:*

$$\{\ (t, t^2) \mid t \in \mathbb{F}_q\}$$

By definition, a parabola of order q admits q points since the first coordinate can attain q different values, and the second one is determined by the first, yielding exactly q pairs.

Definition 3.6 *A non-vertical line is defined to be tangent to a parabola on the affine plane if they have exactly one point in common.*

One can show, analogously to the Euclidean plane, that a parabola and a line can share at most two common points, and the lines of constant type $[c]$ (corresponding to vertical lines on the Euclidean plane) always have exactly one common point with the parabola.

When the order of the plane is an odd prime power, to all points of the parabola belongs exactly one tangent, and in all non-vertical parallel classes of lines, exactly one line is tangent to the parabola.

We can use the above in the following application:

3.5 Round-robin scheduling in a soccer tournament

When organizing a round-robin soccer tournament, the task is to achieve that each participating team play exactly one game against every other team, with games in each round taking place contemporaneously. To accomplish this, one can exploit the parabola described above to provide the scheduling.

Let our tournament consist of $q + 1$ teams, where q is an odd prime power. Consider the affine plane over $\mathbb{F}_q$, and on it, the parabola of order q. This parabola has q points. We can associate each with a team and assign a label to the remaining $(q + 1)^{th}$ team. Exclude the lines of constant type, and for all other parallel classes, associate a round to each $[c]$ slope. The line with that slope either intersects the parabola in two points, then, the teams corresponding to the intersection points play a game; or is tangent to the parabola, then pair up the one team corresponding to the point of tangency with the pre-labelled point. Parallelity renders it impossible for two such lines with different slopes to intersect, so no team plays more than one game within a round. However, all teams are involved in the construction, and games are conducted simultaneously, ensuring the preset conditions of the scheduling are all met.

We remark that for this construction to work, it is necessary for q to be an odd prime power, as in the case where $q = 2^k$, the tangents to the parabola - surprisingly - become concurrent, and the reasoning provided loses its significance.

This method is indeed utilized when organizing tournaments with one team more than an odd prime power, including the current Hungarian National League involving 12 teams ($q = 11$).

4 Pencils of Circles

We may now proceed back to the classical Euclidean plane where we initiate our investigation into pencils of circles, a special case of pencils of conics which - whilst inheriting their properties - serve as a simplified approach to understanding them, especially when introduced. This section relies on sources [3] and [5].

4.1 Power of a point with respect to a circle

To introduce the concept of the power of a point with respect to a circle, let us recall a theorem from elementary geometry concerning secants drawn to a circle.

Theorem 4.1 (Intersecting Secants Theorem) *Let us draw secants from point P on the plane to a fixed circle. Then, the product of the distances measured from P to the points of intersection are independent of the location of P.*

Proof We prove the theorem by showing that given an arbitrary pair of secants, the aforementioned products are equal.

Let us suppose that P is incident with the circle. Then, at least one of the segments is of length 0 (in fact, both are if the secant is in a marginal position i.e. it is a tangent). That implies that their product is necessarily 0, the theorem trivially holds.

Let P lie outside the circle, and let the secants intersect it at points A, B, R , and S as shown below.

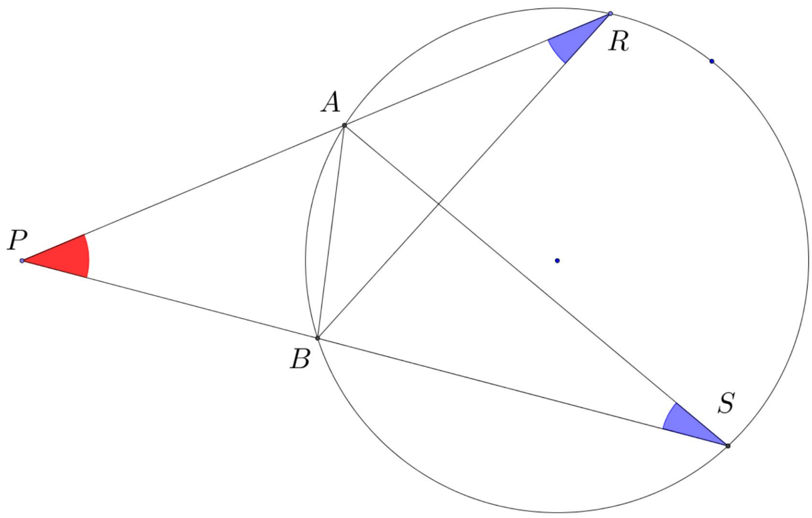

Figure 8: Intersecting secants I.

Then, $\angle ARB = \angle ASB$ as they are inscribed angles belonging to arc AB. Triangles PBR and PAS share $\angle APB$, and have another angle in common, thus, they are similar, implying

$$\frac{PA}{PS} = \frac{PB}{PR},$$

which yields

$$PA \cdot PR = PB \cdot PS.$$

Let P be located inside the circle as depicted.

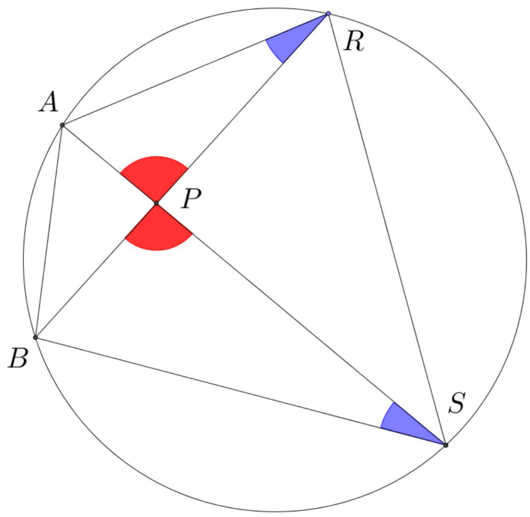

Figure 9: Intersecting secants II.

Similar reasoning leads to triangles PAR and PBS being similar, resulting in:

$$\frac{PA}{PB} = \frac{PR}{PS},$$

which turns out as

$$PA \cdot PS = PB \cdot PR.$$

Definition 4.2 *In accordance with the above, the product of segments of secants is a unique constant dependent only on the circle and the point. We hereby define this constant as the power of a point with respect to a circle.*

It is worth noting that the above theorem may also be extended to directed segments, by which the power of a point with respect to a circle is positive if the point is external to the circle, negative if internal to the circle, and 0 if incident with it.

Below, we introduce two new characterizations of the power we defined.

Theorem 4.3 *The power of point P with respect to a circle of center O and radius r is $d^2 - r^2$ where $PO = d$.*

Proof Let us consider the special secant passing through the center of the circle.

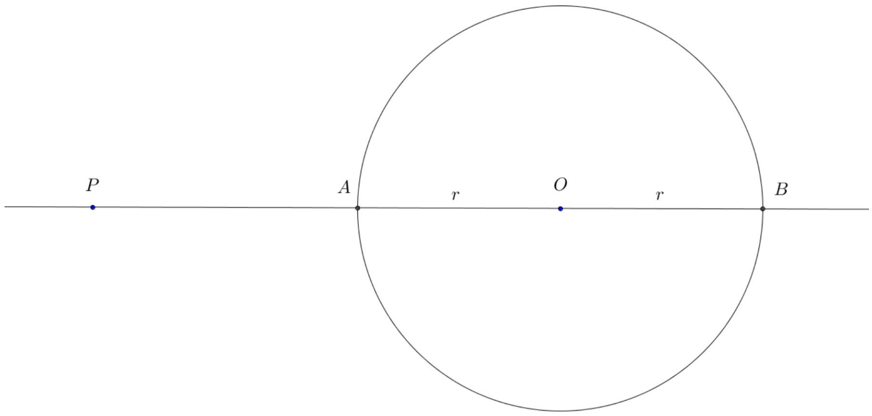

Figure 10: Power of a point I.

Taking into account the signed distances, $PO = d$, $OA = -r$, $OB = r$, that makes $PA = d - r$, $PB = d + r$, resulting in the power:

$$PA \cdot PB = (d + r)(d - r) = d^2 - r^2.$$

Theorem 4.4 *The power of an external point with respect to a circle is the square of the length of tangent drawn to it from the same point. We remark that it also proves that the power of an external point is always positive.*

Proof Let us further consider the secant passing through center O, and let the tangent drawn from P touch the circle at point T.

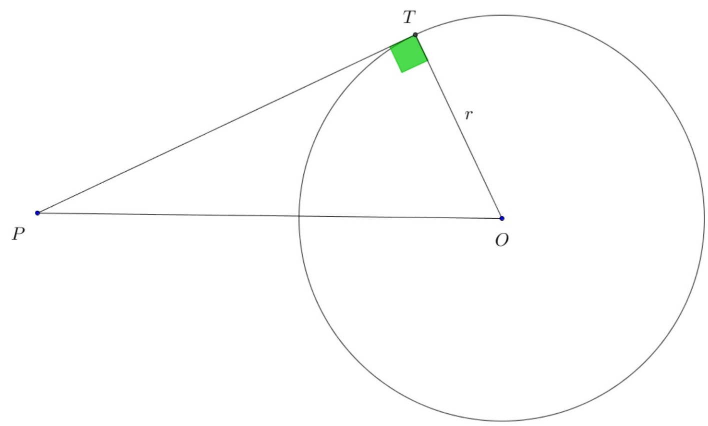

Figure 11: Power of a point II.

It is known that the radius is perpendicular to the tangent at the point of tangency, thus, triangle PTO is a right-angled triangle, invoking the Pythagorean theorem:

$$PT^2 = PO^2 - TO^2 = d^2 - r^2,$$

which is the power as desired.

Definition 4.5 *For generality, let us define the power of point P with respect to a degenerate circle C to be $CP^2 = d^2$ in accordance with the previous definition in the limiting case when the radius is 0.*

Let us transition to an analytic geometric approach for which we consider the normal equation K of a circle of radius r centered around point $O(u, v)$:

$$K:\ (x-u)^2 + (y-v)^2 - r^2 = 0.$$

Substituting the coordinates of $P(a, b)$ into the equation gives the power of P with respect to the circle:

$$K(P) = (a-u)^2 + (b-v)^2 - r^2 = PO^2 - r^2 = d^2 - r^2.$$

4.2 Power line

Definition 4.6 *The locus of points on the plane the power of which with respect to two fixed circles coincides, is called the power line or radical axis of these circles. We will soon see that this is indeed a line.*

In the case of two non-identical concentric circles, this set is empty, as the powers of a point with respect to two concentric circles of different radius are never equal: $d^2 - r_1^2 = d^2 - r_2^2$.

We remark that if the circles were identical, the power line would obviously cover the whole plane - this case is disregarded.

Theorem 4.7 *The power line of two non-concentric circles is indeed a line.*

Proof In line with our analytic approach, for the points lying on the power line defined by the normal equations of $K_1 = 0$ and $K_2 = 0$ of two circles,

$$K_1(P) = K_2(P),$$

holds, therefore they satisfy

$$(K_1 - K_2)(P) = 0.$$

The previous expression is consequently the equation of the power line. $K1$ and $K2$ being circles, the coefficients of the quadratic terms in both are equal to 1, and thus cancel out in the difference, making it either a linear or a constant expression. If it were a constant, then the equations of the circles would appear as K_1, $K_1 + c$, making them concentric, a case excluded on approach. From this we may deduce that the equation represents a line.

Theorem 4.8 *If two circles have a common point, it must lie on their power line.*

Proof Our proof may be prompt: the common point is incident with both circles, consequently, its power with respect to both circles is 0, therefore equal.

A straightforward corollary of the previous theorem is the fact that the power line of two intersecting circles the line joining the intersection points. This may be generalized by the following theorem.

Theorem 4.9 *The power line of two circles is perpendicular to their central (the line joining their centers) as shown below.*

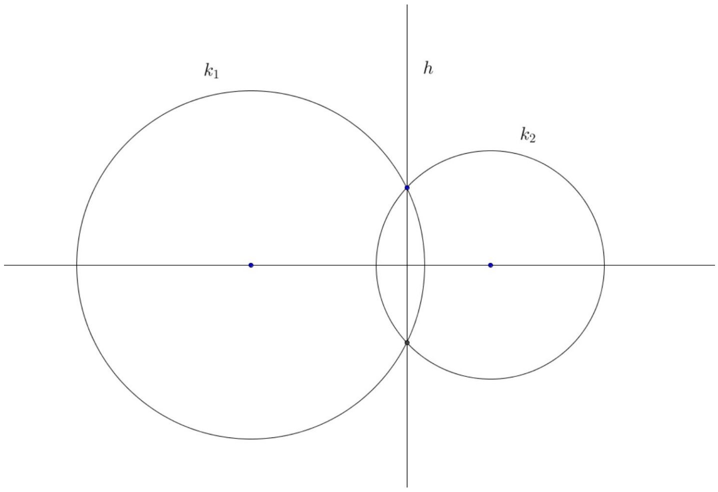

Figure 12: Power line of intersecting circles

Proof Reflecting a point to the central of the circles results in a point the power of which with respect to the circles is unchanged as the axis of reflection coincides with the central, and the distances from the centers are left fixed.

This means that an arbitrary point on the power line reflected to the central also lies on the power line, i.e. the power line is symmetric to the central. That restricts us to the fixed points of the reflection to the central, namely: lines perpendicular to the central and the axis itself.

It suffices to prove that the central cannot form a power line, for which it is enough to prove that the central contains a point not incident with the power line. Such a point can easily be found: there must be a point lying on the inside of one circle and the outside of the other, whose powers are clearly of different sign. This implies that the power line is orthogonal to the central.

Combining the previous theorems yields that the power line of tangent circles is their tangent line itself, and, specially, when one of the circles shrinks to a point on the other, the power line is the tangent to the other at this distinct point.

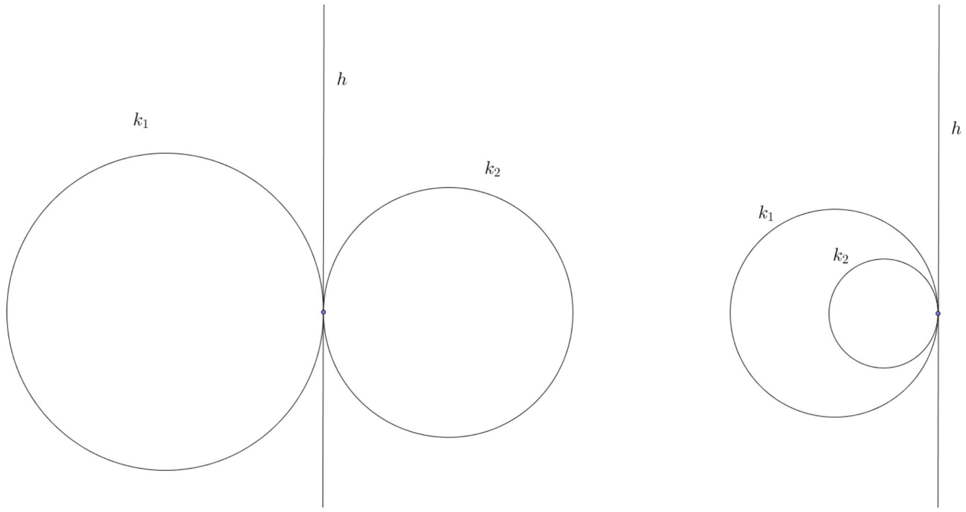

Figure 13: Power line of tangent circles

We may now proceed to the case with three circles involved.

Definition 4.10 *A set of lines on the plane is called a pencil of rays if all its elements pass through a single point. When this is a regular point, the pencil is named concurrent, and when this is an ideal point, the pencil is defined to be parallel.*

Theorem 4.11 *If three circles do not contain a pair of concentric ones, then their power lines belong to a pencil of rays.*

Proof If there are no two concentric ones amongst them, three circles determine $\binom{3}{2} = 3$ power lines. Let their normal equations be $K_1 = 0$, $K_2 = 0$ and $K_3 = 0$. Their equations are then the following:

$$K_1 - K_2 = 0,\ K_2 - K_3 = 0,\ K_3 - K_1 = 0.$$

As the left hand sides sum up to 0, they span a set of lines concurrent at a single point or parallel to each other, forming a pencil of rays.

Theorem 4.12 *If the centers of the three circles are not collinear, then there exist a point the power of which with respect to all three circles is equal, this point is defined to be the power point of the circles.*

Proof As the three centers are not collinear, they cannot contain two corresponding ones, meaning no two of the circles are concentric. Invoking the previous theorem, the power lines belong to a pencil of rays, i.e. they pass through a single point or are parallel.

We are aware that the power lines are perpendicular to the centrals, but the centrals are extensions of the sides of a non-degenerate triangle, therefore, they cannot be parallel, that leaves them to be concurrent.

We remark that when the centers are collinear, then the power lines are parallel lines orthogonal to the common central, in which case there is no power point (albeit one can define one, the ideal point on the projective plane).

4.3 Pencils of circles

Definition 4.13 *Let $K_1 = 0$ and $K_2 = 0$ be the equations of two circles or a line and the circle, but not two lines. Let us consider the figures satisfying*

$$\lambda_1 K_1 + \lambda_2 K_2 = 0$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. The set of these figures is defined to be the pencil of circles and the figures determined by (λ_1, λ_2) are elements of the pencil.

It is obvious that the determining objects themselves are elements of the pencil. The left side of the above equation is a quadratic expression in which the coefficients of x^2 and y^2 are equal, perhaps 0, therefore, every element of the pencil is either a circle or a line. We remark that, in a broad sense, lines may be regarded as circles with centers at infinity, thus, for the sake of simplicity, we sometimes refer to lines as bases of a pencil of circles.

Theorem 4.14 *Every pencil of circles contains at most one line.*

Proof Indirectly, let us suppose that the pencil contains two different lines, the equations of which are $L_1 = 0$ and $L_2 = 0$. Then, for some α_i, β_j coefficients, one obtains:

$$\begin{cases} L_1 = \alpha_1 K_1 + \beta_1 K_2 \\ L_2 = \alpha_2 K_1 + \beta_2 K_2 \end{cases}$$

As the two lines are different, $\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \neq 0$. But then, for some γ_i, δ_j coefficients, one gets:

$$\begin{cases} K_1 = \gamma_1 L_1 + \beta_1 L_2 \\ K_2 = \gamma_2 L_1 + \beta_2 L_2 \end{cases},$$

However, figures K_1 and K_2 are linear combinations of lines, therefore lines themselves, contradicting our proposition that one of the equations represents a circle.

Theorem 4.15 *A pencil of circles may be determined by any two of its elements.*

Proof We only provide a proof sketch as the reasoning greatly resembles the previous ones. Any two elements of the pencil are a linear combination of the equations of the base circles. Nevertheless, as the base figures are non-degenerate, invertibility of the matrix of the system of equations describing the pencil is implied. This means that the two base figures may also be written as the linear combination of any two arbitrary elements, asserting that picking two arbitrary elements produce the same pencil.

As a consequence, two pencils of circles must have at most one element in common, otherwise they would be identical.

Theorem 4.16 *If two different elements of a pencil of circles have a point of intersection, then this intersection point is contained by all other elements of the pencil.*

Proof Let base circles K_1 and K_2 intersect at P. These two circles, as seen, determine the pencil, thus, every element admits an equation of the form

$$\alpha K_1 + \beta K_2 = 0.$$

However, as P is incident with K_1 and K_2,

$$K_1(P) = K_2(P) = 0$$

holds, but also

$$\alpha K_1(P) + \beta K_2(P) = 0,$$

meaning that P is contained in every element in the pencil.

Definition 4.17 *A point which all elements of a pencil of circles passes through is called a base point of the pencil.*

Theorem 4.18 *A pencil of circles contains at most two base points.*

Proof Three points uniquely determine a line or a circle, thus, if a pencil of circles contained three base points, as all elements pass through them, it would only consist of this line or circle.

The following propositions connect pencils of circles with power lines:

Theorem 4.19 *If the power line of any two elements of a pencil of circles exists, then it is contained in the pencil.*

Proof Let $K_1 = 0$ and $K_2 = 0$ be the normal equations of the two circles. Then, for $\lambda_1 = 1$, $\lambda_2 = -1$, figure $K_1 - K_2$ is also contained in the pencil, but that is the equation of the power line.

Theorem 4.20 *The power lines of any two circles in a pencil - if they exist - correspond.*

Proof If a pencil contains two concentric circles, then all its elements are concentric, not admitting a power line.

In light of that, a pencil of circles is either concentric, or an arbitrary pair of circles has a power line. We have shown that this line is an element of the pencil, but also that the pencil is allowed to contain at most one line, therefore, this line is necessarily the power line of any two arbitrarily selected circles in the pencil.

We will next be classifying pencils of circles according to the number of base points they have, and whether they contain a line or not.

Let us remind ourselves that a pencil of circles can contain at most two base points and one line.

Definition 4.21 *If the centers of all circles in a pencil coincide, we call the pencil concentric.*

Obviously, such a pencil does not contain a base point as concentric circles lack common points, neither a line, otherwise it would be the power line of concentric circles which is non-existent. It is thereby clear that we only have to classify pencils containing a line with respect to the number of base points.

Definition 4.22 *If a pencil of circles containing a line admits no base points, it is named elliptic, if one, parabolic, if two, hyperbolic.*

The following figure illustrates the above classification:

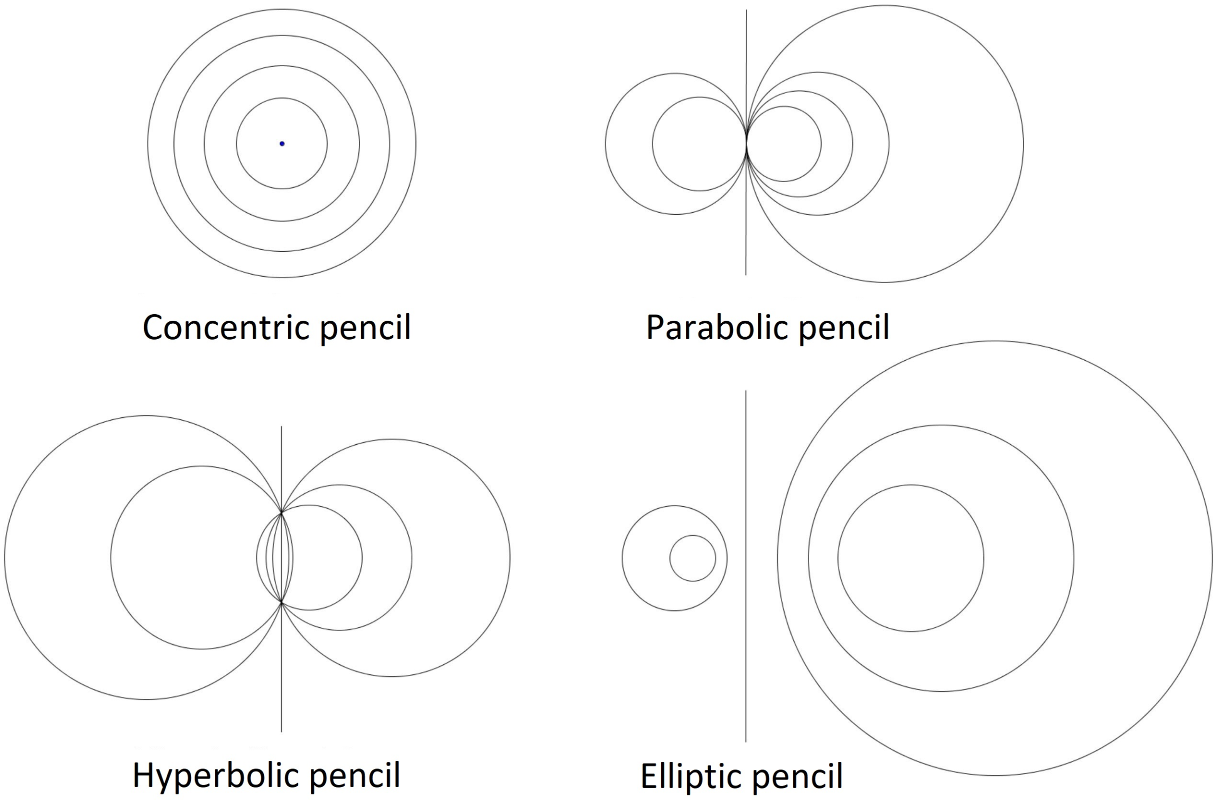

Figure 14: Types of pencils of circles

We now proceed to investigate orthogonality of pencils of circles.

Definition 4.23 *We call two intersecting circles orthogonal if the tangents drawn to them at the intersection points are orthogonal.*

We remark that this definition easily extends to the notion of the angle of two intersecting smooth curves, their angle is the angle of their tangents at the intersection point.

It is a widely known fact that this holds if and only if radii drawn at the intersection points are perpendicular. We now introduce another, interesting characterization equivalent with this one.

Theorem 4.24 *Two circles are orthogonal if and only if the power of the center of one circle with respect to the other equals the square of the radius of this circle.*

Proof If two intersecting circles are orthogonal, as illustrated in the following figure, the orthogonality of the radii allows us to apply the Pythagorean theorem:

$$r_1^2 = O_1O_2^2 - r_2^2,$$

meaning that the power of the center of one circle with respect to the other circle is r_1^2, and backwards, if the power is r_1^2, then rearranging the terms yields:

$$r_1^2 + r_2^2 = O_1O_2^2,$$

which, by the converse of the Pythagorean theorem, implies orthogonality of the radii.

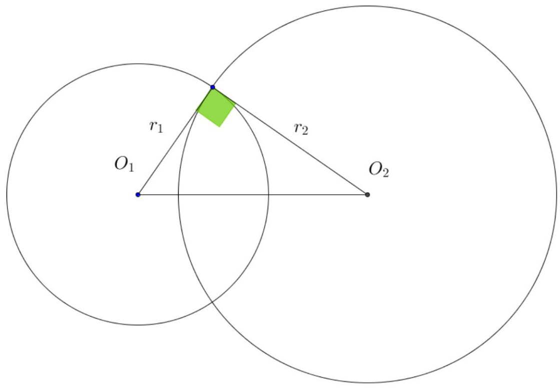

Figure 15: Orthogonal circles

Theorem 4.25 *If a circle k is orthogonal to two elements of a pencil of circles, then it is orthogonal to all of its elements.*

Proof Let us suppose that the two intersected elements are two circles with normal equations $K_1 = 1$ and $K_2 = 0$. Let the center of k be denoted by C, and radius by r. As orthogonality holds, $K_1(C) = K_2(C) = r^2$, but this also means that C lies on the power line of the two circles. Combining this result with the previous, one obtains that the power line is orthogonal to k. Nevertheless, this is the common power line of all circles in the pencil, to which k is orthogonal, implying that k is orthogonal to all other elements of the pencil.

Suppose the two intersected elements are a line and a circle (it cannot be two lines). Then, this line is the common power line of the pencil. Orthogonality implies that C lies on the power line, therefore, its power with respect to all circles is equal. Yet, as orthogonality holds, we are aware that this power is r^2, meaning k is orthogonal to all other elements of the pencil.

A similar theorem holds for lines:

Theorem 4.26 *If a line is orthogonal to two elements of a pencil of circles, then, it is orthogonal to all elements of the pencil.*

Proof A line is orthogonal to a circle if and only if it passes through its center. Therefore, if the two intersected elements are circles, then the intersecting line coincides with the central of the circles, thereby orthogonally intersecting all other circles in the pencil, and even the power line due to a previously proven statement.

If one of the intersected elements is the power line itself, the other is a circle, then the line passes through the center of the circle and is orthogonal to the power line of the pencil, meaning it passes through all other centers, implying orthogonality to all of them.

Definition 4.27 *Two pencils of circles are called conjugate if all elements of one orthogonally intersect all elements of the other.*

The state of two pencils of circles being conjugate, as expected, admits symmetry with respect to the pencils due to the symmetry of orthogonality.

Based on the theorems established above, for the orthogonality of two pencils of circles to hold, it suffices to show that two arbitrarily selected elements from each pencil are pairwise orthogonal.

In terms of conjugation, the following interesting statements are true:

Theorem 4.28 *The conjugate of a parabolic pencil of circles is also a parabolic pencil with the same base point, and its power line is the line spanned by the centers of the original pencil.*

Theorem 4.29 *The conjugate of a hyperbolic pencil of circles is an elliptic pencil whose two point circles are the base points of the original pencil, and whose power line is the line spanned by the centers of the original pencil.*

Theorem 4.30 *Owing to the symmetry of conjugation, the conjugate of an elliptic pencil of circles is a hyperbolic pencil such that its base points are the two point circles of the original pencil, and its power line is the line connecting them.*

We may complement the above with a beautiful characterization of pencils of circles by stereographic projection, a transformation widely used by geographers due to being an invertible, circle-preserving, conformal mapping of the sphere onto the plane.

Theorem 4.31 *Consider the image of pencils of circles containing lines with respect to inverse stereographic projection. This yields a bundle of spheric circles. The planes that cut out these circles from the sphere intersect in a line in space, furthermore,* *this line cuts through the sphere if the pencil is hyperbolic, is tangent to the sphere if the pencil is parabolic, and fully evades the sphere if the pencil is elliptic.*

If the pencil is concentric, the pre-image is a set of spheric circles generated by a bundle of parallel dissecting planes which intersect in an ideal line of the projective space.

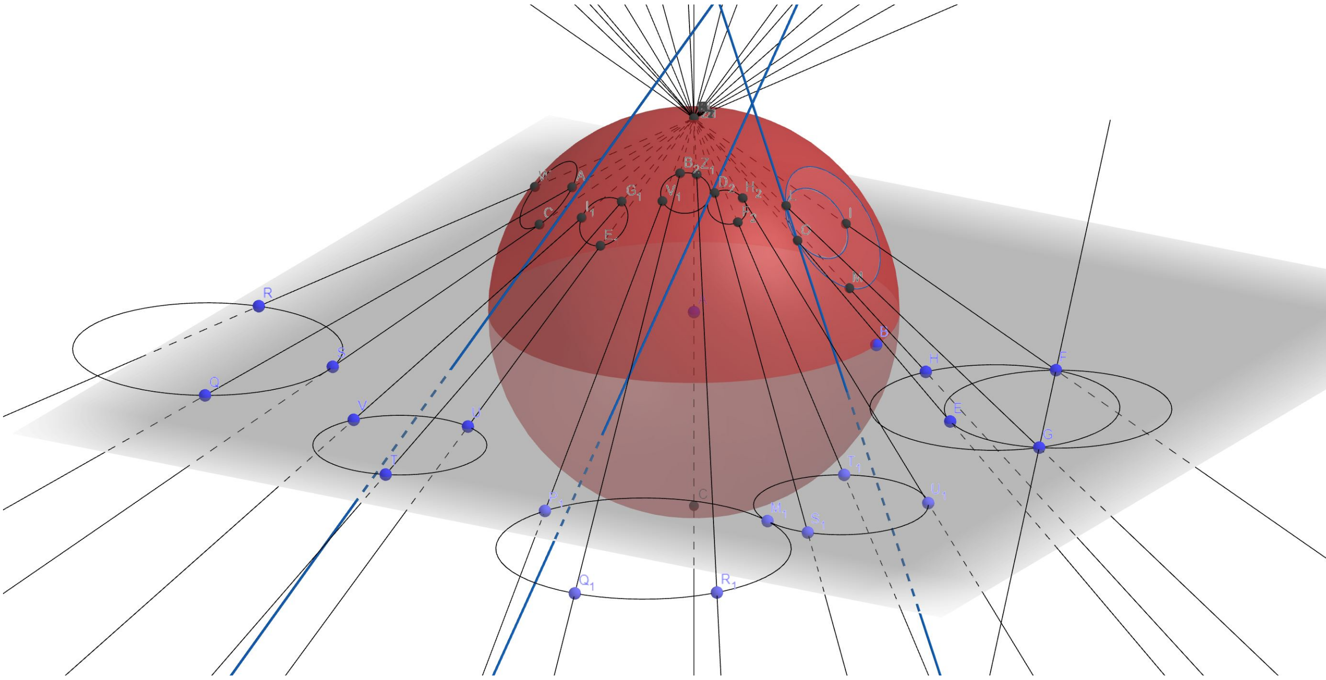

Figure 16: Stereographic projection of pencils

5 Frégier's Theorem

We hereby grab the opportunity to show off at the end of this chapter a less known yet intriguing theorem regarding conic sections, Frégier's theorem. To achieve that, we need to establish basis for a powerful tool at our disposal many proofs in geometry rely upon: polar correspondence. We hereby acknowledge that this chapter has heavily exploited source [1].

5.1 Inversion

In this section, we introduce inversion with respect to a circle, a simple but powerful tool extensively used in geometry and complex analysis.

Definition 5.1 *We define the inversive plane (or Möbius plane) to be the Euclidean plane supplemented by a single point at infinity [10].*

Definition 5.2 *Given a circle on the inversive plane with center O and radius r, the inverse of a point A with respect to this circle is A' such that A' lies on the ray OA and $OA \cdot OA' = r^2$. Clearly, this can only hold when $A \neq O$. For the case when A and O coincide, the image of the center, also called the pole, is defined to be the point at infinity.*

We remark that the inversive plane is closed under inversion with respect to a generalized circle (regular circles and circles of infinite radius that pass through the point at infinity), hence its name.

Obviously, points on the circle are fixed points of inversion. Furthermore, lines passing through the center of the circle, viewed as sets, are left fixed too. We now take a look at what happens to general circles under inversion.

Theorem 5.3 *Inversion maps circles not passing through its center into circles, and circles passing through its center into lines.*

This claim is provided without proof as it is part of the general university geometry curriculum, we hereby cite [3] as its source.

Roughly said, as a result of this theorem, inversion preserves circles. We remark that it is also a conformal mapping, i.e. it preserves angles spanned by tangents of smooth curves.

5.2 Polar Correspondence with Respect to a Circle

Definition 5.4 *Let k be a circle with center O and radius r. Let A be a point on the plane such that $A \neq O$. The line a is called the polar of A with respect to k if it is perpendicular to ray OA and it cuts OA in a point A' such that A' is the inverse of A with respect to the circle, i.e. $OA \cdot OA' = r^2$. Conversely, A is called the pole of a with respect to k. Furthermore, if $O = A$, its polar defined to be the line at infinity, and the polar of a point at infinity is the extended diameter of the circle that is perpendicular to the parallel lines intersecting at this point.*

We now make an exhaustive list of cases of how to construct poles of lines and polars of points according to their position in reference to the circle.

We describe poles of secants, tangents and lines external to the circle, polars of internal and external points, and last but not least, points lying on the circle itself.

Theorem 5.5 *The construction process differentiating all cases goes as follows:*

(a) If a point lies outside the circle, then its polar is the line joining the points of tangency generated by tangents drawn from this point to the circle.

(b) If a point lies on the circle, its polar is the tangent drawn to the circle at this point.

(c) If a point A lies inside the circle, erect a perpendicular through it to ray OA. Let this line intersect the circle in T. (There are two such T-s but due to symmetry, we can use the one of our choosing.) Let ray OA intersect the tangent drawn to the circle in T be called A'. Then the perpendicular to ray OA erected through A' produces the polar of A.

(d) If a line intersects the circle in two points, its pole is the intersection of tangents drawn to the circle at the original intersection points.

(e) If a line is tangent to the circle, its pole is the tangency point itself.

(f) If a line a evades the circle, draw a line perpendicular to a through the center O. Let the foot of this perpendicular be called A'. Draw tangents to the circle from A'. Let the line joining the tangency points intersects ray OA' at A. Then A is the pole of a.

Proof By definition, polar correspondence is involutive, i.e. the pole of the polar of a point is the point itself and vice versa. Hence, it suffices to prove the claims for points and their polars only, the rest is straightforward after reversing the roles of pole and polar.

(a) Let A denote the external point (See figure below). Draw tangents to the circle touching it in points P and Q. Let ray OA intersect secant PQ at A'. Clearly, $OA \perp PQ$ and $OP \perp PA$ because of the tangency property. Triangles PAO and $PA'O$ share $\angle POA$, and $\angle OPA = \angle OA'P = \frac{\pi}{2}$, therefore, they are similar. Consequently,

$$\frac{OA'}{r} = \frac{r}{OA}.$$

Rearranging the terms yields

$$OA \cdot OA' = r^2$$

as desired. Points P, A' and Q are collinear which implies that line PQ is the polar.

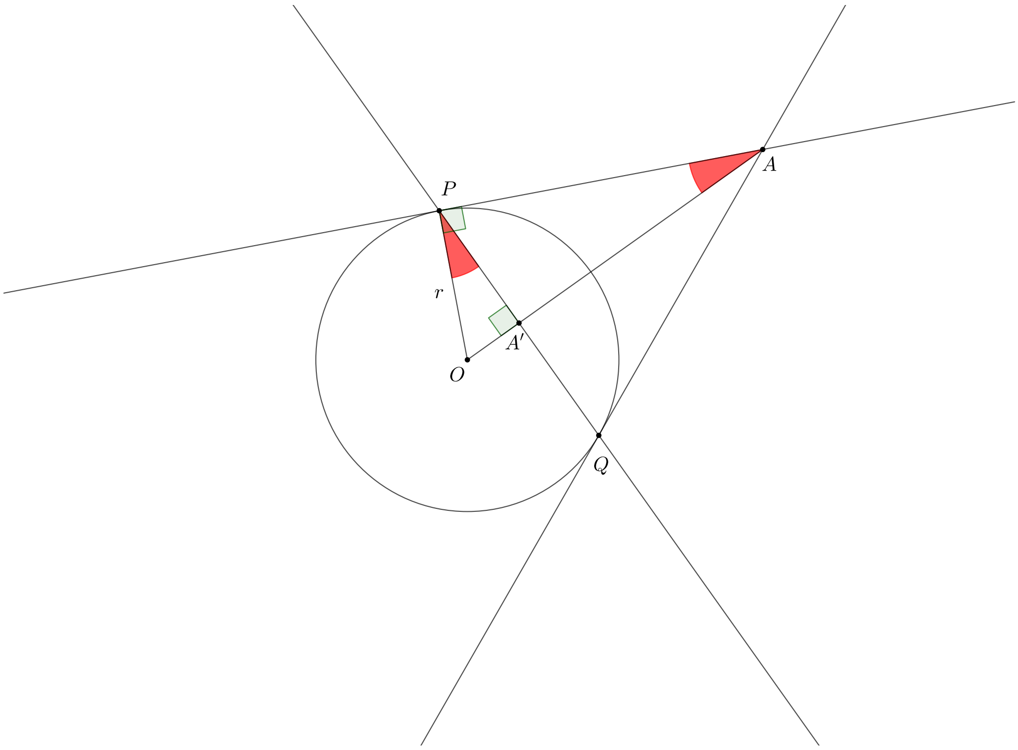

Figure 17: Polar of an external point

(b) It is easy to see that the tangent satisfies the requirements for it to be the polar: it is perpendicular to the ray drawn at the point of tangency, and as points on the circle are fixed under inversion, the tangency point corresponds with its own inverse.

(c) The configuration is depicted in the following figure. Because of tangency and the construction of AT, triangle OTA' is a right triangle, and AT is the altitude belonging to the hypotenuse. Invoking the leg theorem for right triangles yields:

$$OA \cdot OA' = r^2,$$

which means that A' is the inverse of A, meaning the perpendicular passing through it is necessarily the polar.

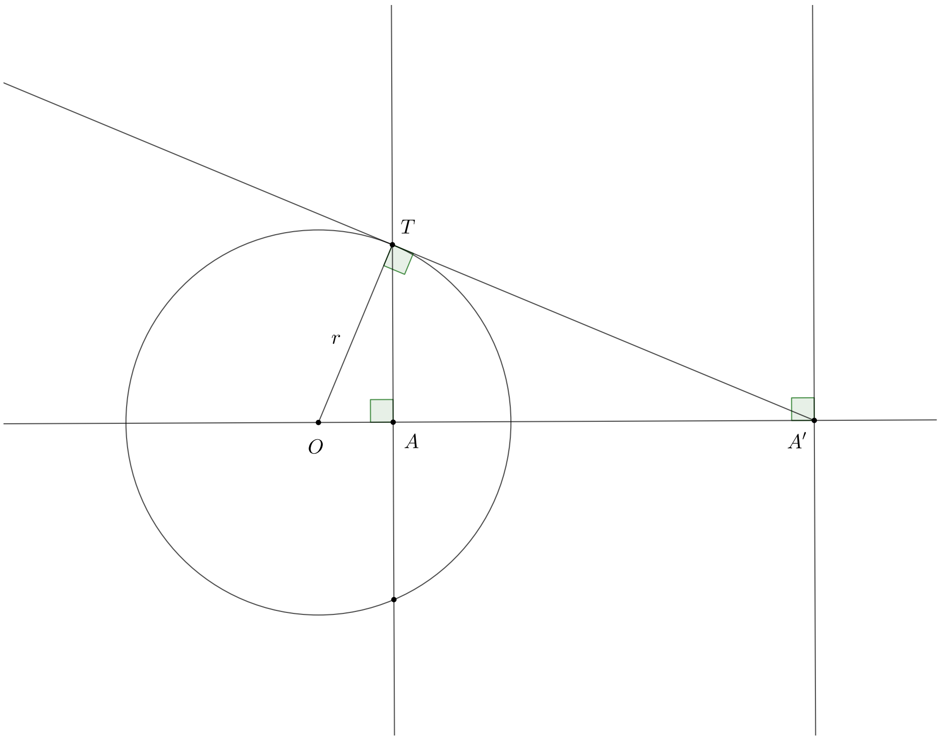

Figure 18: Polar of an internal point

Theorem 5.6 *Let a be the polar of point A. Then, for every point $B \in a$, its polar b passes through A.*

Proof Let A' and B' be the inverses of A and B respectively, as seen on the next figure. By definition, this means that that

$$r^2 = OA \cdot OA' = OB \cdot OB'.$$

Rearranging the terms yields:

$$\frac{OA'}{OB} = \frac{OB'}{OA},$$

implying that triangles OBA' and OAB' are similar. As B lies on the polar of A, $BA' \perp OA'$. Consequently, triangle OBA' is a right triangle, and so is OAB', meaning $OB' \perp B'A$. Therefore, $B'A$ is necessarily the polar of B and passes through A as proposed.

We remark that dually, if point A is the pole of line a, then for every line b passing through A, its pole B lies on a.

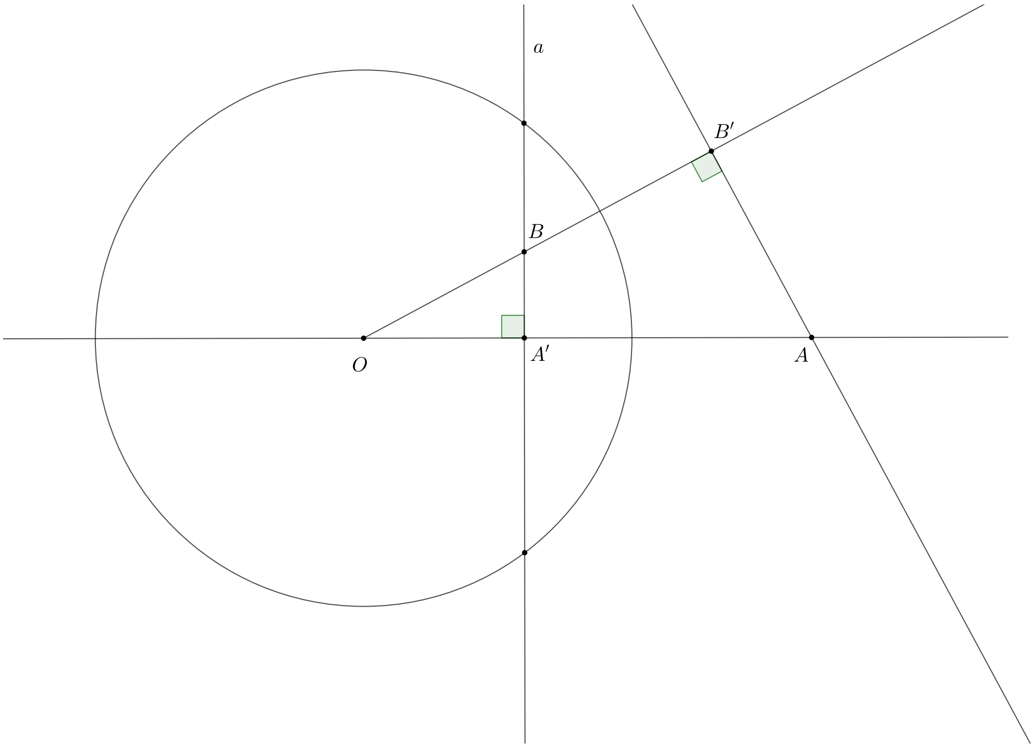

Figure 19: Concurrency of polars

The following useful lemma that helps us construct poles and polars is a direct corollary of the previous theorem.

Lemma 5.7 *Let a be a line, and X and Y points lying on it. Let the polars of X and Y intersect at A. Then A is the pole of a; and conversely, given two lines, x and y, intersecting at point A, the line joining their poles, X and Y is the polar of A.*

Proof According to Theorem 5.6, the polars of X and Y pass through the pole of a, thus, their intersection automatically determines the pole. The converse follows from interchanging the roles of pole and polar.

5.3 Conics and Projectivity

In investigating conics inverted with respect to a circle, one comes across a major technical drawback: it maps circles into circles but not conics into conics. It produces much less known and less easily treatable curves such as lemniscates and cardioids. To overcome this issue, we will introduce the so called polar transform that does bear this property, it preserves conics.

To lay the foundation of polar correspondence with respect to a conic, first we have to define projectivity [9] (also widely known as projective transformation, projective collineation or homography), a key concept in projective geometry.

Definition 5.8 *A projectivity is a bijective mapping between two projective planes that preserves lines (hence the name collineation).*

This notion helps us through the drawback just mentioned in that it preserves conics.

Theorem 5.9 *The projective image of a conic is also a conic.*

Proof of this fundamental claim is omitted due to shortage of space yet a heuristic analogy is provided to outline the intuition behind the construction.

This notion is merely incentivized by considering the intersection of a cone and a plane in space centrally projected from the apex of the cone. The projections are determined by the tilt of the dissecting plane. Basically, this is how a conic is transformed when seen in perspective from different angles by the human eye.

Varying the tilt of the plane yields different conic sections but one can be transformed into another by rotating the plane. The existence of such a projection transforming conics into each other is of prime importance and is exploited in defining polar correspondence with respect to a conic. The realization of this projection is outlined in the following figure. ω_2, ω_3 and ω_4 are projected images of a circle, denoted by ω_1, from point P, the vertex of the cone. They are an ellipse, a parabola and a hyperbola respectively, all indifferentiable from a projective point of view. We remark that if dissecting a two-sided cone, the plane intersects only one side of it except when it is parallel to exactly two generating lines of the cone (a line joining any point on the surface of the cone with its apex), in which case the intersection is a hyperbola whose second branch manifests itself on the other side.

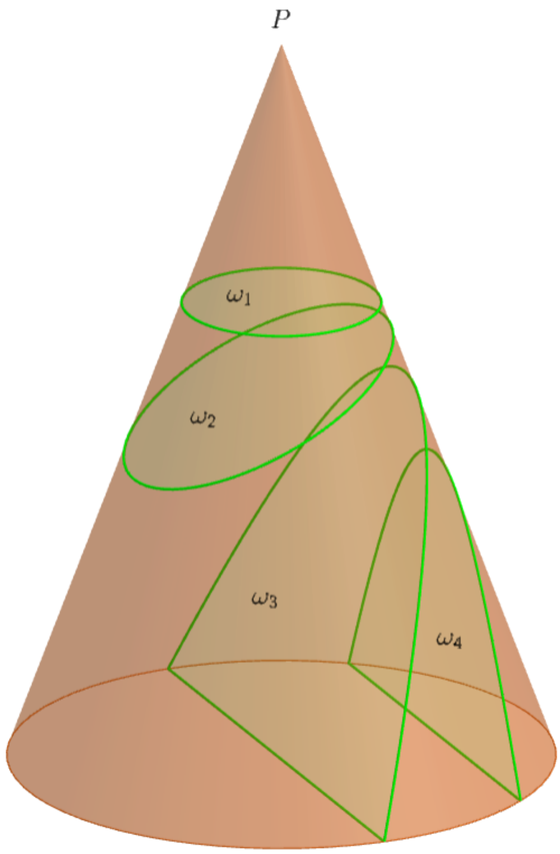

Figure 20: Projection of conics

5.4 Polar Correspondence with Respect to an Arbitrary Conic

Let α be a conic and A a point. Let Φ be a projective transformation mapping α into a circle (such a projectivity exists). Define A' as the image of A under this transformation, i.e. $A' = \Phi(A)$ and a' as the polar of A' with respect to this circle. Let a be the image of a' under the inverse of this transformation, i.e. $a = \Phi^{-1}(a')$.

Definition 5.10 *We hereby define a to be the polar of A with respect to the conic.*

Obviously, as Φ^{-1} is also a projectivity and a' is a line, $a = \Phi^{-1}(a')$ must also be a line itself. As a' is the polar of A' with respect to the circle, it is the line connecting the tangency points of tangents drawn from A' to the circle. Obviously, projective transformations preserve incidence (i.e. intersections and tangencies of lines and conics in this case), thus, $a = \Phi^{-1}(a')$ is the polar of A with respect to the conic, as illustrated in the following figure.

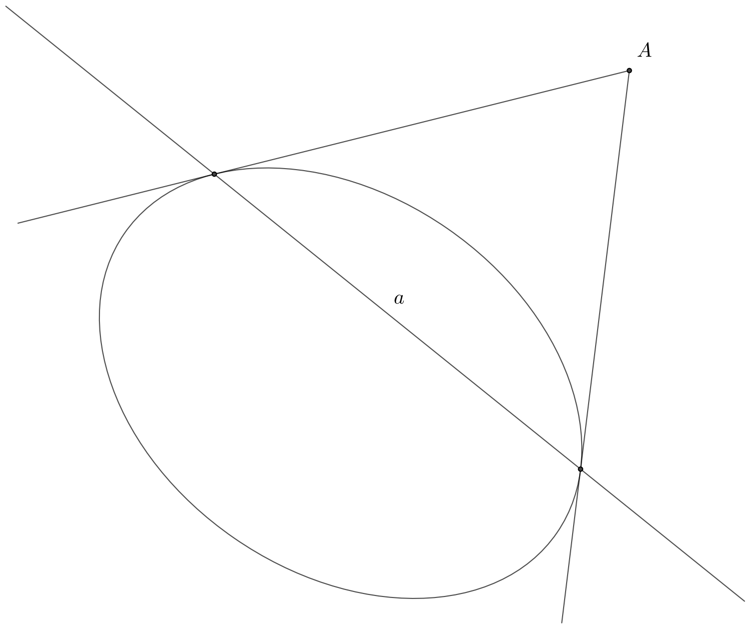

Figure 21: Polar with respect to a conic

It is easy to see that the definition of polar correspondence is independent of the circle we choose to map the conic into, therefore, the pole and polar are well defined. One can also effortlessly establish that polar correspondence with respect to a conic inherits all properties we proved regarding polar correspondence with respect to a circle.

We now proceed to define the so called polar curve of a general planar curve with respect to a conic with the aid of polar correspondence defined above.

Definition 5.11 *Let γ be a smooth planar curve. Let Γ be the set of polars associated with each point of γ with respect to a fixed conic. The polar curve of γ is defined to be the curve $R(\gamma)$ enveloped by Γ, i.e. each element of Γ is tangent to $R(\gamma)$.*

Existence and uniqueness of such a curve is guaranteed, but proof of this would reach far beyond the extent of this thesis.

The polar transform of a curve with respect to a conic also bears the involutory property as stipulated in the following theorem.

Theorem 5.12 *Let γ be a curve and $R(\gamma)$ its polar curve. Then, $R(R(\gamma)) = \gamma$.*

Upon shortage of space, proof of this claim is omitted, but without indulging in technicalities, this may easily be proven using the involutory property of the foundation of this transform, polarity with respect to a circle.

Polar correspondence has an important property we would like to include, albeit without proof: it maps conics into conics in the following way.

Theorem 5.13 *The polar curve of a conic with respect to another conic is also a conic.*

It is a widely known fact that for any five points on the plane in general position (i.e. all are distinct and no three of them lie on a line), there exists exactly one conic passing through them. Dually, for any five lines in general position (all distinct and non-concurrent), there exists a unique conic tangent to them. Therefore, it is sufficient to map five points of a conic into their polars, it will determine the image.

5.5 Frégier's Theorem

We hereby finally unveil the jewel in the crown, Frégier's theorem, a simple, concise, yet impressive result in the geometry of conics that is not widely known.

Theorem 5.14 *Let P be a point on a conic denoted by ω. Then the chords seen from P at a right angle are concurrent (as shown in the following figure).*

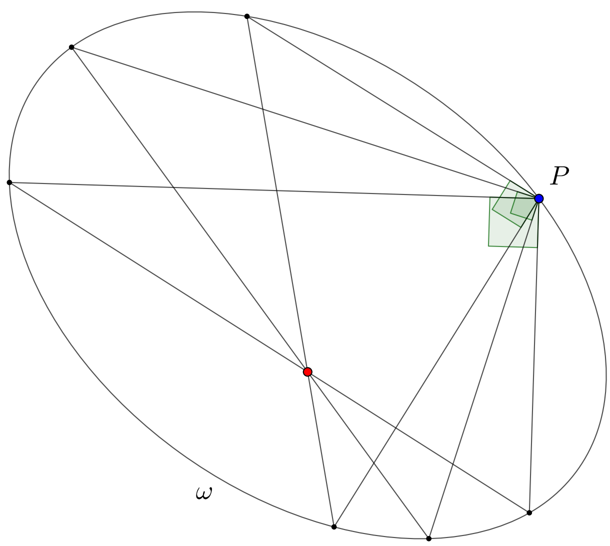

Figure 22: Frégier's Theorem

We remark that if the conic is a circle, Frégier's theorem qualifies as the generalization of the converse of Thales' theorem: the chords seen at a right angle from a point on a circle pass through its center.

Proof Let k be a circle centered at P as displayed in the following configuration. Consider the polar transform of ω, denoted by $R(\omega)$, with respect to k. According to Theorem 5.13, $R(\omega)$ is also a conic. As ω passes through the center of k, the polar of P with respect to k, a tangent to $R(\omega)$, is the ideal line by definition. A conic tangent to it, as is this polar curve, must therefore have a unique common point with the ideal line of the plane. This implies that $R(\omega)$ contains exactly one ideal point. As a consequence, $R(\omega)$ is a parabola.

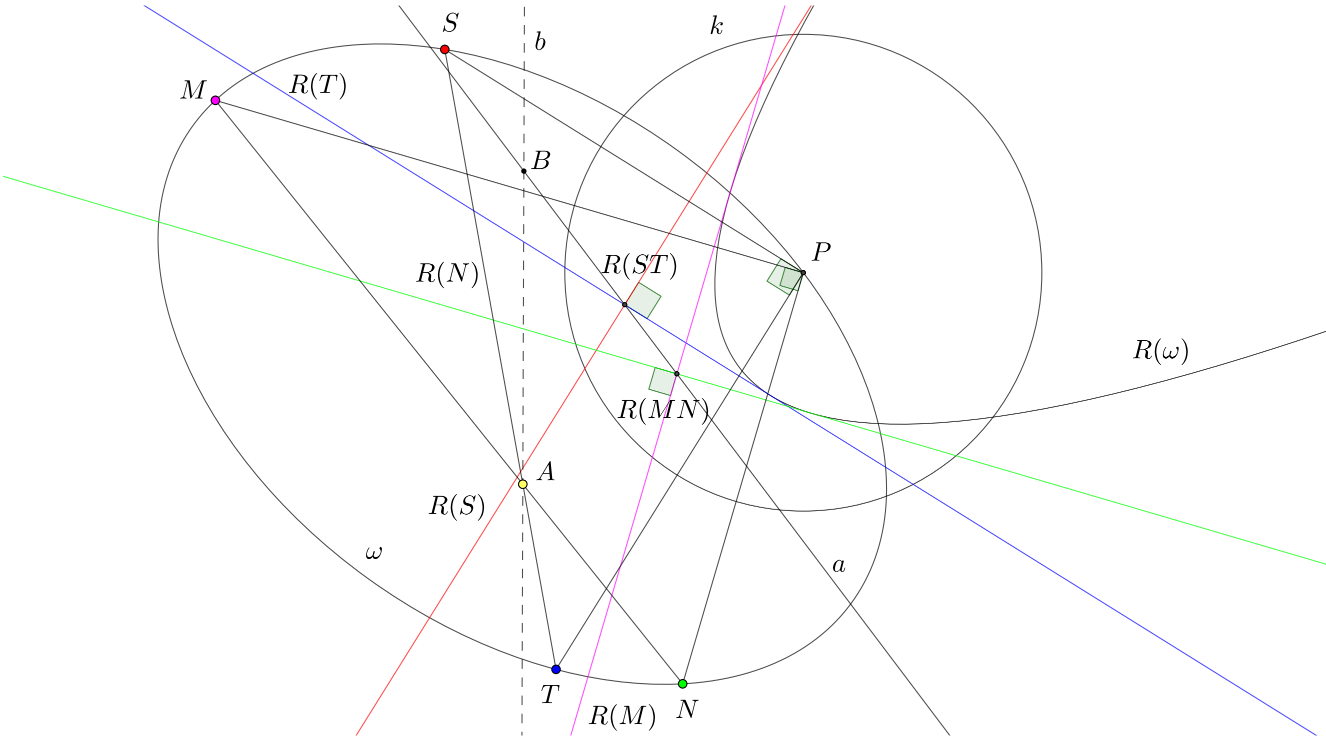

Figure 23: Proof of Frégier's Theorem

Consider a chord ST seen from P at a right angle. Obviously, as SP and TP pass through P, the center of k, and $SP \perp TP$, the pole of any one of the two lines will be the ideal point of the other.

Points S and T lie on the conic, consequently, their polars, denoted by $R(S)$ and $R(T)$ are tangents to the parabola. As S and T are endpoints of connecting perpendicular chords, their polars must also be perpendicular. Thus, $R(S)$ and $R(T)$ are perpendicular tangents to the parabola.

On the side, we present a method to construct this parabola from its tangents remarking that this construction does not contribute to the proof in any way but played a role in accurately displaying the parabola. The construction is illustrated in the following figure.

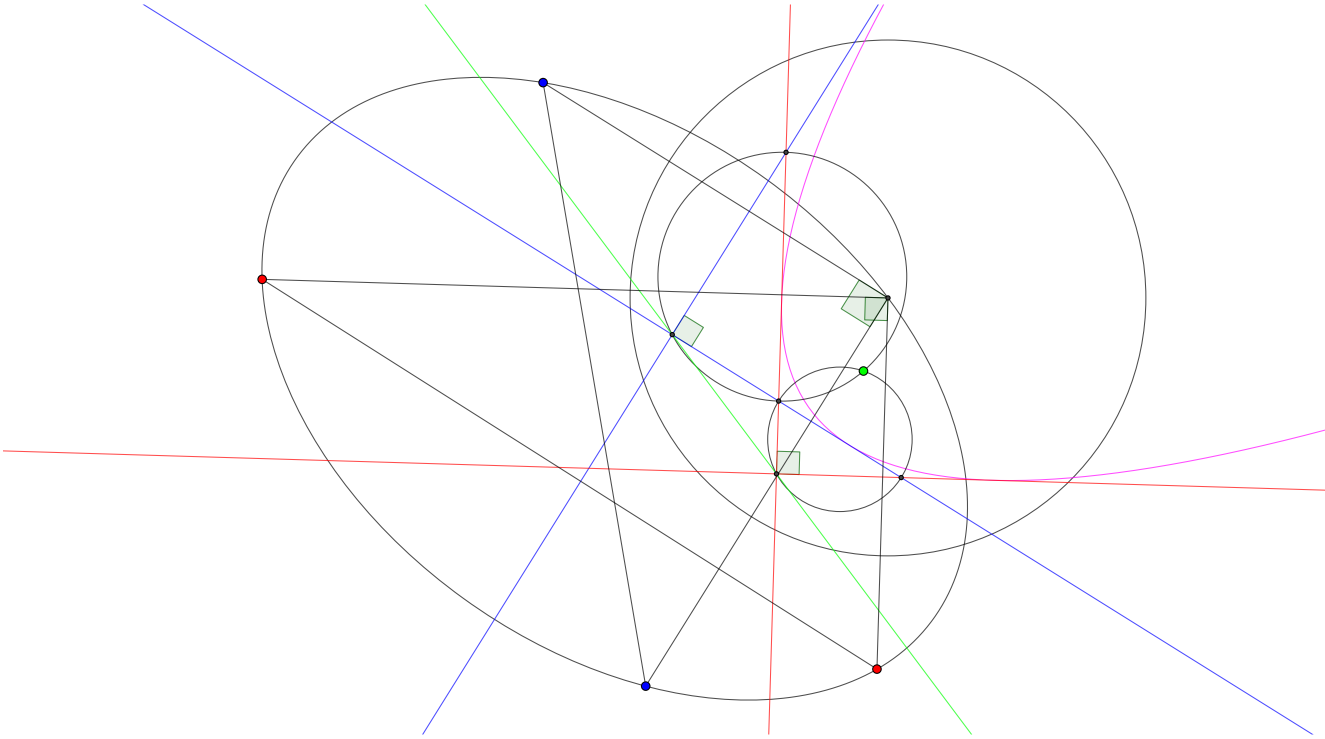

Figure 24: Construction of the polar parabola

As a parabola is uniquely determined by its focus and directrix, we need to find these two. This addition has partly been recycled from my BSc thesis [2], where I also dealt with conic sections and included a theorem asserting that the circumcircles of tangential triangles to a parabola all pass through its focus (as found in [1]). The four lines produced by taking the polars of the endpoints of two chords in Frégier's theorem, in light of the previous paragraph, are tangents to this parabola, and thereby determine tangential triangles to it. The intersection of their circumcircles is necessarily the focus of the parabola.

To hunt down the directrix, we can now proceed with the construction in three directions. Included in my thesis were two theorems we can build on. One claims that the reflection of a focus of a conic with respect to an arbitrary tangent lies on the directrix (cited from [7]). It is therefore sufficient to reflect the focus with respect to two tangents, the reflected points will span the directrix. Another approach invokes a theorem stating that the orthocenters of tangential triangles to a parabola also lie on the directrix (sourced from [1]), thereby warranting that it suffices to obtain two tangential triangles, and their orthocenters will automatically determine the directrix. There is a third way which may also be utilized to obtain the directrix, and that is where the proof and the construction connect in a way:

To move further with the proof, we characterize perpendicular tangents to a parabola by the following lemma.

Lemma 5.15 *The locus of points from which tangents drawn to a parabola are perpendicular is its directrix.*

Proof Let F denote the focus of the parabola as shown in the following figure. Let P be an arbitrary point on the directrix. Draw tangents from P to the parabola. Let these tangents touch the parabola in points R and S. In accordance with the previous construction, the reflections of the focus with respect to the tangents lie on the directrix. Let M and N be the reflections of F onto PR and PS. Because of the reflection, triangle MFP is equilateral, thus, $\angle RPF = \angle RPM$. For the same reason, $\angle SPF = \angle SPN$. As a consequence,

$$2\angle RPF + 2\angle SPF = \angle MPN = \pi,$$

or equivalently,

$$\angle RPF + \angle SPF = \frac{\pi}{2}.$$

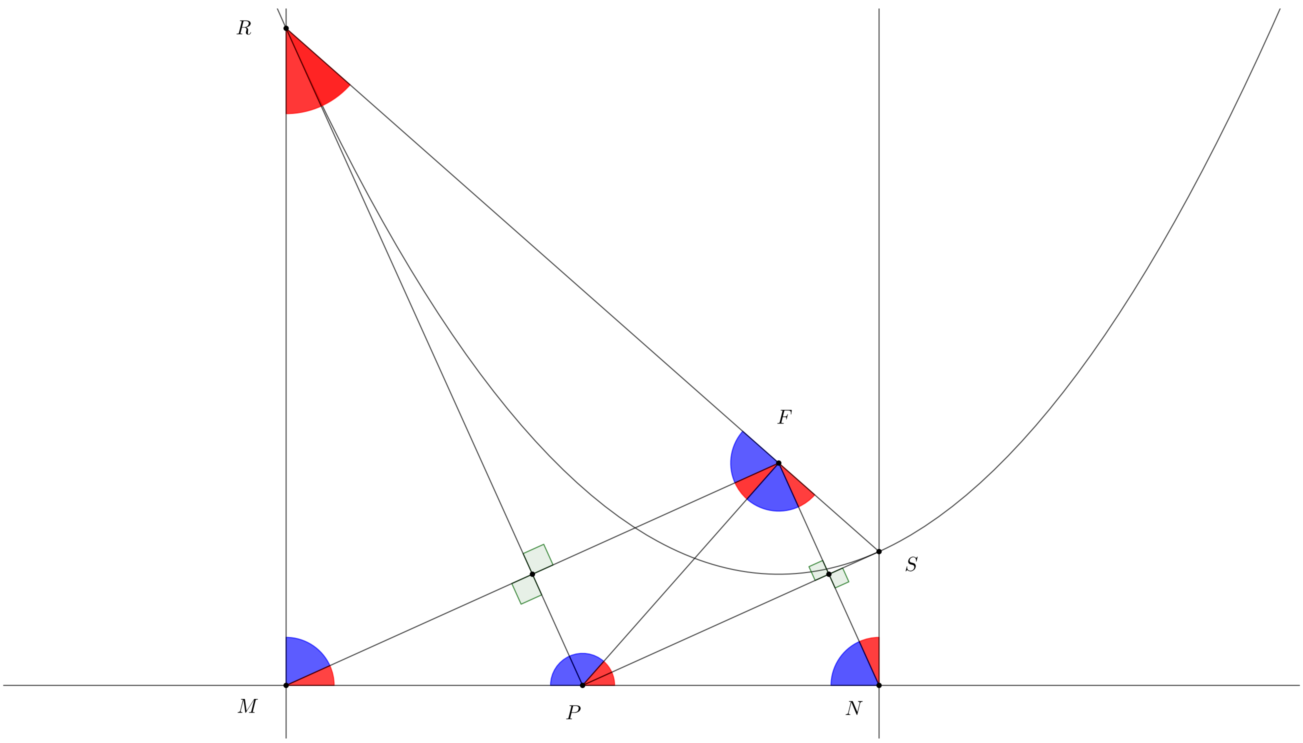

Figure 25: Perpendicular tangents of a parabola I.

One can also seamlessly establish another intriguing property from this figure as a corollary impressive in itself: PR and PS are seen at a right angle from F, therefore, the focus is contained in the chord spanned by the tangency points.

We managed to show that tangents drawn to the parabola from a point on the directrix are perpendicular. We have yet to prove that if there exists a point Q from which the tangents are seen at a right angle, it must lie on the directrix. Let us suppose to the contrary that it does not. Consider the configuration in the following figure. Obviously, it is wrong yet it will help us construct a proof. Let the tangents drawn from Q touch the parabola at points X and Y. By assumption, $\angle XQY = \frac{\pi}{2}$. Let T denote the intersection of line XQ with the directrix (due to symmetry, we could also consider the intersection with YQ). TX is already a tangent to the parabola. Draw the other tangent from T. Let this line be tangent to the parabola at K. In accordance with the property we just proved,

$$\frac{\pi}{2} = \angle XTK = \angle XQY.$$

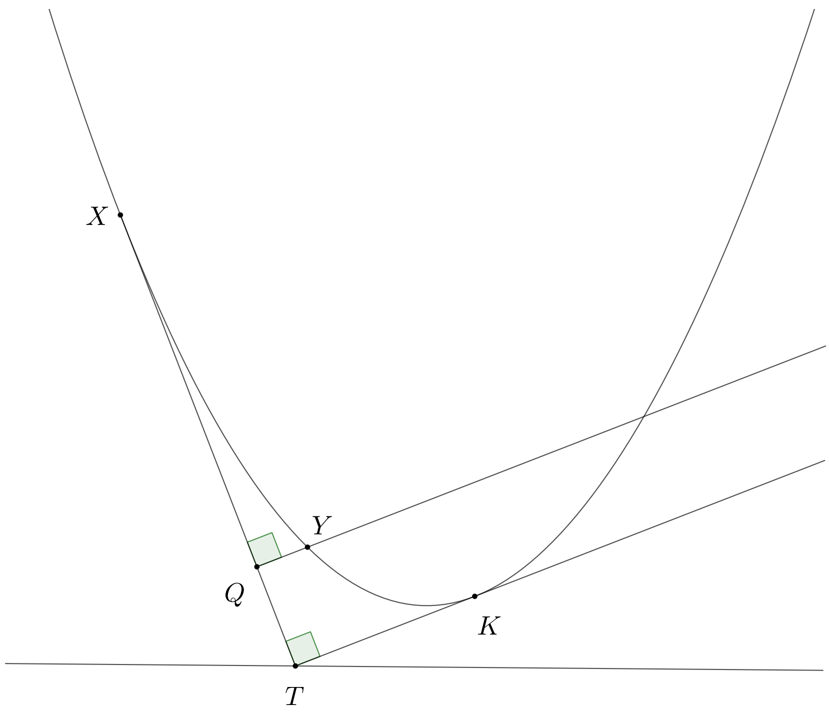

Figure 26: Perpendicular tangents of a parabola II.

This means that lines QY and TK are parallel tangents. From a parallel class of lines, at most one can be tangent to a parabola, as a result, QY and TK coincide, and so do T and Q, i.e. $T \equiv Q$, contradicting the assumption that Q lies outside the directrix.

This proves that one can draw perpendicular tangents from a point to the parabola if and only if the point lies on the directrix as proposed.

To recapitulate, we have already shown that $R(S)$ and $R(T)$ are perpendicular tangents to the parabola. In light of the aforementioned lemma, their intersection lies on the directrix.

Apply Lemma 5.7 to our setting: substitute X and Y with the endpoints of a chord. It follows that the intersection of their polars is the pole of the chord.

Consider another chord of the conic with endpoints M and N, seen from P at a right angle. Let it intersect ST at A. The pole of this chord, $R(MN)$, along with pole $R(ST)$, must then also lie on the directrix. In fact, we are too short on space to show that, but the poles of such chords make up the whole directrix. Therefore, such chords are produced by taking the intersection of the conic and the polars of all points on the directrix.

Apply the lemma again to points $R(ST)$ and $R(MN)$ to obtain that the intersection of their polars, A is the pole of the line joining them, i.e. the directrix of the parabola.

Let us invoke Theorem 5.6 again which claims that if line a is the polar of point A, then, for every point B on a, its polar b (dotted) passes through A. Let a denote the directrix, and A the intersection of the two chords, ST and MN in accordance with our notation. a is the polar of A, hence, for all $B \in a$, its polar b, which corresponds to a chord seen from P at a right angle, passes through A. As the points of the directrix are bijectively mapped to the perpendicular chords, this property holds for all such chords, rendering them concurrent at A as proposed.

List of Figures

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