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PORTFOLIO OPTIMIZATION WITH THE BLACK–LITTERMAN MODEL

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Contents

1 Introduction

The topic of investment decision making can be interpreted as a mathematical problem several ways: maximizing return, minimizing volatility, calculating likelihood, using statistical analysis to predict future returns. Although it seems to be a rational process, most investors are not quite rational, and they rely on their own views besides recommendations and historical data. On the contrary, most models do not take views into account, not even considering an intelligent investor (for example a portfolio manager) who's views can be profitable due to their experience.

Prior to the research about financial models, most investors invested in terms of the historical performance of individual assets and their current performance. The first researcher, who combined mathematical modeling with financial optimization was Harry Markowitz, the author of *"Portfolio Selection"*, an article first published in the March 1952 *Journal of Finance*. His angle was innovative, because he took the relationship between the asset's returns into consideration, and introduced the concept of diversification.

In 1958, Charles Tobin introduced the theory of the efficient frontier in the article titled *"Liquidity Preference as Behavior Towards Risk"*. The theory claims, that taking the volatilities and returns into considerations, the efficient frontier (a curve, where efficient portfolios are represented) can be drawn.

In 1964, William Sharpe complemented Markowitz's and Tobin's work by introducing the CAPM-model, and the concept of the Sharpe ratio, which linked return and volatility, and made it possible for investors to compare possible investment opportunities, and determine whether an asset is fairly priced. Later, other risk-adjusted metrics emerged, several of them are discussed below in this manuscript.

At this point, investors had a model, and metrics to use when making decisions, but a new challenge appeared: the Markowitz model relied on historical data, and built portfolios only on the basis of historical correlation, volatility, and returns. On the contrary, the first line a person can read in any investment brochure says: *"Past performance is no guarantee of future returns"*. Economical factors can alter, and in the constantly changing environment, it is not profitable to make investment decisions by only considering past data, but creating a vision about the future, and incorporate it into the investment decisions. Intelligent investors can form unique views based on their own analysis, and they would like to take advantage of their ideas, but views about the future performance of the market is not taken into consideration in the Markowitz model.

In 1990, Analyst Bob Litterman at Goldman Sachs also faced the problem, and with mathematician Fischer Black (who can also be familiar from the Black–Scholes model) developed and published the Black–Litterman model, which took investor views into account. In 2003, Litterman collected all his information and philosophy about investment management in the book *"Modern investment management - an equilibrium approach"* , which was my guideline while understanding the concept and the model. In this detailed book, not just Litterman, but other professional investors contribute to their strategy beyond using the Black–Litterman model while allocating assets.

In this manuscript, I aim to derive the Black–Litterman model, and introduce it through a real-life example (with actual data). Although the model itself does not advise on constructing the views about the future returns and volatility, I present a common solution for forecasting future returns by performing linear regression.

In Section 2 and 3 the economical and mathematical framework are built. After that in Section 4 and 5 we explore the Markowitz model and the Capital Asset Pricing Model, which need to be understood to discuss the Black–Litterman model. In the brief Section 6, the equilibrium approach is explained, which is the core of Litterman's philosophy of investing. In Section 7 the general Black–Litterman model is derived on the basis of the bayesian approach. In the final Section 8, I present a real-life problem, and give solution by using the Black–Litterman model.

2 Economical framework

2.1 Overview

This section aims to define the basic economic concepts such as return and risk to help the understanding of the basic laws of financial markets.

The term financial markets refers to any marketplace, where securities can be traded. There are many types of financial markets, some examples are bond market, stock market or foreign exchange market. These securities can be listed on regulated exchanges, or can be traded over-the-counter (these OTC transactions occour when two parties conduct a purchase without the contribution of a third party person or institution).

A portfolio is the collection of the investor's assets. It can include traditional assets like stocks or bonds, but broadly speaking, we consider physical gold, property or artwork as assets in the portfolio. In this manuscript, we only include traditional securities in the financial modeling.

The risk free rate is a theoretical zero-risk return. This rate represents the return, which is expected by an investor from an absolutely risk free investment over a specified period of time. The risk free rate is usually equal to the yield of the zero coupon bond issued by the investors government in the investors base currency. The concept of the risk free rate is the core of all models concerning optimal portfolios, because investors usually measure their returns against the risk free rate (or in some case, inflation, which is also discussed below).

The risk free rate is influenced by central banks through the changes in the target interest rates. In the decision process, they asses monetary features and key indicators (such as inflation or supply and demand conditions). For example, to curb inflation, central banks usually raise the interest rate to cool down the overheating economy, or lower yield to stimulate borrowing and economic activity.

Inflation is the main factor investors consider during decision-making, because their ultimate goal is to protect their purchasing power. Inflation is a gradual loss of purchasing power. It is calculated by the price change of a fixed purchasing basket of goods and services. High inflation means that prices are increasing quickly, while low inflation means prices are increasing more slowly. Deflation can also occur when prices decline and purchasing power is rising.

Return is a change in a price of an asset over time plus any cashflow during that period, usually represented in percentage change. Excess return means the difference between the risk free rate and the return of the investment. This shows the investor the return which is added by taking risks. For example, if the US risk free rate is 4% and a given stock performed 10% over a specified period of time, the excess return is 6%. Historically, an investor only invested in risk free assets cannot beat the inflation (although many governments offer floating rate bonds linked to the inflation rate), because of that, many consider taking risks.

In portfolio management we interpret risk as the probability that the actual results will differ from the expected returns, also known as the volatility of the returns. The potential return of the investor is getting greater, as they are willing to take more risk. Risk has various types, and investors need to be compensated fairly for investing in a risky asset. For example, let us assume that the U.S. Treasury rate is 5% (equal to the risk free rate). It is considered to be the safest investment (companies are more likely to go bankrupt than the U.S. government, because governments can always print money to meet their obligations), so compared to corporate bonds, it provides a lower return (because the default risk of a corporate bond is higher, investors offered a higher return). Risk can occur many ways, such as business risk, interest rate risk, credit risk, political risk, liquidity risk etc., but this topic will not be discussed in this manuscript. (The article "Understanding Financial Risk Management" blog article by Leavy School of Business Santa Clara University discusses the topic further.)

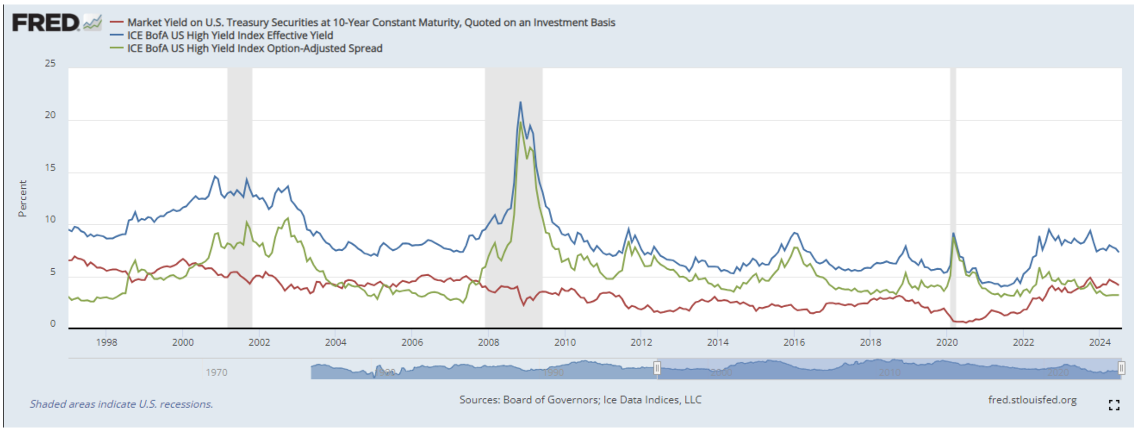


Figure 1: U.S. Treasury yields, High yield bond effective yields, High yield bond spreads 1998-2024. (source: Federal Reserve Economic Data, St. Louis Fed)

The core of investment management is the optimalization of risk and return through diversification, however these concepts can only be discussed linked to each other. For this reason, we need to define some indicators, which will contain these two metrics.

2.2 Risk and return

2.2.1 Systematic and non-systematic risk

Definition 2.1 (Systematic risk)**.** *Systematic risk refers to a macro risk across the entire market or a market segment. It is also referred as undiversifiable risk. It impacts the whole market, not just any particular asset.*

A common example of the systemic risks is that the main cause is an outside event, like pandemics, wars, regulatory changes, or supply chain disruption.

Definition 2.2 (Idiosyncratic risk)**.** *Idiosyncratic risk (or unsystematic risk) refers to the risks which are endemic to a specified asset (or, broader, asset classes).*

Examples of idiosyncratic risk include poor management decisions on regulatory issues, geographical location, company culture or employee strikes. Certain securities naturally have a higher idiosyncratic risk than other, for example, new industries, where at this point, regulations are weak, can experience high volatility. For example, cryptocurrencies are mostly driven by idiosyncratic factors which have limited correlations with traditional asset classes.

The main difference between idiosyncratic and systemic risk is that idiosyncratic risk can be lowered by diversification. When investments with negative or zero correlation are picked, idiosyncratic risk can converge to zero, if (approximately, as a rule of thumb) 30 assets are chosen.

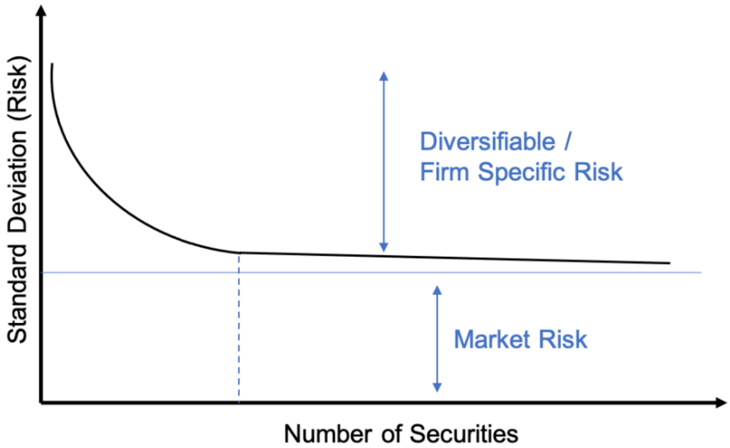


Figure 2: Standard deviation of a portfolio in relation to the number of securities

2.3 Metrics and indicators

This section strongly relies on the term-defining articles of investopedia.com, a global financial media site, which provides investment dictionaries and news.

To follow with the indicators, we need to understand to concept of risk-adjusted return. The risk-adjusted return measures the portfolio return against the risk (also known as volatility) which the assets represented throughout a period. For example, if we analyze two assets, A and B with the same return, the one with the lowest risk will be the better according to the risk-adjusted return.

We have several risk-adjusted measures, the most commonly used ones are Sharpe ratio, Treynor ratio, information ratio, alpha, beta, and beyond you can interpret risk through standard deviation and R-squared also.

Definition 2.3 (β)**.**

$$\beta = \frac{Cov(R_m, R_i)}{Var(R_m)}$$

where

$R_i = expected\ or\ actual\ return\ of\ investment\ i$

$R_m = return\ of\ the\ market$

Alpha is an investment's return in relation to the return of a benchmark (a set of assets, which we use to measure our investments against). Alpha is measured during a specified time period.

Definition 2.4 (α)**.**

$$\alpha = R - R_f - \beta \cdot (R_m - R_f)$$

where

$R = Return\ of\ the\ portfolio$

$R_f = Risk\ free\ rate$

$R_m = Market\ return,\ per\ the\ benchmark$

$\beta = Systematic\ risk\ of\ the\ portfolio$

The Sharpe ratio measures the profit of the investment that exceeds the risk free rate per unit of standard deviation. It is calculated by the return of the investment subtracting the risk and dividing this result by the investment's standard deviation.

Definition 2.5 (Sharpe ratio)**.**

$$SR_i = \frac{r_i - r_f}{\sigma_i}$$

where

$r_i = expected\ or\ actual\ return\ of\ investment\ i$

$r_f = return\ of\ the\ risk\ free\ investment$

$\sigma_i = standard\ deviation\ of\ r_i$

The Treynor ratio is calculated the same way as the Sharpe ratio, except it divides by the beta of the market, thus it measures the investments relative return compared to the market return.

Definition 2.6 (Treynor ratio)**.**

$$TR_i = \frac{r_i - r_f}{\beta_i}$$

where

$r_i = expected\ or\ actual\ return\ of\ investment\ i$

$r_f = return\ of\ the\ risk\ free\ investment$

$\beta_i = Beta\ of\ r_i,\ discussed\ above$

To discuss the information ratio, we have to understand the role of benchmarks in the valuation of investments. The risk free rate operates as a benchmark, but we can get more data and conclusions from using various benchmarks depending on the situation. For example, when valuing a U.S. stock, the $S\&P500$ index may be a more suitable benchmark.

The information ratio standardizes the returns by dividing the difference in their performances, known as their expected active return by their tracking error.

Definition 2.7 (Information ratio)**.**

$$IR_i = \frac{r_i - r_b}{Trackingerror}$$

where

$r_i = expected\ or\ actual\ return\ of\ investment\ i$

$r_b = return\ of\ the\ benchmark$

Tracking error: The standard deviation of the difference between portfolio and benchmark returns

Return of the benchmark (r_b) and return of the market (r_m) are only equal, when investors choose a market index (for example the MSCI World Index) to the benchmark of their portfolios. The term market refers to a fixed index (allocation of securities) which is not chosen by the investor, but benchmarks are chosen to represent the allocation in the portfolio.

When deciding whether a certain stock will be a profitable investment, it is useful to calculate the valuation ratios of the company. The two most used ratios are the P/E (price-to-earning) and P/S (price-to-sales).

Definition 2.8 (P/S)**.**

$$P/S = \frac{MVR}{SPS}$$

where

$MVR =$ *market value per share*

$SPS =$ *sales per share (total revenue earned by the company, per share)*

Definition 2.9 (P/E)**.**

$$P/E = \frac{MVR}{EPS}$$

where

$MVR =$ *market value per share*

$EPS =$ *earnings per share (total income minus dividends per the number of outstanding stocks)*

To interpret the standard deviation and R^2 as risk measurement indicators, we need to combine the economic concepts with the mathematical framework.

2.4 Valution

Subsection 2.4 heavily relies on the article series of Investopedia.com, titled *"Practical look on microeconomics"*.

Since investment portfolios are held by human beings, with goals other than maximizing returns, the valuation of investment decisions should consider several other factors, such as the risk aversion of the investor, or the utility of the investment. This topic can be rather psychological, but this manuscript will only touch on that a bit after establishing the main principles and definitions of the topic.

2.4.1 Risk aversion

Risk aversion, according to psychology is *"a preference for a sure outcome over a gamble with higher or equal expected value"*[14]. In economics, risk aversion means *"the tendency of people to prefer outcomes with low uncertainty to those outcomes with high uncertainty, even if the average outcome of the latter is equal to or higher in monetary value than the more certain outcome."* [14]

Individuals have different attitudes towards risk, they can be:

- risk averse (or risk avoiding)
- risk neutral
- risk seeking

2.4.2 Utility

The utility of the consumed goods depends on not just the goods itself, but somehow the circumstances: the first consumed good usually has a higher utility, because it satisfies a higher need then the following units. Common example is, when being hungry, the first unit of food satisfies a higher need, than the last one. The basis of marginal utility is, that the additional benefit (utility) a consumer derives from buying an additional unit of a good, is inversely related to the number of product that they already own.

Research of consumer behaviour shows that utility functions can look several ways, because goods differ (some goods even have a negative marginal utility, where consuming more can be harmful).

Utility formulas offer a solution to calculate utilities and make comparisons between them.

Definition 2.10 (Linear form of the utility function)**.** *$U = a + bX$, where a represents the utility of no consumption, b denotes the marginal utility per unit increase in consumption of the good or service, and X corresponds to the quantity consumed from the goods or services.*

Definition 2.11 (Exponential form of the utility function)**.** *$U = 1 - e^{-aW}$, where W is wealth and a is the risk-aversion parameter.*

The exponential form of the utility function typically used in models concerning risk or uncertainty.

Definition 2.12 (Logarithmic form of the utility function)**.** *$U = ln(W)$, where W is wealth.*

This formula is used to represent risk aversion and diminishing marginal utility of wealth.

Definition 2.13 (Cobb-Douglas utility function:)**.** *$U = X_1^{\alpha} X_2^{1-\alpha}$, where X_1, X_2 are quantities of two goods and α indicates the preferences.*

This form is optimal for consumer demand analysis and production theory.

In portfolio theory, when referring to the utility functions, we mean the exponential form, where c equals consumption or wealth, and a represents the degree of risk preference.

$$u(c) = \begin{cases} (1 - e^{-aW})/c & c \neq 0 \\ c, & c = 0 \end{cases}$$

Since (as discussed above), risk is the source of return (above the return of the risk free rate), every additional amount of risk has an exact utility it adds to the portfolio, thus all three types of investor have a different type of utility function.

- risk averse (or risk avoiding) utility function: $\delta > 0$
- risk neutral utility function: $\delta = 0$
- risk seeking utility function: $\delta < 0$

Most models assume that investors are risk-averse.

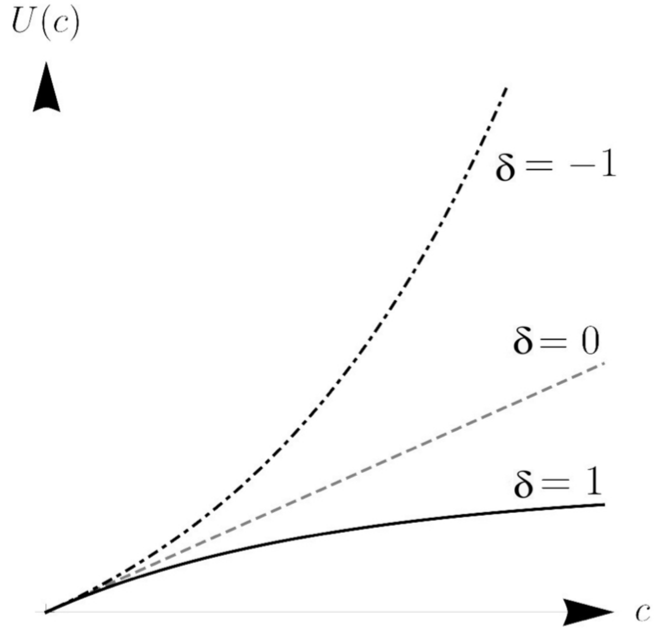


Figure 3: Utility functions for different δ-values

3 Mathematical framework

3.1 Additivity of risk and return

When building portfolios, we need to be familiar with how risk and return behave when a new asset is added to the portfolio. The following intuitive approach was detailed by Litterman in Modern Investment Management. [6]

The intuition of the additivity of risk and return can be seen geometrically in the following diagrams. A new asset affects the risk of the portfolio in the same way as the addition of a side to a line segment changes the distance of the end point to the origin.

The length of the original line segment A represents the risk of the original portfolio. We add a side to this segment, the line B represents the volatility of the investment added, and C represents the new portfolio. The distance from the origin is clearly determined by the angles of the two lines, similarly to the portfolio risk determined by the correlation of the original portfolio and the new asset added.

Correlations range between -1 and 1, in the figure represented by 0 ° to 180 °. The case of 0 correlation corresponds to the 90° angle. Positive correlations correspond to angles between 90° and 180°, and negative correlations correspond to angles between 0 ° and 90 °.

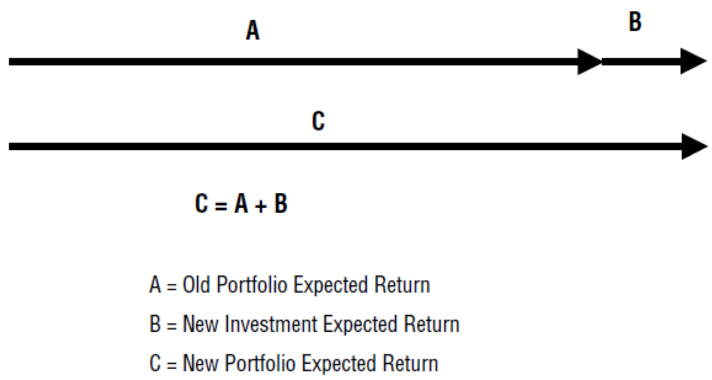


Figure 4: Expected returns sum linearly [6]

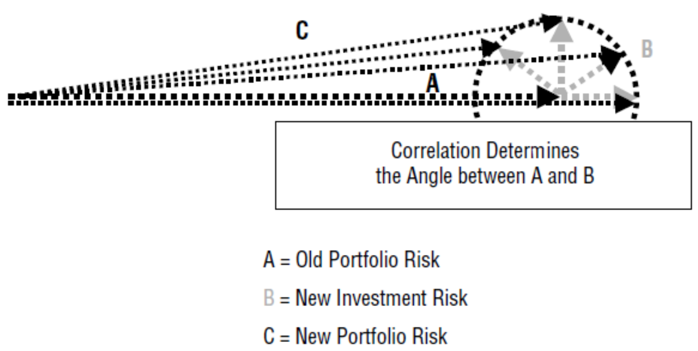


Figure 5: Summation of risk depends on the correlation [6]

3.2 Laws of large numbers

Since investing is taking bets (or staying neutral, which I also consider a bet) on possible outcomes, probability theory and statistics are needed to analyse the possibility of certain outcomes. Expected returns can be easily interpreted as the expected value of random variables, volatilities as their standard deviation. According to my theory, this interpretation includes the "randomness" and unpredictable nature of the assets, as well as opportunity of using predictions based on objective matters.

In this following section, we will discuss the Laws of large numbers, and the Central limit theorem.

In order to make wise decisions, analysts frequently examine the distribution of the stock market returns. Analysts compute sample mean and standard deviation of an index or a portfolio, without determining each asset's own distribution. It also states that if the individual assets of the portfolio are identically distributed and independent, the overall portfolio's distribution tend to be closer to normal distribution while increasing the number of assets in the portfolio. This leads to several applications in portfolio optimizing.

Theorem 3.1 (Weak law of large numbers)**.** *Let X_1, $X_2 \ldots$ be a sequence of mutually independent and identically distributed random variables. Let us assume that $D(X_1) < \infty$. Then for any $\varepsilon > 0$*

$$P(|\overline{X_n} - E(X_1)| > \varepsilon) \to 0$$

$$n \to \infty.$$

Meaning that X_1 converges in probability to $E(X_1)$, where $\overline{X_n}$ equals the sample mean ($\frac{X_1+...+X_n}{n}$).

Theorem 3.2 (Strong law of large numbers)**.** *Let X_1, $X_2 \ldots$ be a sequence of mutually independent and identically distributed random variables. Let us assume that $m = E(X_1) < \infty$. Then*

$$\overline{X_n} = \tfrac{X_1+X_2+...+X_n}{n} \to E(X_1) = m \text{ with probability 1.}$$

$$n \to \infty.$$

3.3 Central limit theorem

Theorem 3.3 (Central limit theorem)**.** *Let X_1, X_2... be a sequence of mutually independent and identically distributed random variables. Let us assume that $m = E(X_1)$ and $D(X_1) < \sigma$. Then for any real number t we have that*

$$P(\tfrac{X_1+X_2+...+X_n - n\cdot m}{\sigma\cdot\sqrt{n}} \le t) \to P(Z \le t)$$

$$n \to \infty.$$

where Z has standard normal distribution:

$$P(Z \le t) = \Phi(t) =$$

$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)\, dx.$$

Loosely speaking, the Central limit theorem states that the sample mean distribution of a random variable will assume a normal or near-normal distribution if the sample size is large enough. This comes useful when analyzing large data sets, like stock price history, index price history, returns or volatility.

It is worth noting that daily and weekly stock returns are usually not normal, but aggregation to monthly return rates produces normality as would be expected. The following figures and data are from the *Financial Valuation and Econometrics* written by Kian Guan Lim, and the normality of the returns are checked with the Jarque–Bera test. [5]

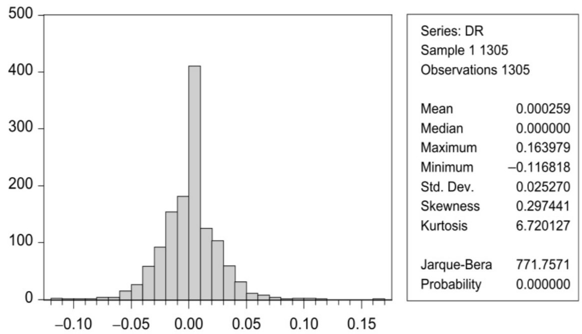


Figure 6: Histogram and statistics of daily OCBC stock return rates [5]

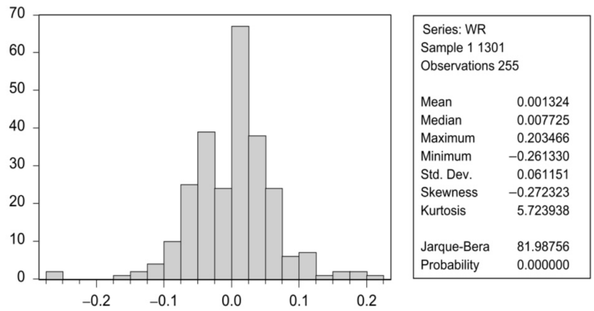


Figure 7: Histogram and statistics of weekly OCBC stock return rates [5]

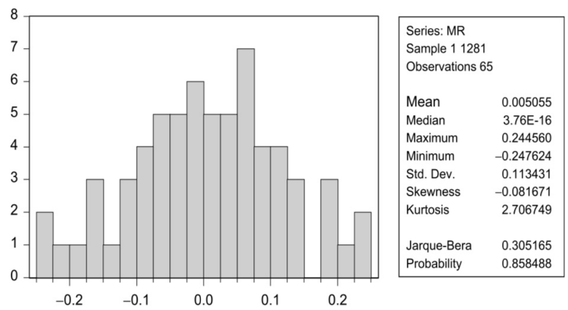


Figure 8: Histogram and statistics of monthly OCBC stock return rates [5]

3.4 Bayesian statitistics

Bayesian statistics is a method to apply probability theory in statistic problems. We may have prior beliefs about events, which may change when new events occour, and this method allows us to update and incorporate our posterior beliefs. Bayesian statistics treats parameters as random variables with probability distribution. [12]

Theorem 3.4 (Bayes' theorem)**.** *Let $A_1, A_2 \dots A_n$ be events, A_i events are an exhaustive partition of the sample space, where all events have nonzero probability. Then for any event (with nonzero probability) B:*

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{k=1}^{n} P(B|A_k)P(A_k)}.$$

Let us write Bayes' theorem in the following form (events A_i are an exhaustive partition of the sample space A):

$$P(B) = \sum_{A_i \in A} P(B \cap A_i),$$

We get the following equation by substituting the definition of conditional probability:

$$P(B) = \sum_{A_i \in A} P(B \cap A_i) = \sum_{A_i \in A} P(B|A_i)P(A_i) \tag{1}$$

By substituting equation (1) into the original Bayes' theorem, we get the formula:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{A_k \in A} P(B|A_k)P(A_k)} \tag{2}$$

Definition 3.1 ($P(D|\theta)$)**.** *[12] The probability of observing data D, under a particular value of θ.*

Definition 3.2 ($P(\theta|D)$)**.** *[12] The probability of the event, when the distribution parameter is θ, if the observed data is D, called the posterior distribution.*

The connection between the two is given by the Bayes' theorem:

Theorem 3.5 (The rule of Bayesian Inference)**.** *[12]*

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}, \text{ where:}$$

- *$P(\theta)$ is the prior (the strength in our belief of θ without considering the information D).*
- *$P(\theta|D)$ is the posterior, the refined strength in our belief of θ with evidence D taken into account.*
- *$P(D|\theta)$ is the likelihood, the probability of seeing data D, generated by a model with the parameter θ.*

- *$P(D)$ is the probability of the data as determined by summing across all values of θ, weighted by the probability of the values occouring.*

Hence the Black–Litterman model updates current portfolios with views about asset classes, the bayesian inference is a suitable framework for the model.

3.5 Linear regression

During linear regression, we aim to approximate the function $f(x) = y$ which is known in points x_1, x_2, ..., x_n. Since the $f(x_i)$ values are the dependent variables, linear regression can provide data about the relationship of the x_i and $f(x_i)$ values.

The simplest method, for linear regression is the least squares method which works by minimizing the sum of the offsets or residuals of points from the plotted curve.

Definition 3.3 (Linear regression)**.** *Let (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) be given pairs of numbers. We are looking for the coefficients a and b, which minimize the following scalar:*

$$h^2 = \frac{1}{n}\sum_{i=1}^{n}[y_i - (ax_i + b)]^2.$$

The solution for a and b are the following:

$$\hat{a} = \frac{\sum_{i=1}^{n}(x_i - \overline{x_n})(y_i - \overline{y_n})}{\sum_{k=1}^{n}(x_k - \overline{x_n})^2},$$

$$\hat{b} = \overline{y} - \hat{a}\overline{x}.$$

The linear model fits differently for different data sets. The accuracy of the model can be understood by residuals (the difference between the estimated and observed value $(Y_i - \hat{a}X_i - \hat{b})$).

Definition 3.4 (Total sum of squares)**.** $\sum_{j=1}^{n}(Y_j - \overline{Y})^2$

Definition 3.5 (Residual sum of squares)**.** $\sum_{j=1}^{n}(Y_j - \hat{a}X_j - \hat{b})^2$

By subtracting the ratio of the residual sum of squares and the total sum of squares from 1, we get the coefficient of determination, also called R^2:

$$R^2 = 1 - \frac{\sum_{j=1}^{n}(Y_j - \hat{a}X_j - \hat{b})^2}{\sum_{j=1}^{n}(Y_j - \overline{Y})^2}$$

in an other form:

Definition 3.6 (Coefficient of determination)**.** $R^2 = \dfrac{[\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})]^2}{[\sum_{k=1}^{n}(X_k - \overline{X})^2][\sum_{k=1}^{n}(Y_k - \overline{Y})^2]}.$

The value of R^2 is between 0 and 1, the higher R^2 is, the better-fitted the model is. R^2 is quite sensitive for outliers.

Theorem 3.6 (Distribution of the regression parameter $\hat{a}$). *[11]*

Under the assumptions of the linear model :

$$\hat{a} \sim N(a, \frac{\sigma^2}{n})$$

Proof:

Recalling that, that by applying the Maximum likelihood estimation,

$$\hat{a} = a = \overline{Y},$$

where the responses Y_i are independent and normally distributed, with parameters:

$$Y_i \sim N(\alpha + \beta(x_i - \overline{x}), \sigma^2)$$

The expected value is $E(\hat{a}) = a$, because

$$E(\hat{a}) = E(\overline{Y}) = \frac{1}{n}\sum E(Y_i) = \frac{1}{n}\sum E(\alpha + \beta(x_i - \overline{x})) = \frac{1}{n}[n\alpha + \beta\sum(x_i - \overline{x})]$$

since $\sum(x_i - \overline{x}) = 0$:

$$\frac{1}{n}[n\alpha + \beta\sum(x_i - \overline{x})] = \frac{1}{n}n\alpha = \alpha$$

By using the knowledge about the variance of the sample mean, the variance of $\hat{\alpha}$ can be directly calculated:

$$Var(\hat{\alpha}) = Var(\overline{Y}) = \frac{\sigma^2}{n}$$

Since the linear combination of independent normal random variables is also normally distributed, we have:

$$\hat{a} \sim N(a, \frac{\sigma^2}{n})$$

as to be proved. The other parameters' distribution can be similarly derived.

Theorem 3.7 (Prediction interval). *[11]*

A $(1-\alpha)$ confidence-level prediction interval for the value of Y_{n+1}, when the predictor is $x = x_{n+1}$ is:

$$\left(\hat{y}_{n+1} - t_{\alpha/2,n-2} \cdot \sqrt{\hat{\sigma}}\sqrt{1 + \frac{1}{n} + \frac{(x_{n+1}-\overline{x})^2}{\sum(x_i-\overline{x})^2}}, \hat{y}_{n+1} + t_{\alpha/2,n-2} \cdot \sqrt{\hat{\sigma}}\sqrt{1 + \frac{1}{n} + \frac{(x_{n+1}-\overline{x})^2}{\sum(x_i-\overline{x})^2}}\right)$$

where $t_{\alpha/2,n-2}$ is the critical value of the two-tailed t-probe with range value α.

Proof:

Recalling the following informations:

- $Y_{n+1} \sim N(\alpha + \beta(x_{n+1} - \overline{x}), \sigma^2)$
- $\hat{\alpha} \sim N(\alpha, \frac{\sigma^2}{n})$ (Theorem 3.6)
- $\hat{\beta} \sim N(\beta, \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2})$
- $\frac{n\hat{\sigma^2}}{\sigma^2} = \frac{(n-2)\hat{\sigma}}{\sigma^2} \sim \chi^2_{n-2}$

are independent, therefore

$$W = y_{n+1} - \hat{y}_{n+1} = y_{n+1} - \hat{\alpha} - \hat{\beta}(x_{n+1} - \overline{x})$$

is a linear combination of independent variables with mean:

$$\begin{aligned} E(W) &= E[y_{n+1} - \hat{\alpha} - \hat{\beta}(x_{n+1} - \overline{x})] \\ &= E(Y_{n+1}) - E(\hat{\alpha}) - (x_{n+1} - \overline{x})E(\hat{\beta}) \\ &= \alpha + \beta(x_{n+1} - \overline{x}) - \alpha - \beta(x_{n+1} - \overline{x}) \\ &= 0 \end{aligned}$$

and variance:

$$\begin{aligned} Var(W) &= Var[y_{n+1} - \hat{\alpha} - \hat{\beta}(x_{n+1} - \overline{x})] \\ &= Var(Y_{n+1}) - Var(\hat{\alpha}) - (x_{n+1} - \overline{x})^2 Var(\hat{\beta}) \\ &= \sigma^2 + \frac{\sigma^2}{n} + \frac{(x_{n+1} - \overline{x})^2 \sigma^2}{\sum(x_i - \overline{x})^2} \\ &= \sigma^2 \left[1 + \frac{1}{n} + \frac{(x_{n+1} - \overline{x})^2}{\sum(x_i - \overline{x})^2} \right] \end{aligned}$$

By putting it all together:

$$W = (y_{n+1} - \hat{y}_{n+1}) \sim N\left(0, \sigma^2 \left[1 + \frac{1}{n} + \frac{(x_{n+1} - \overline{x})^2}{\sum(x_i - \overline{x})^2} \right] \right)$$

The definition of a T random variable tells that:

$$T = \frac{\dfrac{y_{n+1} - \hat{y}_{n+1} - 0}{\sqrt{\sigma^2 \left[1 + \frac{1}{n} + \frac{(x_{n+1} - \overline{x})^2}{\sum(x_i - \overline{x})^2} \right]}}}{\sqrt{\dfrac{n\hat{\sigma^2}}{\sigma^2}/(n-2)}} = \frac{y_{n+1} - \hat{y}_{n+1}}{\sqrt{\hat{\sigma}} \sqrt{\sigma^2 \left[1 + \frac{1}{n} + \frac{(x_{n+1} - \overline{x})^2}{\sum(x_i - \overline{x})^2} \right]}} \sim t_{n-2}$$

since the numerator and the denominator are independent, the numerator is normally distributed with parameters $N(0,1)$, and the denominator is the square root of χ^2_{n-2}:

$$P = \left(-t_{\alpha/2,n-2} \leq \frac{y_{n+1} - \hat{y}_{n+1}}{\sqrt{\hat{\sigma}}\sqrt{\sigma^2\left[1 + \frac{1}{n} + \frac{(x_{n+1}-\bar{x})^2}{\sum(x_i-\bar{x})^2}\right]}} \leq t_{\alpha/2,n-2} \right) = 1 - \alpha,$$

By manipulating the quantity inside the parenthesis, we get the $(1-\alpha)$ confidence level prediction interval for Y_{n+1}:

$$\left(\hat{y}_{n+1} - t_{\alpha/2,n-2} \cdot \sqrt{\hat{\sigma}}\sqrt{1 + \frac{1}{n} + \frac{(x_{n+1}-\bar{x})^2}{\sum(x_i-\bar{x})^2}}, \hat{y}_{n+1} + t_{\alpha/2,n-2} \cdot \sqrt{\hat{\sigma}}\sqrt{1 + \frac{1}{n} + \frac{(x_{n+1}-\bar{x})^2}{\sum(x_i-\bar{x})^2}} \right)$$

3.5.1 Forecasts in linear models

In finance, one of the most examined variables are stock prices in relations of several indicators. Analysts use this method to estimate future prices. In this section, I present an example for calculating the expected yearly return of the MSCI World Index.

The data set includes the monthly closing prices of the MSCI World index, and the weighted P/S ratios of the stocks from the index. For example, the first 5 rows are the following:

Dates	P/S	Price (USD)
1996.03.29	1,0533	761,18
1996.04.30	1,0027	777,93
1996.05.31	0,9652	777,44
1996.06.28	0,955	780,2
1996.07.31	0,9767	751,45
⋮	⋮	⋮

The data includes 339 ratios-price pairs (28 years).

As far as we know, valuation indices forecast stock prices long time ahead, so I constructed the forward 10-year-return for every date. For example the forward 10-year-return calculated for 1996.03.29. is the price of the index on the day 2006.03.31. divided by the price on 1996.03.29. After that, the yearly return is calculated, which is the tenth root of the previous ratio, thus we have 219 observations. The outcome is the following:

P/S	Yearly return
1,0533	5,78%
1,0027	5,85%
0,9652	5,45%
0,955	5,40%
0,9767	5,85%
⋮	⋮

The data points:

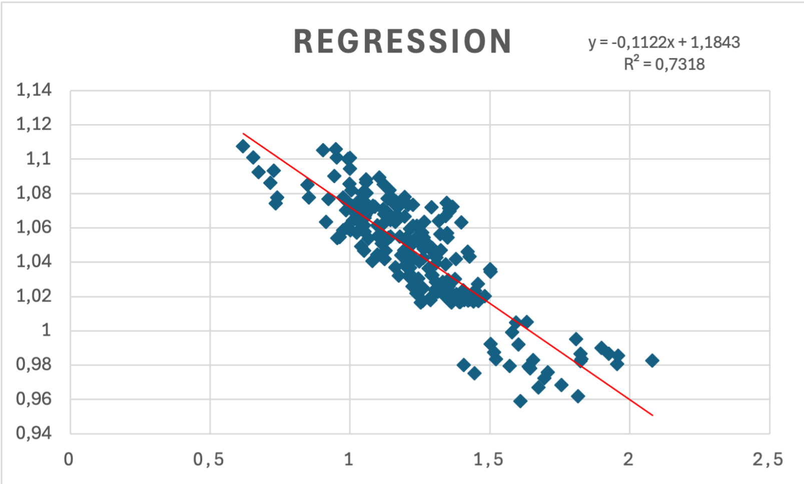


Figure 9: Data points

By conducting linear regression, we get the following summary:

Regression Statistics	
Multiple R	0,8555
R^2	0,7318
Adjusted R^2	0,7306
Standard error	0,017
Observations	219

$\hat{a} = 0,0689$

$\hat{b} = 0,9336$

The regression equation is $y = -0,1122x + 1,1843$.

The value at the date of the study (2024.05.31) of the P/S ratio is $2,2187$, substituted into the regression equation: $y = -0,1122 \cdot 2,2187 + 1,1843 \Rightarrow y = 0,935$.

3.5.2 Interpreting R-squared as risk measurement

R^2 measures the relationship between a portfolio and its benchmark index. Deviations from the benchmark tend to raise risks, thus higher R^2 usually leads to lower risk (in this case, the fund is passive, the portfolio manager mimics the benchmark), lower R^2 leads to higher risk (these active funds are absolute return funds, where the portfolio manager aims to profit for the investors, regardless of the direction of the market, and outperform their benchmark).

4 Markowitz Model - The Modern Portfolio Theory

4.1 Main considerations

The fundamental idea of the modern portfolio theory is that all investors have limited capacity for taking risks, hence risk should be treated as a scarce, valuable resource, investors should not avoid in order to realize return, however they differ in their risk-aversion.

In a portfolio, each asset has their own expected return and risk. Litterman argues that the bottom line is: *"...in constructing their portfolios, investors need to look at the expected return of each investment in relation to the impact that it has on the risk of the overall portfolio."* [**?**]

The primary determinant of an investment's contribution to portfolio risk is not the risk of the investment itself, but rather their covariance (which is the correlation times the volatility of each return, and returns are normally distributed [5]). Independent assets have zero, or near-zero correlation, thus correlation multiplied by the volatility are equal to zero, when the assets are not correlated. Thus, independent assets have zero covariance.

With the understanding of covariances we can achieve an increased return by recognizing situations in which adjusting the sizes of risk allocation would improve the expected return of the overall portfolio. For example, an independent investment, even with high risk, can add relatively low risk to the portfolio.

In the optimal portfolio, according to this theory, represented assets are allocated in the way that the ratio of expected excess return to the marginal contribution to portfolio risk is the same for all assets. If they differ, it means we have a chance to eliminate the item with the lower expected-return-to-contribution, and this way we could get a better performing portfolio (a portfolio with the same risk and higher expected return, or with the same expected return at lower risk).

Let us look further into this concept:

Definition 4.1 (Δ)**.** *The marginal contribution to the risk of the portfolio on the last unit invested in an asset. The value of Δ can be measured by calculation the portfolio risk for a given asset allocation and then measure what happens if we change the allocation.*

Theorem 4.1 (Risk of the portfolio)**.** *For a portfolio, containing two assets Asset 1 and Asset 2, a_1 and a_2 represent the amounts for each assets, with volatilities σ_1^2 and σ_2^2, and covariance $Cov(a_1,a_2)$. Then the risk of the portfolio is:*

$$Risk(a_1,a_2) = \sqrt{a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_1\sigma_2 Cov(a_1,a_2)}$$

Definition 4.2 (Δ_1)**.** *Marginal contribution to portfolio risk of Asset 1, defined as:*

$$\Delta_1 = \frac{Risk(a_1+\delta,a_2) - Risk(a_1,a_2)}{\delta}, \delta \to 0.$$

Δ_2 is defined similarly to Δ_1.

At this point, the question is whether we improve the portfolio by selling Asset 1 and buying Asset 2 instead. The ratio of marginal contribution to risk is $\frac{\Delta_1}{\Delta_2}$. The expected excess returns are e_1 and e_2.

We suppose that

$$\frac{\Delta_1}{\Delta_2} > \frac{e_1}{e_2}.$$

The rate of change in risk from the sale of Asset 1 is $-\Delta_1$ per unit sold. In order to keep the previous risk level we need to purchase $\frac{\Delta_1}{\Delta_2}$ units of Asset 2. The effect on expected return to the portfolio is e_1, per unit sold on Asset 1, and $+\frac{\Delta_1}{\Delta_2}e_2$ from the purchase of an amount of Asset 2 which leaves risk unchanged. If now expected return is increased, then we should continue to increase the allocation to Asset 2, if it decreased, then we should sell Asset 2, and buy Asset 1 back. The only case when the expected return of the portfolio cannot be increased while holding the risk level constant, is when the following condition is true:

$$-e_1 + \frac{\Delta_1}{\Delta_2}e_2 = 0.$$

By rearranging, we get:

$$\frac{e_1}{\Delta_1} = \frac{e_2}{\Delta_2}.$$

More generally, we can consider the reallocation of any two assets in the portfolio. In this context, let the risk function $Risk(w)$ give the risk for the vector w, which contains the weights for all assets. $Risk_m(w, \delta)$ gives the risk of the portfolio with weights w, and a small change, δ to the weight asset in asset m.

Definition 4.3 (Δ_m)**.** *Marginal contribution to portfolio risk of Asset m, defined as:*

$$\Delta_m(\delta) = \frac{Risk_m(w, \delta) - Risk(w)}{\delta}, \delta \to 0.$$

Then, as above, in an optimal portfolio, it must be the case that for every pair of assets m and n, the following condition must hold:

$$\frac{e_m}{\Delta_m} = \frac{e_n}{\Delta_n}.$$

4.2 Theory

When developing the model, the following assumptions were made by Markowitz:

1. Risk of a portfolio is based on the variability of returns from said portfolio.
2. An investor is risk averse.

3. An investor prefers to increase consumption.
4. The investor's utility function is concave and increasing, due to their risk aversion and consumption preference.
5. Analysis is based on a single period model of investment.
6. An investor either maximizes their portfolio return for a given level of risk or minimizes their risk for a given return.
7. An investor is rational in nature.

4.3 Model

According to the model ([9]) the following steps should be made to reach the optimal portfolio:

1. Determination of the set which contains the efficient portfolios.
2. Selection of the best portfolio from the set of efficient portfolios.

4.3.1 Notation

The following notation is used:

1. w: the column vector of portfolio weights
2. w^*: Markowitz's optimal portfolio
3. σ^2: the variance of the portfolio
4. r_i: expected return of asset number i
5. r_f: risk free rate
6. r_p: expected return of the portfolio
7. w_f: weight of the risk free asset in percent of the whole portfolio
8. μ: the column vector of expected (excess) return
9. Σ: the covariance matrix (of the asset returns, which are considered as random variables)
10. δ: risk aversion parameter (stated by the investor, in reflection of the trade-off ratio between risk and return, discussed in subsection 2.4)
11. k: number of assets in the portfolio

We set:

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{bmatrix}$$

$$e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Hence we get:

$$e \times r_f = \begin{bmatrix} r_f \\ r_f \\ \vdots \\ r_f \end{bmatrix}$$

4.3.2 Determination of the set of efficient portfolios

We can approach the determination of the attainable portfolios we can either:

- minimize the variance of the portfolio in a given level of expected return

$$\begin{cases} min_w w^T \Sigma w \\ w^T r = r_p \end{cases} \tag{3}$$

- maximize the expected return for a certain level of risk (variance)

$$\begin{cases} max_w w^T r \\ w^T \Sigma w = \sigma^2 \end{cases} \tag{4}$$

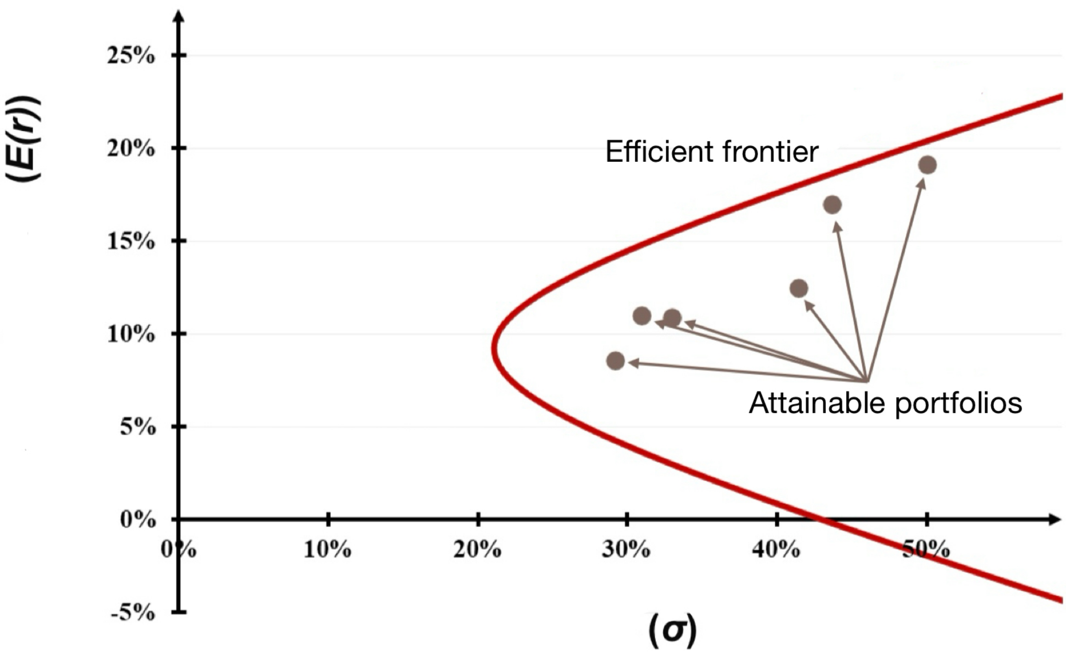


Figure 10: Set of attainable portfolios

Let us now include a risk free asset. Assuming that we have d risky assets, the weight of the risk free asset in the portfolio is:

$$w_f = 1 - e^T w$$

Then r (expected return of the portfolio) is:

$$r_p = w^T r + w_f r_f$$

Let us expand the return as:

$$r_p = w^T r + (1 - w^T e) r_f = w_t (r - e r_f) + r_f$$

Now we define μ as the expected excess return as:

$$\mu \equiv r - e r_f = \begin{bmatrix} r_1 - r_f \\ r_2 - r_f \\ \vdots \\ r_d - r_f \end{bmatrix}$$

4.3.3 Selection of the best portfolio

Let us introduce the risk aversion parameter δ. Now the following problem needs to be solved:

$$max_w r_f + w^T \mu - \frac{\delta}{2} w^T \Sigma w$$

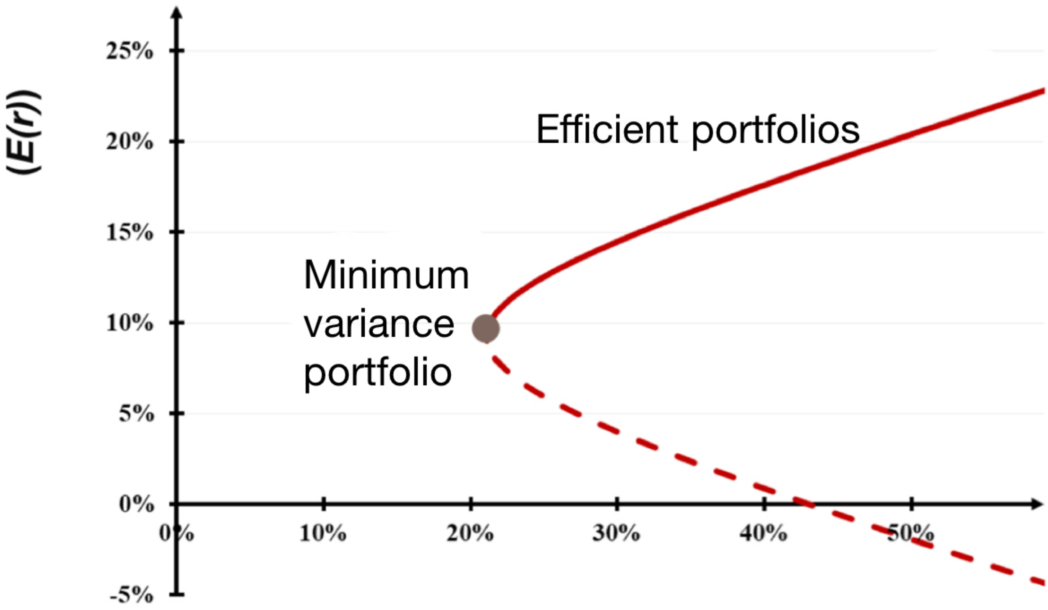


Figure 11: Set of efficient portfolios

Since r_f is a constant, we can have the same result by removing it from the problem:

$$max_w w^T \mu - \frac{\delta}{2} w^T \Sigma w$$

Let us set:

$$e_k^T = [0...010...0], \text{number of elements equals number of assets, 1 when entry } k$$

Let us differentiate the function and set it equal to zero:

$$e_k^T \mu - \frac{\delta}{2} w^T \Sigma w - \frac{\delta}{2} w^T \Sigma e_k = 0$$

$$e_k^T (\mu - \delta \Sigma w) = 0$$

This is true for all $k = 1...d \Rightarrow w^* = (\delta\Sigma)^{-1}\mu$, where w^* represents the Markowitz optimal portfolio given the risk aversion coefficients, covariance matrix and the vector of expected returns by the investor.

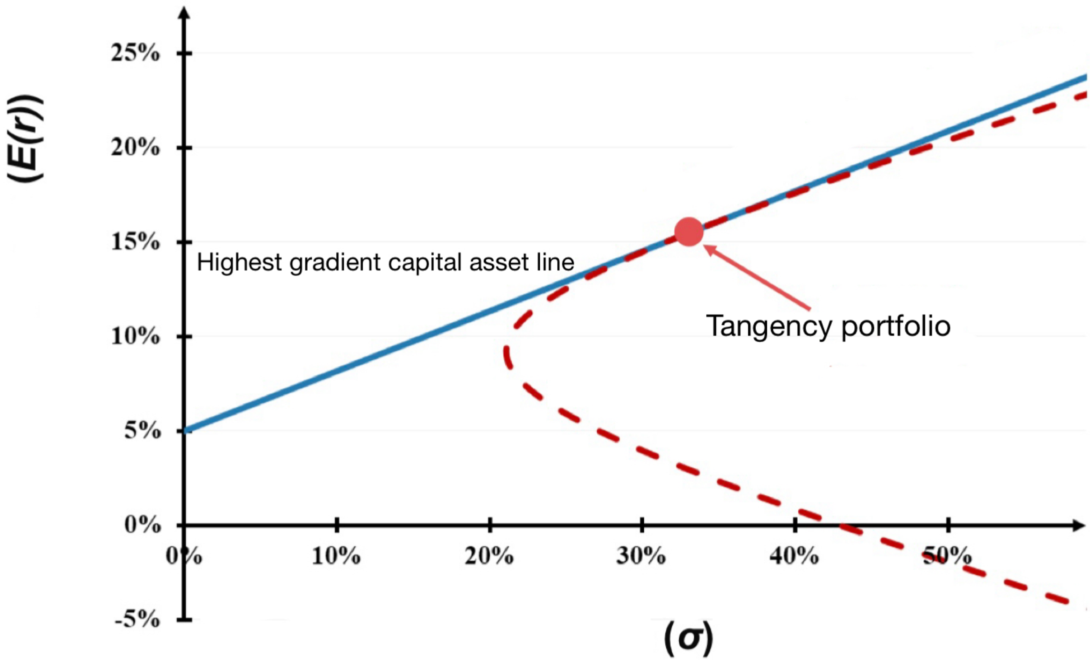


Figure 12: Tangency portfolio

4.4 Deficiencies

The Markowitz model might seem appealing from a theoritical point of view, but several problems arise when using in real-life situations, thus the model is rather interpreted as a framework of modeling markets than actual solution for portfolio optimization. His research in the topic earned Markowitz (and fellow researcher William F. Sharple and Merton Miller) a Nobel Prize in 1990.

The most important problems in using the Markowitz model are the following:

1. The model maximizes errors according to Michaud [10]. The model usually overweights high expected returns and low correlations, and underweights low expected return and positive correlation.
2. The model does not count for asset's market capitalization. It often suggests high allocation in low capitalized assets, which is actually a problem, when adding a short constraint (short constraint means that taking short positions is prohibited, investors can not use the technique to sell assets with plans to buy it back later at a lower price).
3. The model does not differentiate within the uncertainty of the inputs of the model.
4. Mean-variance models are often unstable, meaning that a small change in the input might dramatically change the portfolio [3].
5. The model often suggests large negative weights in assets, but fund managers are usually permitted to take short positions. If a short constraint is added, the model gives zero weights to many of the assets, and high weights only to a small number of assets, witch lead to a non-diversified portfolio.
6. Estimating covariances between assets is problematic. In a portfolio, containing k assets, k variances and k expected returns are calculated, but the number of covariances that need

to be estimated is $\frac{k(k-1)}{2}$. According to Markowitz *"in portfolios involving large numbers of correlated securities, variances shrink in importance compared to covariances"* [8].

These deficiencies led to further research, which resulted in the Capital Asset Pricing Model, mainly developed by Sharpe.

5 The Capital Asset Pricing Model

The Capital Asset Pricing Model (further referred as CAPM) aims to determine an assets (theoritically) appropriate return to its risk, with consideration of the assets sensitivity to the systematic risks. This section will provide the model for defining the equilibrium state of the market, which will be the starting point for Black–Litterman model.

5.1 Model

The main difference between the Markowitz model and the CAPM is that the CAPM model calculates the appropriate asset prices, it does not have suggestions about possible allocations of assets in the portfolio. However, the Markowitz model assigns weights to said assets in the portfolio.

The following assumptions are made:

1. Investors aim to maximize economic utility.
2. Investors are risk averse and rational.
3. Portfolios are diversified across a range of investments.
4. Investors are price takers, meaning they cannot individually influence prices.
5. Investors can lend and borrow any amount under the risk free rate.
6. Investors can trade without taxation or transaction costs
7. All assets are divisible and liquid.
8. Investors have homogeneous expectations about the market.
9. All information is available at the same time for all investors.

The CAPM states, that the fair price of an investment is the risk free rate plus the markets excess return above the risk free rate weighted by β. (Definition 2.3).

Definition 5.1 (The Capital Asset Pricing formula)**.** *[13]*

$$E(R_i) = R_f + \beta_i(E(R_m) - R_f)$$

where:

$E(R_i)$ *= expected return of investment i*

R_f*= risk-free rate*

β_i*=* β *of investment i*

$(E(R_m) - R_f)$*= market risk premium*

Definition 5.2 (The Capital Market Line)**.** *[13]*

The capital market line represents portfolios where risk and return are optimally combined. The line calculated as:

$$R_p = R_f + \frac{R_T - r_f}{\sigma_T}\sigma_p$$

where:

R_p= *return of the portfolio*

r_f= *risk-free rate*

R_T= *market return*

σ_T= *standard deviation of the market returns*

σ_p= *standard deviation of the portfolio*

The intercept point of the Efficient Frontier (the curve where efficient portfolios have taken place) and the Capital Asset Line results in the most efficient portfolio, the tangency portfolio.

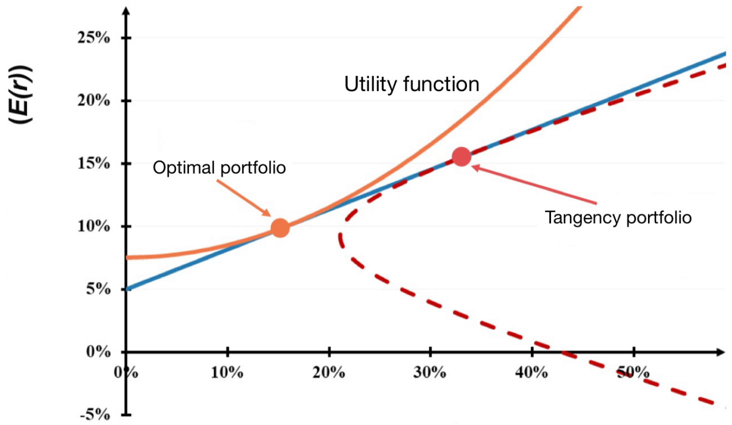


Figure 13: Markowitz's optimal portfolio

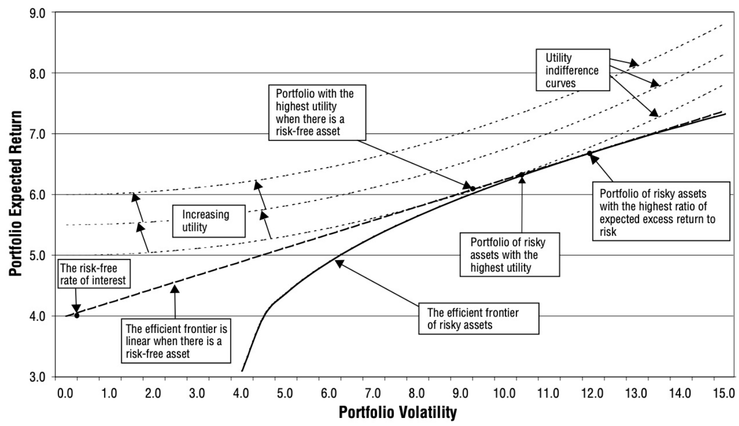


Figure 14: Markowitz's optimal portfolio [6]

5.2 Deficiencies

1. The CAPM model uses future returns for estimating future prices.
2. The model does not assume that investors can rebalance their portfolios over time.

6 About the equilibrium approach

There are several ways to approach investing, but the Black-Litterman model is founded on the equilibrium approach. In dynamic systems, equilibrium is an idealized point, where forces are balanced. In the terms of financial markets, equilibrium is a state, when supply equals demand. Although Litterman admits that this state never actually holds, the natural forces of financial markets (arbitrageurs and highly intelligent investors who can take advantage of certain unique situations), will always push the market back to the equilibrium state, thus it behaves like a centre of gravity.

Please note, the fact that from now on, we try to model the world-wide financial market, but ignore the fact, the risk and return looks different for different nationalities, because of currency exchange rates. In this manuscript, investors are investing in the same currency, and investments are denominated in the same currency.

Litterman suggests that the CAPM equilibrium is a great starting point for the Black–Litterman theory.

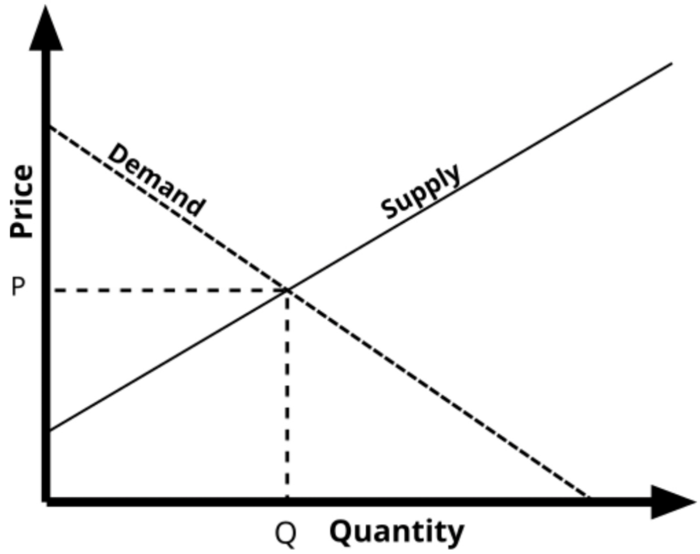


Figure 15: The state of economic equilibrium

7 Black–Litterman Model

7.1 Main considerations

Holding equilibrium portfolios might be a good idea for an investor with no additional financial background, but highly intelligent investors (for example professionals, like portfolio managers) usually have expectations about the market, and they would like to utilize them to have a higher expected return. Thus, they need a financial model, which can handle views, and the probability signed to the occurring of the expected situation in question. This problem concerned Litterman at Goldman Sachs, in the same time that Fischer Black finished his paper *"Universal hedging"* [4]. The two economists worked together developing the Black–Litterman model to find a real-life working solution for portfolio optimization.

7.1.1 Bayesian and sampling approach

The model can be approached two different ways: through sampling theory and bayesian statistics. The bayesian approach is the most commonly used (Litterman used this in his book, *"Modern investment management" [6]*), the sampling theory is not well-known, and has a little bibliography. The most detailed one is *"The Black–Litterman Model - Towards its use in practice"*, a PhD dissertation by Charlotta Mankert [7]. According to Mankert, the two methods have nearly the same results. In the following sections, when referring to the Black–Litterman model, we mean the original bayesian approach.

7.2 Theory

This chapter is strongly based on the article *"A step by step guide to the Black–Litterman model - Incortporating user specified confidence levels"* by Thomas M. Idzorek. [1]

When developing the model, Litterman had many assumptions, both about financial modeling and portfolio modeling. The following list contains the usual assumptions when modeling financial markets and portfolios:

1. Investors are rational.
2. Returns are normally distributed.
3. Arbitrage is absent.
4. Wealth has a decreasing marginal utility.
5. Investors are risk-averse.
6. Increased expected return is considered a positive outcome.
7. There is a trade-off between expected return and risk.
8. Capital markets are efficient, meaning: prices of securities reflect all available information, and prices of individual securities adjust very rapidly to new information.
9. Each investment has a probability distribution of expected returns over a specified holding period.

10. The two factors considered, when investing are risk and return.

11. Investors will choose a portfolio with the highest expected return at a given level of risk, or the portfolio with the lowest level of risk, with a given expected return.

12. The portfolio's risk can be measured by calculating future variances and covariances of all assets.

13. Taxes and transaction costs are not taken into account.

The following list contains the assumptions specifically applying to the Black–Litterman model:

1. Investors have views about all assets, that they believe will lead to better performing portfolios.

2. The market is not entirely efficient (Litterman, 2003.)

3. Portfolios are evaluated according to a benchmark portfolio.

4. To every view, investors calculate a rate of (un)confidence.

7.3 Model

7.3.1 Starting point - Equilibrium returns

The Black–Litterman formula uses the equilibrium returns as a neutral starting point. The equilibrium returns are derived as:

$$\Pi = \lambda \Sigma w_{mkt}$$

where:

- N is the number of assets in the portfolio
- Π is the implied excess equilibrium return vector ($N \times 1$ column vector)
- λ is the risk aversion coefficient
- Σ is the covariance matrix of excess returns ($N \times N$ matrix)
- w_{mkt} is the market capitalization weight of the assets ($N \times 1$ column vector), where for every w_{mkt_i} is the market capitalization of Asset i divided by the total market capitalization

By rearranging the formula and substituting μ (representing any vector of excess return) for Π leads to the formula:

$$w = (\lambda \Sigma)^{-1}$$

which is the solution of the maximization problem $max_w w^T \mu - \frac{\delta}{2} w^T \Sigma w$, which we got in the Markowitz model. If μ does not equal Π, w will not equal w_{mkt}.

The Implied equilibrium return vector (Π) is the market-neutral starting point of the Black–Litterman model.

7.3.2 Views

More often than not, professionals have a specific views of some assets in the portfolio, which differ from the Implied equilibrium return.

When using the Black–Litterman model, two different kinds of views can be expressed: absolute and relative views.

An absolute view only expresses expectation about one asset, not taking the other assets into consideration (for example: *"US Bonds will perform* 5%*."*. A relative view contains 2 or more asset classes, and the view expresses some level of difference between their performance (for example: *"US high yield bonds will outperform US treasury bonds by* 4%*."*.)).

Each view has a rate of confidence assigned to them.

Let us see a combined view: *"US Small Cap Equity and US Value will outperform US Large Cap Growth by 3%, with 70% confidence."*. As we can see, the number of underperforming assets does not need to equal the number of outperforming assets. The assets in the view form two separate mini-portfolios, a long portfolio and a short portfolio. The relative weight of each nominally outperforming asset proportional to the asset's market capitalization divided by the sum of sum of the market capitalization of the other nominally outperforming assets of that particular view, and the same applies to underperforming assets. The net long positions less the net short positions equal zero.

7.3.3 The Black–Litterman Formula

Theorem 7.1 (Black–Litterman formula)**.** $E[R] = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q]$

where:

N	*is the number of assets in the portfolio;*
K	*is the number of views expressed;*
$E[R]$	*is the new (posterior) Combined return vector ($N \times 1$ column vector);*
τ	*is a scalar;*
Σ	*is the covariance matrix of excess returns ($N \times N$ matrix);*
P	*is the matrix that identifies the assets involved in the views ($K \times N$ matrix);*
Ω	*is the diagonal covariance matrix of error terms ($K \times K$ matrix)*
Π	*is the Implied equilibrium return vector ($N \times 1$ column vector);*
Q	*is the view vector ($K \times 1$ column vector);*

7.3.4 Building the input

The model does not require expressing views for all assets (however, theoretically, by expressing one view, and taking the correlations into account, we express indirect views about certain assets).

The uncertainty of the views result in a random, unknown, independent, normally-distributed variable ε, called the Error term vector, with a mean of 0, and a covariance matrix Ω. Thus, a view has a form of $Q + \varepsilon$.

$$Q+\varepsilon = \begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}$$

The Error term vector (ε) does not goes directly into the model, however the variance of each error term (ω: the absolute difference from the error term's expected value of 0) does enter the formula. The off diagonal elements of Ω are 0's, because the model assumes that the views are independent of one another. The variances of the error terms (ω) represent the uncertainty of the views. Larger ω represents larger uncertainty of the view.

$$\Omega = \begin{bmatrix} \omega_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_k \end{bmatrix}$$

Each expressed view results in a $1 \times N$ row vector, thus K views result in a $K \times N$ matrix.

$$P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{k,1} & \cdots & p_{k,n} \end{bmatrix}$$

After defining P, we can calculate the variance of each individual view portfolio.

According to Idzorek: *"Conceptually, the Black–Litterman model is a complex, weighted average of the Implied Equilibrium Return Vector (Π) and the View Vector (Q), in which the relative weightings are a function of the scalar (τ) and the uncertainty of the views (Ω). "* [1]

The scalar τ is approximately inversely proportional to the relative weight given to Π. More or less every economist using the model calibrates the scalar in a different way, and has different reasons. For example Lee typically sets τ between 0,01 and 0,05, but both Black, Litterman and Lee argues that the scalar is close to zero. Black and Litterman usually sets τ by calibrating the confidence of views so that the ratio of τ/ω equal to the variance of the view portfolio. When Ω is calculated this was, the actual value of τ is irrelevant, only the ratio τ/ω enters the model.

$$\Omega = \begin{bmatrix} (p_1 \Sigma p_1)\tau & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (p_k \Sigma p_k)\tau \end{bmatrix}$$

Now all of the inputs of the Black–Litterman model are set.

7.3.5 Calculating the new Combined return vector $E[R]$

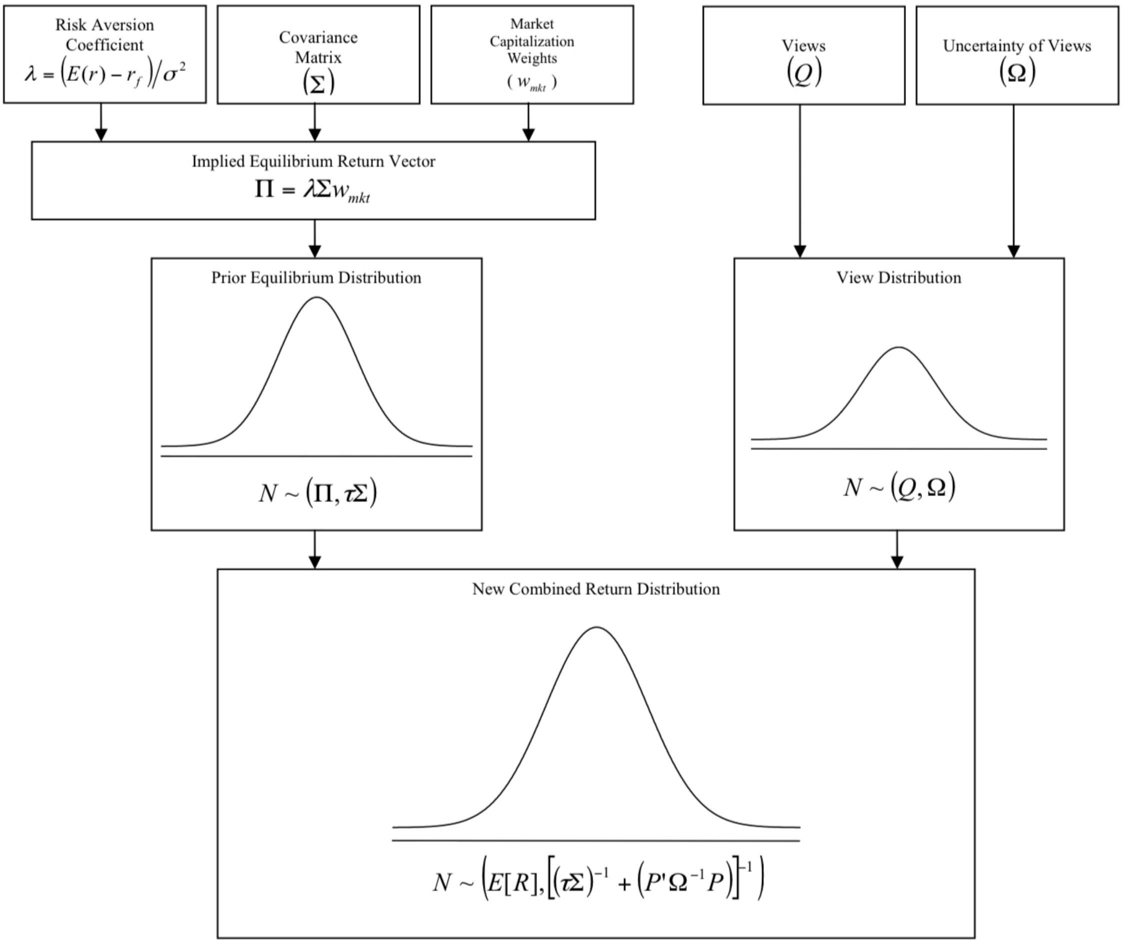


Figure 16: Deriving the new combined return vector [1]

7.4 Deficiencies and further research

7.4.1 The matrix Ω is complicated to determine

According to Litterman, *"representing the uncertainty of the views, is a common question without a universal answer"*. Some methods (like the presented linear regression) naturally suggest metrics to determine their punctuality (in the case of linear regression, the length of the confidence interval is a great indicator), but in other, rather economical, not statistical methods, uncertainty is harder to determine. In article mentioned above, Idzorek suggests a way to simplify the determination of the matrix Ω. [1]

7.4.2 The value of τ needs adjusting based on previous experience

On the value-setting of τ there are only suggestions. Researchers advise investors to set the parameter close to zero, and approximately between 0.01 and 0.05, but the investor need to incorporate previous experience when building the input, to determine their own τ. [1]

7.4.3 Reliance on the basis models

Since the Black–Litterman model relies on several other models, their assumptions most hold, which creates more possible deficiencies, especially when markets are not near equilibrium for a short period of time. [1]

8 Black–Litterman in use

In the following section I aim to walk the reader through a concrete example of using the Black–Litterman model. Let us imagine the situation where analysts are interested in whether the emerging market stocks will outperform the developed market stocks, and if yes, how should investors alter their positions to benefit from this specific view.

8.1 Input construction

8.1.1 Parameters r_f, λ, τ

- risk free rate = 5%
- risk aversion (λ) = 1,5
- scalar (τ)=0,01

8.1.2 Equilibrium allocation Π

Most investors use the MSCI World Index as their benchmark. The index geographically allocates as the following:

Region	Allocation (%)
North-America	65,4%
Pacific Region	8,7%
Developed Europe	14,5%
Emerging Asia	8%
Latin-America	2%
Eastern-Europe (ex Russia)	1,4%

Each region is represented by a selected ETF (exchange traded fund), this way we can measure performance:

Region	Respresenting ETF
North-America (NA)	iShares MSCI North America UCITS ETF USD (Dist)
Pacific Region (PR)	iShares Core MSCI Pacific ETF
Developed Europe (DE)	iShares Core MSCI Europe ETF
Emerging Asia (EA)	iShares MSCI Emerging Markets Asia ETF
Latin-America (LA)	iShares MSCI EM Latin America UCITS ETF USD D
Eastern-Europe (ex Russia) (EE)	Amundi MSCI Eastern Europe Ex Russia UCITS ETF

All index and ETF prices and information are from Bloomberg. Since we have the representing ETF's, we can calculate correlations from the historical data.

8.1.3 Covariance matrix Ω

Calculating from the historical data, the correlation matrix and the volatility vector are the following:

Correlation	NA	PR	DE	EA	LA	EE
NA	1	0,805	0,818	0,616	0,577	0,672
PR	0,805	1	0,859	0,802	0,529	0,730
DE	0,818	0,859	1	0,708	0,654	0,861
EA	0,616	0,802	0,708	1	0,444	0,622
LA	0,577	0,529	0,654	0,444	1	0,683
EE	0,672	0,730	0,861	0,622	0,682	1

$$Volatility = \begin{bmatrix} 0,156389 \\ 0,14172 \\ 0,171227 \\ 0,17254 \\ 0,27844 \\ 0,300299 \end{bmatrix}$$

After multiplying the correlation (either from the left and the right side) with the volatility vector of the assets, we divide by the risk aversion parameter, hence we get the covariance matrix:

Covariation	NA	PR	DE	EA	LA	EE
NA	0,000245	0,000178	0,000219	0,000166	0,000251	0,000316
PR	0,000178	0,000201	0,000208	0,000196	0,000209	0,000311
DE	0,000219	0,000208	0,000293	0,000209	0,000312	0,000442
EA	0,000166	0,000196	0,000209	0,000298	0,000214	0,000322
LA	0,000251	0,000209	0,000312	0,000214	0,000775	0,000571
EE	0,000316	0,000311	0,000442	0,000322	0,000571	0,000902

8.1.4 Views Q

In this example, two views are expressed by the investor:

1. In general, emerging markets will outperform developed markets by 3,7%.

2. The eastern european stock market will outperform the Asian stock market by 5%.

Although the Black–Litterman model does not discuss how views are made, I aim to walk the reader through one example while constructing View 1.

As seen above, various indicators are used to predict the performance of the stock market. We will use linear regression to forecast future prices, and the chosen indicator is P/S (price to sales ratio), because compared to other indicators, linear regression with P/S has a relatively high R^2.

Correlation of log valuation ratios with actual subsequent S&P 500 total returns: 1950-2017		
Metric	10-year	12-year
Nonfinancial market cap/ Corporate gross value-added (Hussman 5/18/15)	-0.91	-0.93
Nonfinancial enterprise value/ Corporate gross value-added	-0.89	-0.91
Nonfinancial market cap/ Nominal GDP	-0.89	-0.90
Price/Revenue	-0.89	-0.90
Margin-adjusted CAPE (Hussman 5/5/14)	-0.89	-0.90
Price/normalized forward earn (Hussman 8/2/10)	-0.88	-0.89
Tobin's Q	-0.86	-0.88
Shiller CAPE	-0.83	-0.86
Price/prior record earnings (Hussman 6/22/98 Barron's)	-0.83	-0.83
Price/Book value	-0.77	-0.79
Price/Forward operating earn (imputed prior to 1980)	-0.78	-0.77
Price/Dividend	-0.77	-0.75
Price/Earnings (trailing 12-mo)	-0.76	-0.75
Enterprise value/Cash flow	-0.70	-0.72
Fed Model (FOE yield-10yr UST)	0.33	0.30
Hussman Strategic Advisors		

Data: Federal Reserve Economic Database, Robert Shiller, Standard & Poors

Figure 17: Commonly used metrics and their R^2

While constructing the views, our tactic is to forecast the expected ten year return, and then calculate the yearly return back, because while linear regression approximate future prices very well in the long run is tends to mistake in short term, as below:

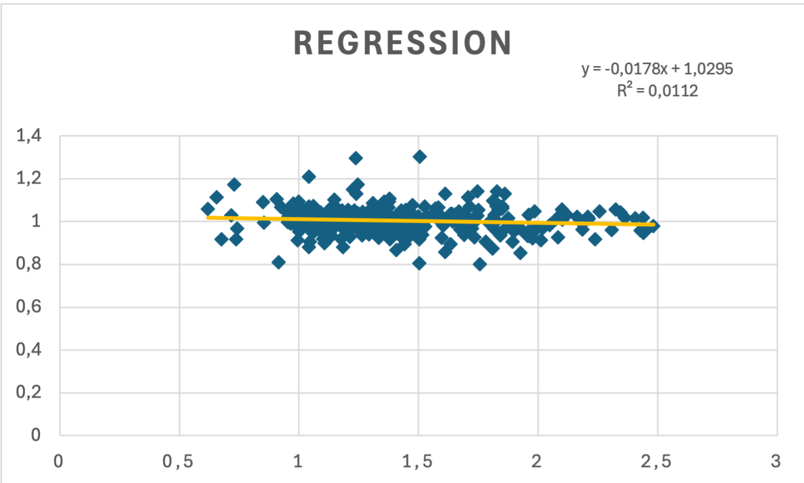


Figure 18: Predicted and actual 12 month return

On the contrary, on the long run, valuation of prices (interpreted through P/S or P/E) explain return.

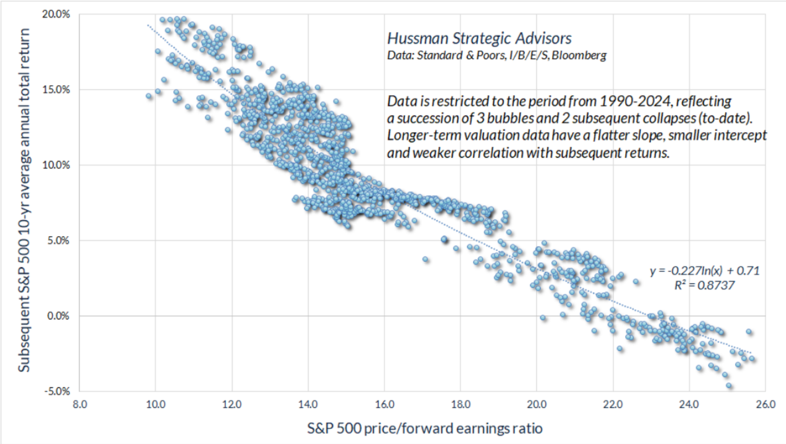


Figure 19: Long run linear regression for forecasting stock market return

The original data contains monthly P/S metrics for both indexes (running back to 1996) and the index's P/S ratio for set dates. The data point for regression are constructed in the way as in section 3, the mathematical framework. After that, we calculate two ratios:

1. The explanatory variable: $\frac{P/S_{emerging}}{P/S_{developed}}$
2. The dependent variable: $\frac{r_{emerging}}{r_{developed}}$

The solution of the linear regression is the following:

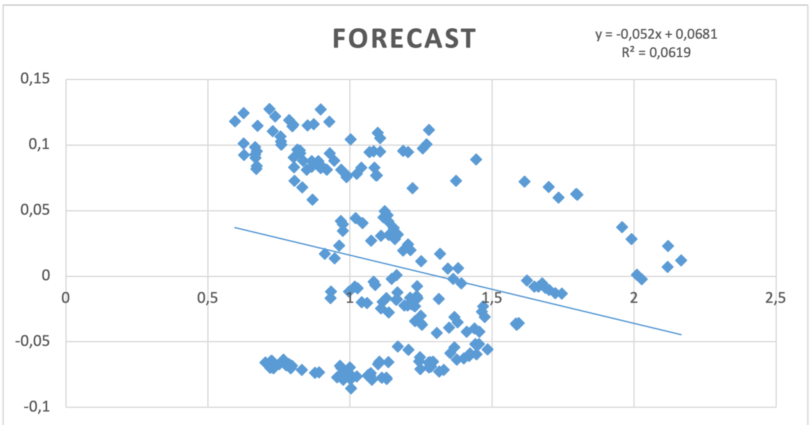


Figure 20: Price difference in relation to the ratio of P/S's

Regression Statistics	
Multiple R	0,321757
R^2	0,103528
Adjusted R^2	0,099397
Standard error	0,058562
Observations	219

The value of R^2 is rather low, due to the emerging market component (the forecast for only developed markets perform $R^2 = 0,97$), which is a more hectic and volatile market due to several government restriction for money markets, and political risk.

The forecasted excess return of emerging market according to the regression is $3,7\%$., hence the view vector is:

$$Q = \begin{pmatrix} 0,05 \\ 0,037 \end{pmatrix}$$

8.2 New Black–Litterman allocation

With the parameters set as above, the model suggests the following allocation:

Region	Equilibrium allocation	Black–Litterman allocation	Difference
North-America	65,4%	63,919%	1,48%
Pacific Region	8,6%	7,219%	1,48%
Developed Europe	14,5%	13,019%	1,48%
Emerging Asia	8%	9,23%	-1,23%
Latin-America	2%	3,48%	-1,48%
Eastern-Europe (ex Russia)	1,4%	3,13%	-1,73 %

8.2.1 Interpreting results

The model suggests a new, Black–Litterman allocation, with moderate changes in the equilibrium portfolio. The regions we expressed positive views about, got higher allocations, and the regions we expressed negative views about, got lower allocations. The sum of the value of the alterations equal to zero, since there is no new cash or short positions in this case.

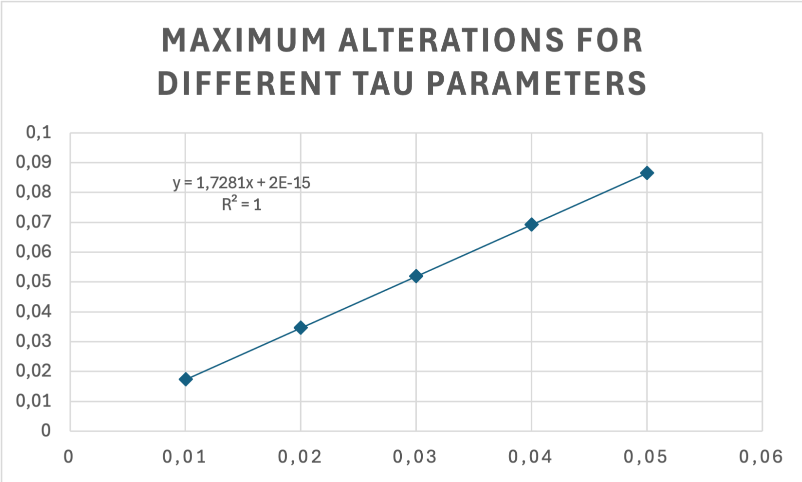
8.3 Parameter alterations and allocations

Since the parameters τ and risk-aversion are set by the investor, and not calculated from actual data, we can examine how their alterations affect the results in this case.

Since views alter the equilibrium portfolio in the way that when positive views expressed, allocations can be higher, and in the case of negative views, allocations are lower, we do not need to examine the directions of alterations, rather the size of alterations suggested in the portfolio. Intuitively, if markets are efficient, investors do not alter their positions largely, but by setting the parameters, the model can suggest bigger movements on the portfolio.

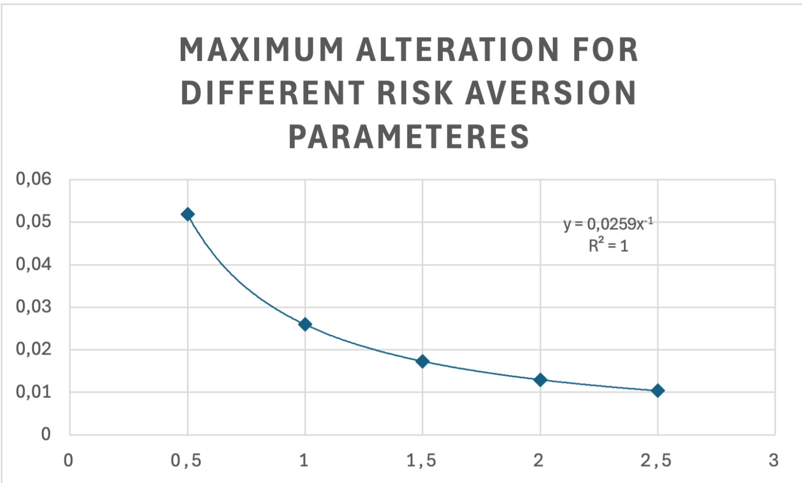
8.3.1 τ alteration

Since the solution is linearly related to the value of tau, there is no surprise, that τ and the maximum alteration are linearly dependent, with $R^2 = 1$.



8.3.2 Risk aversion alteration

The risk aversion parameter is calculated in the Implied Equilibrium Vector (Π), which is in inverted in the Black–Litterman formula, hence the maximum alteration is inversely proportional to the value of the risk aversion parameter, and the quadratic regression equation $y = 0,0259x^{-1}$ fits the data point with $R^2 = 1$.



9 Summary

In this thesis I aimed to introduce the Black–Litterman model, which (unlike other financial models) can handle the situation, when an intelligent investor has independent views about future market performance, hence suitable to manage and constantly alter existing portfolios.

Since the Black–Litterman model relies on many other models and assumptions, I considered introducing them necessary: in the first sections, basic economical concepts are derived, besides the mathematical framework, where I aimed to demonstrate the key theoretical background, on which the model relies.

After the mathematical and economic framework, the two most important models are viewed, on which most investment strategies rely: the Markowitz model and the Capital asset pricing model.

In the short section 6, the brief summary of the equilibrium approach is detailed, which is the main philosophy of the Black–Litterman model.

At this point, all previous concepts are derived to the Black–Litterman model. In section 7, I introduced the main elements of the model along with the formula, and the detailed description of the input parameters. In the end of the section, some deficiencies are listed.

In section 8, I walk through the reader a real-life example of using the model with actual market data. This example was actually an inspiration for considering the Black–Litterman model as a topic for my thesis, because valuations across regions suggests major alterations on investment portfolios, due to the high, above average valuation of developed market stock markets, and lower valuation of emerging market stock markets, but since the emerging market's weight in the market (equilibrium) portfolio is lower, in spite of the fact that return expectations are higher, allocating too much to this regions adds higher volatility to the portfolios. I consider the model especially useful in these scenarios, when an already managed investment portfolio needs adjusting to the current circumstances.

Besides deriving the model, I constructed one from the two example views, View 1, by performing linear regression. Firs, I confirmed, that the selection of the explanatory variable is adequate by citing other research with actual data, and then I estimated a 3,7 % yearly overperformance in emerging markets on a 10 year horizont. The model suggested a subtle allocation change from the developed market to the emerging markets.

After interpreting the results, I analyzed how the alteration of the two investor-dependent parameters, τ and risk aversion change the size of the possible suggested alterations.

Possible future research may include taking currency rates, and views about them into consideration.

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10 Appendix

The data, and the analysis performed are accessible in THIS Github folder.

Black–Litterman model_final: The first sheet, the Black–Litterman sheet, the input cells are marked with green, and the result cells are market with blue to help the reader.

View_final: Contains the data for linear regression, the regression itself, and the View 2. estimation.

Black–Litterman model_final_input_alteration: In the third file, the first sheet is the same, as in the model, it is suitable for altering parameters. On the other sheets, the comparison of outputs takes place.

linearregression_example_1year_10year: In the fourth file, the example regression in the mathematical framework is derived.