Financial bubbles from an epistemic viewpoint

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A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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Egri Máté

a hallgató aláírása

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Abstract

In the realm of games of incomplete information, modeling the beliefs among multiple agents presents significant challenges, especially when considering belief hierarchies — agents' beliefs about others' beliefs, and so forth. Harsányi's (1967) introduction of types offers a notion to navigate these complexities, representing agents' beliefs through a type function. This research aims to adapt the type space model to the phenomenon of financial bubbles, a concept that, surprisingly, lacks a universally accepted rigorous mathematical definition in existing literature. We seek to formally define financial bubbles and establish a common prior that facilitates the emergence of such a type space. By analyzing the conditions necessary and sufficient for a financial bubble to form, our study seeks to provide insights into the machinery underlying speculative bubbles within the context of type spaces.

Chapter 1

Introduction

1.1 Financial Bubbles

Financial bubbles[1] are characterized by the escalation of asset prices beyond their intrinsic values. That is, the investors are willing to make decisions based on speculating about each other's beliefs, rather than their own valuation of an asset as the sum of discounted future cashflows. Explaining the phenomenon of financial bubbles presents significant difficulties, mainly because they are driven by a complex web of beliefs and information among investors. So far, there is no universally accepted theory describing the causes of financial bubbles. One common explanation, described for example by Law (2016), is the Greater Fool Theory. It stipulates speculative episodes, which are marked by a mutual reinforcement of overvaluation, where the price increases are fueled by the belief that future buyers will be willing to pay even more. The essence of financial bubbles lies in the interplay of beliefs, making them a quintessential example of a scenario dominated by incomplete information and belief hierarchies.

1.2 Type Spaces

In addressing the challenges posed by incomplete information, Harsányi's (1967) seminal work introduced a groundbreaking approach through the concept of type spaces. Types provide a structured way to model the knowledge-belief spaces, encapsulating not only the

[1]Some well-known examples include the Dutch Tulip Mania emerging in 1634 and collapsing in 1637 and the US Dot-com Bubble starting out in 1995 and peaking in 2000.

agents' information but also their beliefs about other agents. By integrating the notion of types into game theory, Harsányi offered a method to analyze games of incomplete information with clarity and precision that was previously unattainable, paving the way for a deeper understanding of strategic interactions under uncertainty.

A key postulate in Harsányi's work is the existence of a common prior, which suggests that agents' beliefs can only differ because of their private information. Since then, the study of type spaces has had common priors as its core focus. The key breakthrough came when Aumann (1976) showed that the presence of a common prior implies the impossibility of an agreement to disagree. Specifically, if players' posteriors are common knowledge, they must align. This insight sparked a series of "no-trade theorems" by various authors, who expanded on Aumann's results in different contexts. Morris (1994) investigated heterogeneous priors to determine conditions under which trades would occur. Hellman (2013) introduced the concept of ε-close priors as a variation to common priors. Samet (1998a) and Feinberg (2000) characterized priors through the analysis of posterior distributions. Hellman and Pintér (2020) further generalized the relationship between disagreement and common priors to uncountably infinite type spaces.

1.3 Beliefs and Markov chains

Consider the scenario where Romeo contemplates how much he is willing to pay for a gold ring. His personal valuation—his first-order belief—reflects what he initially thinks the ring is worth. If he believes Juliet values the ring more highly, he might pay more than his valuation so that he can sell it to her at a profit. His second-order belief is about what her valuation might be. However. Juliet might be engaging in a similar thought process and willing to pay even more if she believes Romeo values the ring highly. This introduces Romeo's third-order belief, which is his assumption about Juliet's thoughts on his valuation of the ring. He can continue the iterative reflections on mutual beliefs and arrive at a price that neither of them believes to be anywhere close to the actual value. The key insight by Samet (1998a) lies in recognizing that a Markov chain can model the chain of beliefs.

Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces. a map $\kappa\colon \Omega_1 \times \mathcal{A}_2 \to [0, \infty]$ is a **Markov kernel**[2] if:

1. $\omega_1 \mapsto \kappa(\omega_1, A_2)$ is $\mathcal{A}_1$-measurable $\forall A_2 \in \mathcal{A}_2$,

2. $A_2 \mapsto \kappa(\omega_1, A_2)$ is a probability measure on $(\Omega_2, \mathcal{A}_2)$ $\forall \omega_1 \in \Omega_1$.

$t_i\colon \Omega \to \Delta(\Omega, \mathcal{A})$ is the type function of player i, which assigns a probability distribution to any state of the world $\omega \in \Omega$, representing i's beliefs about which state we might be in.

We observe that a type function behaves as a Markov kernel, which allows us to define a Markov process on the state space, effectively capturing the sequential chain of beliefs among the participants.

Based on this, we define the Markov transition M_i for each player i. M_i exhibits several desirable properties:

- The set of priors of player i is exactly the set of invariant distributions of M_i.

- Applying M_i to a random variable yields its expectation with respect to player i's type.

- The elements of i's knowledge partition are the irreducible classes of M_i.

In this setting, we can also define the higher-order belief function of a chain of players $i_1, i_2, \ldots, i_k$, which gives the distribution of the beliefs of i_k about the beliefs of i_{k-1} about the beliefs of, and so on... about the beliefs of i_1 about the state space.

After the necessary definitions and results for the Markov model, we turn to our main objectives.

1.4 Objectives

We propose that financial bubbles have not been adequately explained because they lack a universally accepted rigorous mathematical definition. The main objective of this paper is to apply the type space model to financial bubbles, establishing a definition using Harsányi's structures. By doing so, we seek to unravel the causes that facilitate the emergence and persistence of speculative bubbles. Specifically, we will investigate the

[2]See e.g. Klenke (2014)

establishment of a common prior that enables the construction of a type space prone to financial bubbles. We examine the relationship between financial bubbles and common priors, similar to the no-trade theorems above. Additionally, we identify the conditions necessary and sufficient for the formation of a financial bubble. These results are in Chapter 3 which, in its entirety, is our own contribution.

Chapter 2

Type spaces

2.1 Definitions

In this paper we employ a countable type space that is not necessarily finite (such as the one described by Samet (1998b)), yet not as general as the framework proposed by Hellman and Pintér (2020). It is defined as follows:

Definition 2.1. *A tuple* $(N, (\Omega, \mathcal{A}), (\Pi_i)_{i\in N}, (t_i)_{i\in N})$ *is a* ***type space****, if*

- N *is the non-empty, finite set of players,*
- Ω *is the countable set of the states (of the world),*
- Π_i *is the knowledge partition of player* $i \in N$*, where each element of* Π_i *is a finite set,* $i \in N$*,*
- $\mathcal{A}$ *is the field of events over the state space* Ω *such that for each player* $i \in N$ *we have* $\Pi_i \subseteq \mathcal{A}$*,*
- $t_i \colon \Omega \to \Delta(\Omega, \mathcal{A})$ *is the type function of player* $i \in N$ *meeting the properties*
  - *for every pair* $\omega, \omega' \in \pi \in \Pi_i$ *it holds that* $t_i(\omega) = t_i(\omega')$*,*
  - *for every* $\omega \in \pi \in \Pi_i$ *it holds that* $t_i(\omega)(\pi) = 1$*,*

  for all $i \in N$*.*

The interpretation of type space we have in mind in this paper is that of a framework to model the belief hierarchies of the agents.

The type function t_i of agent i assigns a probability distribution to any state $\omega \in \Omega$, representing her belief in that particular state of the world. The true power of the type function lies in its ability to model not just what an agent believes about the world but also what she believes about other agents' beliefs, what she believes about the others' beliefs about her beliefs, ad infinitum.

Any element $\pi \subseteq \Omega$ of the partition Π_i on Ω is a set containing states which to agent i are indistinguishable, hence t_i is constant on. Denote by $\Pi_i(\omega)$ the element of Π_i containing ω. Π_i embodies the definitive knowledge of player i, as opposed to their beliefs. Agent i in the state ω receives information about the world, so they can determine that the world is in one of the states inside the set $\Pi_i(\omega)$. This marks the boundary of knowledge, any further distinction among the states enters the realm of beliefs about probabilities.

Harsányi (1967) also introduced the pivotal "common prior assumption", which stipulates that the players' beliefs are consistent if their types are derived by updating the same distribution based on each player's unique information partition element. For that, we need the following definitions.

Definition 2.2. *A probability distribution $p_i \in \Delta(\Omega, \mathcal{A})$ is a **prior** for agent i if $\forall\, \pi \in \Pi_i$, $\forall\, \omega \in \pi$, $\forall\, E \in \mathcal{A}$ it holds that $p_i(E \cap \pi) = t_i(\omega)(E)p_i(\pi)$.*

Notice that this condition can be given equivalently as $p_i(E\,|\pi) = t_i(\omega)(E)$ whenever $p_i(\pi) > 0$, motivating the intuition behind this notion.

A prior might be interpreted as a representation of the initial beliefs of an agent before any specific information differentiates her type.

Definition 2.3. *A probability distribution $p \in \Delta(\Omega, \mathcal{A})$ is a **common prior** if it is a prior for every agent $i \in N$.*

One perspective on the notion of a common prior is that its existence implies a possible shared baseline belief among all agents before their private information leads to divergent beliefs.

Definition 2.4. *The **meet** Π_N of $(\Pi_i)_{i\in N}$ is the finest partition of Ω which is coarser than any Π_i.*

The meet will prove to be an important tool for relating the knowledge partitions of the players.

2.2 An example

Something similar to the following simple example was given by Allen and Morris (2001), which demonstrates bubble behavior in a financial setting, due to the interplay between the higher-order beliefs of the agents participating.

Romeo and Juliet are two depositors in a bank, and they have liquidity values $\omega_1, \omega_2 \in \mathbb{Z}$. Both of them know that the difference between their liquidity value is at most 1 (beyond that they assume uniform distribution). If the liquidity value is smaller than 0, the player has to withdraw, otherwise, he can choose between withdrawing and keeping it in the bank with the following payoffs:

	Remain	Withdraw
Remain	110 | 110	0 | 100
Withdraw	100 | 0	100 | 100

We can write this in the type space setting:

- The state space is as follows:

$$\Omega = \{(\omega_1, \omega_2) \in \mathbb{Z}^2 \colon |\omega_1 - \omega_2| \leq 1\}$$

- The player set:

$$N = \{1, 2\}$$

- The knowledge partitions:

$$\Pi_1 = \{\{(\omega_1, \omega_1 - 1), (\omega_1, \omega_1), (\omega_1, \omega_1 + 1)\} : \omega_1 \in \mathbb{Z}\}$$

  and

$$\Pi_2 = \{\{(\omega_2 - 1, \omega_2), (\omega_2, \omega_2), (\omega_2 + 1, \omega_2)\} : \omega_2 \in \mathbb{Z}\}$$

- The type functions: for every $(\omega_1, \omega_2) \in \Omega$

$$t_1(\omega_1, \omega_2)(\{(\omega_1, \omega_1 - 1)\}) = t_1(\omega_1, \omega_2)(\{(\omega_1, \omega_1)\}) = t_1(\omega_1, \omega_2)(\{(\omega_1, \omega_1 + 1)\}) = \frac{1}{3}$$

  and

$$t_2(\omega_1, \omega_2)(\{(\omega_2 - 1, \omega_2)\}) = t_2(\omega_1, \omega_2)(\{(\omega_2, \omega_2)\}) = t_2(\omega_1, \omega_2)(\{(\omega_2 + 1, \omega_2)\}) = \frac{1}{3}$$

Suppose we are in the state $\omega = (\omega_1, \omega_2) = (2, 2)$. From Romeo's perspective, there is no apparent reason to withdraw as not only does he know that Juliet's liquidity is non-negative, but he also knows that Juliet knows that his liquidity is non-negative. Likewise, the same can be stated from Juliet's viewpoint. However, this chain of knowledge doesn't extend to higher degrees of belief, which can cause the phenomenon of a financial bubble. Romeo believes ω_2 to be 1,2 or 3 with probabilities $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, that is

$$t_1(\omega)(\{(\omega_1', \omega_2') \in \Omega \colon \omega_2' = 1\}) = \frac{1}{3},$$
$$t_1(\omega)(\{(\omega_1', \omega_2') \in \Omega \colon \omega_2' = 2\}) = \frac{1}{3},$$
$$t_1(\omega)(\{(\omega_1', \omega_2') \in \Omega \colon \omega_2' = 3\}) = \frac{1}{3}.$$

Juliet believes with probability $\frac{1}{3}$ that Romeo believes ω_2 to be 0, 1, or 2 with probabilities $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.

Juliet also believes with probability $\frac{1}{3}$ that Romeo believes ω_2 to be 1, 2, or 3 with probabilities $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.

Juliet also believes with probability $\frac{1}{3}$ that Romeo believes ω_2 to be 1, 2, or 3 with probabilities $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.

Adding up the probabilities for the cases gives the expressions:

$$t_2(\omega)\left(\left\{\omega' \in \Omega \colon t_1(\omega')(\{(\omega_1'', \omega_2'') \in \Omega \colon \omega_2'' = 0\}) = \frac{1}{3}\right\}\right) = \frac{1}{3},$$
$$t_2(\omega)\left(\left\{\omega' \in \Omega \colon t_1(\omega')(\{(\omega_1'', \omega_2'') \in \Omega \colon \omega_2'' = 1\}) = \frac{1}{3}\right\}\right) = \frac{2}{3},$$
$$t_2(\omega)\left(\left\{\omega' \in \Omega \colon t_1(\omega')(\{(\omega_1'', \omega_2'') \in \Omega \colon \omega_2'' = 2\}) = \frac{1}{3}\right\}\right) = 1,$$
$$t_2(\omega)\left(\left\{\omega' \in \Omega \colon t_1(\omega')(\{(\omega_1'', \omega_2'') \in \Omega \colon \omega_2'' = 3\}) = \frac{1}{3}\right\}\right) = \frac{2}{3},$$
$$t_2(\omega)\left(\left\{\omega' \in \Omega \colon t_1(\omega')(\{(\omega_1'', \omega_2'') \in \Omega \colon \omega_2'' = 4\}) = \frac{1}{3}\right\}\right) = \frac{1}{3}.$$

Multiplying[1] the layers of beliefs, we get the distribution $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$, $\frac{2}{9}$, $\frac{1}{9}$ based on Juliet's second-order beliefs.

We can also continue this line of reasoning to get the following distributions of ω_2 based on the higher-order beliefs:

[1]This is a sensible approach to what weight a player might assign to an event based on a chain of beliefs. This is to be discussed and formalized in Section 2.4.

-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
					$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$					
				$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$				
			$\frac{1}{27}$	$\frac{3}{27}$	$\frac{6}{27}$	$\frac{7}{27}$	$\frac{6}{27}$	$\frac{3}{27}$	$\frac{1}{27}$			
		$\frac{1}{81}$	$\frac{4}{81}$	$\frac{10}{81}$	$\frac{16}{81}$	$\frac{19}{81}$	$\frac{16}{81}$	$\frac{10}{81}$	$\frac{4}{81}$	$\frac{1}{81}$		
	$\frac{1}{243}$	$\frac{5}{243}$	$\frac{15}{243}$	$\frac{30}{243}$	$\frac{45}{243}$	$\frac{51}{243}$	$\frac{45}{243}$	$\frac{30}{243}$	$\frac{15}{243}$	$\frac{5}{243}$	$\frac{1}{243}$	
$\frac{1}{729}$	$\frac{6}{729}$	$\frac{21}{729}$	$\frac{50}{729}$	$\frac{90}{729}$	$\frac{126}{729}$	$\frac{141}{729}$	$\frac{126}{729}$	$\frac{90}{729}$	$\frac{50}{729}$	$\frac{21}{729}$	$\frac{6}{729}$	$\frac{1}{729}$

Notice that the distribution in the last row - based on Juliet's sixth-order beliefs - implies that she maximizes her utility by withdrawing, as

$$\frac{78}{729} \cdot 0 + \frac{651}{729} \cdot 110 < 100.$$

By the same argument, Romeo also withdraws.

The example, although straightforward, illustrates two crucial insights. First, it quantitatively demonstrates how higher-order beliefs can escalate to extreme outcomes in interactive scenarios. Second, it reveals that even when utilizing the type space framework to analyze a relatively simple problem, the process of calculating and articulating the chain of beliefs remains complex and challenging. These observations pave the way for the next section, where we explore higher-order beliefs in a new context, enabling clearer definition of terms and more streamlined proof of results.

2.3 Type spaces as Markov chains

The results of this section are based on Samet (1998a), which first explored finite type spaces in the Markov setting. For our model, we notice that the principal results also extend to our countable state spaces, with identical proofs.

From now on we consider the type space $(N, (\Omega, \mathcal{A}), (\Pi_i)_{i\in N}, (t_i)_{i\in N})$ and fix a linear order over the state space Ω.

Also, from now on, suppose $t_i(\omega)(\omega) > 0 \;\; \forall\, i \in N, \; \forall\, \omega \in \Omega$. This sensible assumption ensures aperiodicity later on.

Note that for any player $i \in N$:

1. For any $E^* \in \mathcal{A}$ the map $\omega \mapsto t_i(\omega)(E^*)$ is measurable.

2. For any $\omega^* \in \Omega$ the map $E \mapsto t_i(\omega^*)(E)$ is a probability measure on $(\Omega, \mathcal{A})$.

These conditions precisely define $t_i(\cdot)(\cdot)\colon \Omega \times \mathcal{A} \to [0,1]$ as a Markov kernel. Thus, we can also define a Markov chain.

Definition 2.5. *Given a player $i \in N$. Then player i's Markov transition matrix M_i is an infinite matrix with dimensions $\Omega \times \Omega$ such that*

$$M_i(\omega, \omega') \triangleq t_i(\omega)(\omega').$$

Hence the rows of M_i are the types of player i: $t_i(\omega)$, $\omega \in \Omega$.

For a distribution p over the state space Ω, the distribution $p\,M_i$ over Ω describes what player i believes given the initial distribution p.

Then $p\,M_{i_1}\,M_{i_2}$ is the distribution over Ω which describes what player i_1 believes about what player i_2 believes given the initial distribution p.

Any $p\,M_{i_1} \ldots M_{i_2}$ can be similarly interpreted.

The following statements can be derived straightforwardly from the definitions of Markov chains and type spaces:

Proposition 2.6. *A probability distribution $p \in \Delta(\Omega, \mathcal{A})$ is a prior of player i if and only if*

$$p = p\,M_i\,,$$

that is, the prior's of i are exactly the stationary/invariant distributions of i's Markov matrix.

Proposition 2.7. *The elements of Π_i are precisely the irreducible communicating classes of M_i.*

Proposition 2.8. *For a random variable f*

$$M_i f(\omega^*) = \int_\Omega f(\omega)\, t_i(\omega^*)(\mathrm{d}\omega) \quad \forall\, \omega^* \in \Omega.$$

The above statements provide the main motivation for the use of Markov models.

Samet (1998b) makes the following observation:

Proposition 2.9. *For any $i \in N$, the set of priors of i is exactly the convex hull of i's types.*

Corollary 2.10. *For any $i \in N$, M_i is an idempotent operator, that is*

$$M_i = M_i\, M_i \quad \forall\, i \in N.$$

Proof. For any distribution $p \in \Delta(\Omega, \mathcal{A})$, $p\, M_i$ is the convex combination of the rows of M_i, which are the types of i, so $p\, M_i$ is a prior for i by Proposition 2.9. Therefore by Proposition 2.6

$$p\, M_i = p\, M_i\, M_i \quad \forall\, p \in \Delta(\Omega, \mathcal{A}),\ \forall\, i \in N,$$

so $M_i = M_i\, M_i \quad \forall\, i \in N$. ■

Idempotence is a sensible, consistent property of the Markov transition. It means that an agent knows their own type/belief.

Now for the rest of this chapter, we explore the properties of the Markov transition obtained by the composition of individual Markov transitions.

Lemma 2.11. *Let $M = M_{i_1} M_{i_2} \dots M_{i_k}$ for some $i_1, i_2, \dots, i_k \in N$ with $\bigcup_{j=1}^{k} \{i_j\} = N$. Then the meet Π_N is a partition of Ω into irreducible, aperiodic classes of M, therefore the restriction of M to $\pi \in \Pi_N$ has a unique stationary distribution.*

Proof. For $\omega, \omega' \in \pi_j \in \Pi_j$:

$$M(\omega, \omega') \geq M_{i_1}(\omega, \omega) M_{i_2}(\omega, \omega) \dots M_{i_{j-1}}(\omega, \omega) M_{i_j}(\omega, \omega') M_{i_{j+1}}(\omega', \omega') \dots M_{i_n}(\omega', \omega') > 0,$$

that is, elements of the same partition element communicate, so if ω is in an equivalence class, $\Pi_i(\omega)$ is a subset of that class $\forall\, i \in N$. Hence each class is a union of elements in Π_N. For any $\omega \in \Omega$ we have $\Pi_j(\omega) \subseteq \Pi_N(\omega)\ \forall\, j \in N$, so any $\pi \in \Pi_N$ is irreducible and therefore an equivalence class. M is aperiodic by the assumption that $t_i(\omega)(\omega) > 0\ \forall\, \omega \in \Omega,\ \forall\, i \in N$. ■

The above lemma enables us to prove the main result of this section:

Proposition 2.12. *The following are equivalent:*

(1) *p is a common prior.*

(2) $p = p\, M_i \quad \forall\, i \in N$.

(3) $p = p\, M_{i_1} M_{i_2} \dots M_{i_k} \quad \forall\, i_1, i_2, \dots, i_k \in N$ *with* $\bigcup_{j=1}^{k} \{i_j\} = N$.

Proof.
(1) $\Leftrightarrow$ (2) : It's by Proposition 2.6.
(2) $\Rightarrow$ (3) : It's clear.
(3) $\Rightarrow$ (2) : Suppose the former and let $i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^{k}\{i_j\} = N$. Then

$$p\, M_{i_1} M_{i_2} \ldots M_{i_k} = p,$$

multiplying by M_1 gives

$$p\, M_{i_1} M_{i_2} \ldots M_{i_k}\, M_{i_1} = p\, M_{i_1},$$

therefore $p\, M_{i_1}$ is a stationary distribution of $M_{i_2} \ldots M_{i_k}\, M_{i_1}$, but we know so is p, and it's unique by Lemma 2.11, so

$$p = p\, M_{i_1}.$$

As i_1 can be chosen arbitrarily, we are done. ∎

Remark 2.13. The often present requirement $\bigcup_{j=1}^{k}\{i_j\} = N$ can be relaxed if we modify our definitions to only concern a subset of players within N, adjusting the conditions and results as needed. However, this approach is similar to viewing N as part of a broader universe of players. In our analysis, it suffices to concentrate specifically on this chosen subset N.

2.4 Higher-order belief functions

Considering the beliefs of a sequence of agents, an event's probability in terms of the agents' types can only be given as a chain of nested probabilities: Romeo believes that with probability p_1 Juliet believes that with probability p_2 Mercutio believes, and so on. However, agents make decisions based on calculating their expected gains. Therefore, it's desirable to construct a merged probability measure with respect to which an expectation can be taken. It should combine the information about the chain of probabilities. For instance, it's an imminently logical aim that if Romeo believes that with probability $\frac{1}{2}$ Juliet believes that with probability $\frac{1}{3}$ the event happens (and for no other combination of probabilities is that true), then the unified measure of Romeo and Juliet should assign probability $\frac{1}{6}$ to the event. Note that from Romeo's point of view, the probability with which Juliet believes the event E to happen is a random variable $t_{\text{Juliet}}(\cdot)(E)\colon \Omega \to [0, 1]$. Therefore, Romeo's beliefs about Juliet's beliefs about the event should be described by the expected value of that random variable with respect to Romeo's type. Continuing this line of reasoning, we arrive at the following definition.

Definition 2.14. *For some state $\omega^* \in \Omega$, the* ***higher-order belief function***[2] $t_{i_1,i_2,\ldots,i_k} : \Omega \to \Delta(\Omega, \mathcal{A})$ *corresponding to the chain of beliefs of agents $i_1, i_2, \ldots, i_k \in N$ (not necessarily distinct) is defined as*

$$t_{i_1,i_2,\ldots,i_k}(\omega^*)(E) \triangleq \int_\Omega \left[\ldots \left[\int_\Omega \left[\int_\Omega t_{i_1}(\omega_1)(E)\, t_{i_2}(\omega_2)(\mathrm{d}\omega_1) \right] t_{i_3}(\omega_3)(\mathrm{d}\omega_2) \right] \ldots \right] t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1})$$

$\forall\, E \in \mathcal{A}$.

Notation 2.15. *For $\ell \in \mathbb{N}$ let $t^\ell_{i_1,i_2,\ldots,i_k} \triangleq t_{i_1,i_2,\ldots,i_k,i_1,i_2,\ldots,i_k,\ldots,i_k}$, that is, the higher-order belief function iterated ℓ times. (Let $\mathbb{N} = \{1, 2, \ldots\}$).*

Lemma 2.16. *$t_{i_1,i_2,\ldots,i_k}(\omega^*)$ defines a probability measure on $(\Omega, \mathcal{A})$ for any state $\omega^* \in \Omega$ and players $i_1, i_2, \ldots, i_k \in N$.*

Proof.

- Non-negativity:

  $$t_{i_1,i_2,\ldots,i_k}(\omega^*)(E) \geq 0 \quad \forall\, E \in \mathcal{A},$$

  because we integrate a non-negative function every iteration.

- The empty set has measure 0:

  $$\begin{aligned} &t_{i_1,i_2,\ldots,i_k}(\omega^*)(\emptyset) \\ &= \int_\Omega \left[\ldots \left[\int_\Omega \left[\int_\Omega 0 \cdot t_{i_2}(\omega_2)(\mathrm{d}\omega_1) \right] t_{i_3}(\omega_3)(\mathrm{d}\omega_2) \right] \ldots \right] t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1}) = 0. \end{aligned}$$

- σ-additivity:

  Note that $t_{i_1,i_2,\ldots,i_k}(\omega^*)$ has finite support, therefore additivity would also suffice. For pairwise disjoint sets $E_1, E_2, \ldots \in \mathcal{A}$:

  $$\begin{aligned} &\sum_{j=1}^\infty t_{i_1,i_2,\ldots,i_k}(\omega^*)(E_j) \\ &= \sum_{j=1}^\infty \int_\Omega \left[\ldots \left[\int_\Omega \left[\int_\Omega t_{i_1}(\omega_1)(E_j)\, t_{i_2}(\omega_2)(\mathrm{d}\omega_1) \right] t_{i_3}(\omega_3)(\mathrm{d}\omega_2) \right] \ldots \right] t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1}) \end{aligned}$$

[2]Samet (1998a) only defines the related notion of iterated expectations of random variables, we extend this into a measure on $(\Omega, \mathcal{A})$.

$$= \int_\Omega \left[\dots \left[\int_\Omega \left[\int_\Omega \sum_{j=1}^{\infty} t_{i_1}(\omega_1)(E_j)\, t_{i_2}(\omega_2)(\mathrm{d}\omega_1) \right] t_{i_3}(\omega_3)(\mathrm{d}\omega_2) \right] \dots \right] t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1})$$

$$= \int_\Omega \left[\dots \left[\int_\Omega \left[\int_\Omega t_{i_1}(\omega_1) \left(\bigcup_{j=1}^{\infty} E_j \right) t_{i_2}(\omega_2)(\mathrm{d}\omega_1) \right] t_{i_3}(\omega_3)(\mathrm{d}\omega_2) \right] \dots \right] t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1})$$

$$= t_{i_1,i_2,\dots,i_k}(\omega^*) \left(\bigcup_{j=1}^{\infty} E_j \right).$$

- The sample space has measure 1:

$$t_{i_1,i_2,\dots,i_k}(\omega^*)(\Omega)$$
$$= \int_\Omega \left[\dots \left[\int_\Omega \left[\int_\Omega 1 \cdot t_{i_2}(\omega_2)(\mathrm{d}\omega_1) \right] t_{i_3}(\omega_3)(\mathrm{d}\omega_2) \right] \dots \right] t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1}) = 1.$$

■

Therefore $t_{i_1,i_2,\dots,i_k}(\cdot)(\cdot)\colon \Omega \times \mathcal{A} \to [0,1]$ is a Markov kernel and the following observation comes from its definition.

Proposition 2.17. *The higher-order belief function $t_{i_1,i_2,\dots,i_k}$ is the Markov kernel corresponding to the Markov transition matrix $M_{i_k} M_{i_{k-1}} \dots M_{i_1}$.*

Now we are ready to state our desired results, which connect the convergence of higher-order beliefs to the common prior assumption.

Lemma 2.18. *For a bounded random variable $f\colon \Omega \to \mathbb{R}$ and $i_1, i_2, \dots, i_k \in N$ with $\bigcup_{j=1}^{k} \{i_j\} = N$ the mapping*

$$\omega^* \mapsto \lim_{\ell \to \infty} \int_\Omega f(\omega)\, t^{\ell}_{i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega)$$

exists $\forall\, \omega^* \in \Omega$. Moreover, it's constant on any $\pi \in \Pi_N$ and it's equal to the expectation of f with respect to the unique invariant measure of $M_{i_k} M_{i_{k-1}} \dots M_{i_1}$ on π.

Proof. By Proposition 2.17 and Proposition 2.8 we have

$$\int_\Omega f(\omega)\, t^{\ell}_{i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega) = (M_{i_k} M_{i_{k-1}} \dots M_{i_1})^{\ell} f(\omega^*).$$

$M_{i_k} M_{i_{k-1}} \dots M_{i_1}$ is irreducible and aperiodic on any $\pi \in \Pi_N$ by Lemma 2.11, so by the Markov ergodic theorem $t^{\ell}_{i_1,i_2,\dots,i_k}(\omega^*)$ converges weakly to its unique invariant measure, therefore, as f is bounded, its expectation also converges to the expectation with respect to the unique invariant measure. ■

Theorem 2.19. *Suppose* $\Pi_N = \{\Omega\}$. *Then there is a common prior* p *if and only if for any bounded random variable* $f\colon \Omega \to \mathbb{R}$ *it holds that*

$$\int_\Omega f(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) \xrightarrow{\ell\to\infty} \int_\Omega f(\omega)\, p(\mathrm{d}\omega)$$

$\forall\, i_1, i_2, \ldots, i_k \in N$ *with* $\bigcup_{j=1}^{k}\{i_j\} = N$.

Proof. By Lemma 2.18, the limit exists and is the expectation of f with respect to the unique invariant measure of $M_{i_k} M_{i_{k-1}} \ldots M_{i_1}$ on Ω. For any bounded f, this is the same for any choice of $i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^{k}\{i_j\} = N$ if and only if the invariant measure is the same for any choice of $i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^{k}\{i_j\} = N$. This, by Proposition 2.12 is equivalent to p being a common prior. ■

Chapter 3

Financial bubbles

Now that we are equipped with the necessary tools, we proceed to define the concepts of unusual market behavior in the type space setting.
As previously, we suppose $t_i(\omega)(\omega) > 0 \ \ \forall\, i \in N,\ \forall\, \omega \in \Omega$ throughout this chapter.

3.1 Illusions

We have observed an interesting phenomenon: even if all players agree that an event is impossible, their higher-order beliefs may not align with this consensus. To explore this, we introduce the concept of an 'illusion' in the context of our model.

Definition 3.1. *An event $E \in \mathcal{A}$ is an **illusion** in the state $\omega^* \in \Omega$ if*

$$t_i(\omega^*)(E) = 0 \ \ \forall\, i \in N$$

but $\exists\, i_1, i_2, \ldots, i_k \in N$ (not necessarily distinct) with $\bigcup_{j=1}^{k}\{i_j\} = N$, such that

$$t_{i_k}(\omega^*)(\{\omega_{k-1} \colon t_{i_{k-1}}(\omega_{k-1})(\ldots(\{\omega_1 \colon t_{i_1}(\omega_1)(E) > 0\})\ldots > 0\}) > 0.$$

That is, every player thinks that the event has probability zero, but there is a sequence of players $i_1, i_2, \ldots, i_k$ for which i_k thinks, that with non-zero probability i_{k-1} thinks, that with non-zero probability i_{k-2} thinks, and so on... that with non-zero probability i_1 thinks that with non-zero probability the event happens.

Example 3.2. Note that in Section 2.2, the event $E = \{(x_1, x_2) \in \Omega : x_1 < 0\}$ is an illusion in any state ω^* with $t_1(\omega^*)(E) = t_2(\omega^*)(E) = 0$.

Proposition 3.3. *Suppose there is a common prior p and $\Pi_N = \{\Omega\}$. Let $\omega^* \in \Omega$ such that $p(\omega^*) > 0$. Let $E \in \mathcal{A}$ and suppose $t_i(\omega^*)(E) = 0 \ \ \forall\, i \in N$. Then E is an illusion in the state ω^* if and only if $p(E) > 0$.*

Proof. Suppose E is an illusion for $i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^{k}\{i_j\} = N$. This means:

$$\int_\Omega \mathbb{1}_E(\omega)\, t_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) > 0.$$

Therefore by Proposition 2.8 and Proposition 2.17:

$$\mathbb{1}_{\omega^*} M_{i_k} M_{i_{k-1}} \ldots M_{i_1} \mathbb{1}_E > 0,$$

where $\mathbb{1}_{\omega^*}$ is the probability distribution that assigns probability 1 to the state ω^* and 0 to all other states. As $p(\omega^*) > 0$, this also means

$$p(E) = p\, \mathbb{1}_E = p\, M_{i_k} M_{i_{k-1}} \ldots M_{i_1} \mathbb{1}_E > 0.$$

Suppose $p(E) = \varepsilon > 0$. By Theorem 2.19, $\exists\, i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^{k}\{i_j\} = N$ and $\exists\, \ell \in \mathbb{N}$ such that

$$\left| \int_\Omega \mathbb{1}_E(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) - \int_\Omega \mathbb{1}_E(\omega)\, p(\mathrm{d}\omega) \right| < \varepsilon.$$

As $\int_\Omega \mathbb{1}_E(\omega)\, p(\mathrm{d}\omega) = p(E)$, this implies $\int_\Omega \mathbb{1}_E(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) > 0$ so we are done. ■

3.2 Bubbles

The key property that defines financial bubbles is that nobody believes the asset to be worth as much as they are willing to pay for it, based on the thought process of the chain of beliefs. As a player keeps on applying Markov transitions, she eventually arrives at a price higher than anybody's personal valuation. We also assume the bubble to be robust, i.e. no matter how many times a player iterates the chain of Markov transitions that lead to the overpricing of the asset, it stays uniformly larger than any individual player's valuation. Formally:

Definition 3.4. *We say that a random variable $f\colon \Omega \to [0, \infty)$ is a* ***financial bubble*** *in the state $\omega^* \in \Omega$ if $\exists\, \varepsilon > 0$, $\exists\, i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^{k}\{i_j\} = N$ such that*

$$\int_\Omega f(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) \geq \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) + \varepsilon \ \ \forall\, j \in N\ \forall\, \ell \in \mathbb{N}.$$

The condition that f is non-negative does not reduce generality significantly, as f could represent a transformation of any financial product.

We say that f is p-integrable if either it is bounded or there exists a sequence of simple functions each dominated by f such that the integral of these simple functions with respect to p goes to infinity, in which case the integral is defined to be infinity.

Theorem 3.5. *Suppose there is a common prior p and $\Pi_N = \{\Omega\}$. A p-integrable random variable $f\colon \Omega \to [0,\infty)$ is a financial bubble in the state $\omega^* \in \Omega$ if and only if*

$$\int_\Omega f(\omega)\, p(\mathrm{d}\omega) > \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) \quad \forall\, j \in N.$$

Proof. Suppose f is a financial bubble in the state $\omega^* \in \Omega$ with $\varepsilon > 0$, $i_1, i_2, \ldots, i_k \in N$. Then $\exists\, \varphi\colon \Omega \to \mathbb{R}$ simple function such that $\varphi(\omega) \leq f(\omega)\ \forall\, \omega \in \Omega$ and

$$\int_\Omega \varphi(\omega)\, t^\ell_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) \geq \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) + \frac{\varepsilon}{2} \quad \forall\, j \in N,\ \forall\, \ell \in \mathbb{N}.$$

We have

$$\int_\Omega \varphi(\omega)\, t^\ell_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) \xrightarrow{\ell \to \infty} \int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega)$$

by Theorem 2.19, therefore

$$\int_\Omega f(\omega)\, p(\mathrm{d}\omega) \geq \int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega) > \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) \quad \forall\, j \in N.$$

Suppose $\int_\Omega f(\omega)\, p(\mathrm{d}\omega) \geq \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) + \varepsilon_0 \quad \forall\, j \in N$ for some $\varepsilon_0 > 0$. That means that $\exists\, \varphi\colon \Omega \to \mathbb{R}$ simple function such that $\varphi(\omega) \leq f(\omega)\ \forall\, \omega \in \Omega$ and

$$\int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega) \geq \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) + \frac{\varepsilon_0}{2} \quad \forall\, j \in N$$

Let $N = \{1, 2, \ldots, n\}$. We know that

$$\int_\Omega \varphi(\omega)\, t^\ell_{1,2,\ldots,n}(\omega^*)(\mathrm{d}\omega) \xrightarrow{\ell \to \infty} \int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega)$$

by Theorem 2.19, therefore $\forall\, \delta > 0\ \exists\, \ell_0 \in \mathbb{N}$ such that

$$\int_\Omega \varphi(\omega)\, t^\ell_{1,2,\ldots,n}(\omega^*)(\mathrm{d}\omega) \geq \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) + \frac{\varepsilon_0}{2} - \delta \quad \forall\, j \in N,\ \forall\, \ell \geq \ell_0.$$

Now choosing $k = l_0 \cdot n$ and $i_1 = 1, i_2 = 2, \ldots, i_n = n, i_{n+1} = 1, i_{n+2} = 2, \ldots, i_{k-1} = n-1, i_k = n$, we have

$$\begin{aligned}\int_\Omega f(\omega)\, t^\ell_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) &\geq \int_\Omega \varphi(\omega)\, t^\ell_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) \\ &\geq \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) + \frac{\varepsilon_0}{2} - \delta \quad \forall\, j \in N,\ \forall\, \ell \in \mathbb{N}.\end{aligned}$$

Choosing $\delta < \frac{\varepsilon_0}{2}$ and $\varepsilon \in \left(0, \frac{\varepsilon_0}{2} - \delta\right)$, we are done. ∎

Remark 3.6. From the proof it's clear that with these assumptions, f is a financial bubble for some choice of players if and only if it's a financial bubble for any choice of players $i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^{k}\{i_j\} = N$.

Corollary 3.7. *Suppose there is a common prior p and $\Pi_N = \{\Omega\}$. Let $\omega^* \in \Omega$ such that $p(\omega^*) > 0$. Suppose $E \in \mathcal{A}$ is an illusion in the state ω^*. Then $\mathbb{1}_E$ is a financial bubble in the state ω^*.*

Proof. By Proposition 3.3:

$$\int_\Omega \mathbb{1}_E(\omega)p(\mathrm{d}\omega) = p(E) > 0 = t_j(\omega^*)(E) = \int_\Omega \mathbb{1}_E(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) \quad \forall\, j \in N,$$

so by Theorem 3.5 we are done. ■

Proposition 3.8. *Suppose there is a common prior p and $\Pi_N = \{\Omega\}$. Suppose $f \colon \Omega \to [0, \infty)$ (p-integrable) is a financial bubble in the state $\omega^* \in \Omega$. Then $\exists\, K \in \mathbb{R}$ such that $\{f > K\}$ is an illusion in the state ω^*.*

Proof. By definition $\exists\, \varepsilon > 0$ such that

$$\int_\Omega f(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) \geq \int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) + \varepsilon \quad \forall\, j \in N,\ \forall\, \ell \in \mathbb{N}.$$

Also, $t_j(\omega^*)$ has finite support $\forall\, j \in N$. This means that $\exists\, K > 0,\ \ \exists\, \delta$ such that

$$t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(f > K) > t_j(\omega^*)(f > K) + \delta \quad \forall\, j \in N,\ \forall\, \ell \in \mathbb{N}.$$

If $t_j(\omega^*)(f > K) = 0 \quad \forall\, j \in N$, then we are done. Suppose for a contradiction that $t_j(\omega^*)(f > K) > 0$ for some $j \in N$. By taking $l \to \infty$, by Theorem 2.19 we have

$$p(f > K) \geq t_j(\omega^*)(f > K) + \delta.$$

Also $p(f \leq K) < t_j(\omega^*)(f \leq K)$, therefore

$$\frac{p(f \leq K)}{p(f > K)} < \frac{t_j(\omega^*)(f \leq K)}{t_j(\omega^*)(f > K)}.$$

However, $t_j(\omega^*)$ is a posterior of p and $t_j(\omega^*)(f \leq K)$ and $t_j(\omega^*)(f > K)$ are both non-zero, therefore the ratio must be equal, so we arrive at a contradiction. ■

The following question presents itself quite naturally: what if repeating a chain of Markov transitions keeps increasing the perceived price, not just above any individual valuation, but above any value? We occasionally see the hyperbolic rise in the price of certain assets, which would indicate it reaching unbounded value in a bounded time interval. That motivates defining the notion of hyperbolic financial bubbles:

Definition 3.9. *We say that a random variable* $f\colon \Omega \to [0,\infty)$ *is a* ***hyperbolic financial bubble*** *in the state* $\omega^* \in \Omega$ *if* $\forall\, K \in \mathbb{R}\ \exists\, i_1, i_2, \ldots, i_k \in N$ *with* $\bigcup_{j=1}^{k}\{i_j\} = N$ *such that*

$$\int_\Omega f(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) > K \quad \forall\, \ell \in \mathbb{N}.$$

Note that for any $f\colon \Omega \to [0,\infty)$, we have

$$\int_\Omega f(\omega)\, t_j(\omega^*)(\mathrm{d}\omega) < \infty \quad \forall\, j \in N,\ \forall\, \omega^* \in \Omega$$

as $t_j(\omega^*)$ has finite support $\forall\, j \in N,\ \forall\, \omega^* \in \Omega$. Hence the following:

Proposition 3.10. *Any hyperbolic financial bubble in the state* $\omega^* \in \Omega$ *is a financial bubble in the state* ω^*.

Also note that since $t_j(\omega^*)$ has finite support $\forall\, j \in N,\ \forall\, \omega^* \in \Omega$, there is a maximum value that any random variable takes with positive probability according to any player's beliefs. Therefore, beyond a certain point, any hyperbolic financial bubble must be an illusion.

Proposition 3.11. *Suppose* f *is a hyperbolic financial bubble. Then* $\exists\, K_0 \in \mathbb{R}$ *such that* $\{f > K\}$ *is an illusion* $\forall\, K \geq K_0$.

Theorem 3.12. *Suppose there is a common prior* p *and* $\Pi_N = \{\Omega\}$. *Let* $f\colon \Omega \to [0,\infty)$ *be a* p*-integrable random variable. Then the following are equivalent:*

(1) f *is a hyperbolic financial bubble in some state* $\omega^* \in \Omega$.

(2) f *is a hyperbolic financial bubble in any state.*

(3) $\int_\Omega f(\omega)\, p(\mathrm{d}\omega) = \infty$.

Proof.
(2) $\Rightarrow$ (1): It's clear.

(1) $\Rightarrow$ (3): Suppose f is a hyperbolic financial bubble in a state $\omega^* \in \Omega$.
Let $K \in \mathbb{R}$ and $\int_\Omega f(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) > K \quad \forall\, \ell \in \mathbb{N}$ for some $i_1, i_2, \ldots, i_k \in N$. Then $\exists\, \varphi\colon \Omega \to \mathbb{R}$ simple function such that $\varphi(\omega) \leq f(\omega)\ \forall\, \omega \in \Omega$ and

$$\int_\Omega \varphi(\omega)\, t^{\ell}_{i_1,i_2,\ldots,i_k}(\omega^*)(\mathrm{d}\omega) > K \quad \forall\, \ell \in \mathbb{N}$$

and

$$\int_\Omega \varphi(\omega)\, t^\ell_{i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega) \xrightarrow{\ell\to\infty} \int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega)$$

by Theorem 2.19, therefore

$$\int_\Omega f(\omega)\, p(\mathrm{d}\omega) \geq \int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega) \geq K \quad \forall\, K \in \mathbb{R},$$

so $\int_\Omega f(\omega)\, p(\mathrm{d}\omega) = \infty$.

(3) $\Rightarrow$ (2):

Suppose $\int_\Omega f(\omega)\, p(\mathrm{d}\omega) = \infty$. That means that $\forall\, K_0 \in \mathbb{R}\ \exists\, \varphi\colon \Omega \to \mathbb{R}$ simple function such that $\varphi(\omega) \leq f(\omega)\ \forall\, \omega \in \Omega$ and $\int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega) \geq K_0$. Let $\omega^* \in \Omega$. We know that

$$\int_\Omega \varphi(\omega)\, t^\ell_{1,2,\dots,n}(\omega^*)(\mathrm{d}\omega) \xrightarrow{l\to\infty} \int_\Omega \varphi(\omega)\, p(\mathrm{d}\omega)$$

by Theorem 2.19, therefore $\forall\, \varepsilon > 0\ \exists\, \ell_0 \in \mathbb{N}$ such that

$$\int_\Omega \varphi(\omega)\, t^\ell_{1,2,\dots,n}(\omega^*)(\mathrm{d}\omega) \geq K_0 - \varepsilon \quad \forall\, \ell \geq \ell_0.$$

Now choosing $k = \ell_0 \cdot n$ and $i_1 = 1, i_2 = 2, \dots, i_{n+1} = 1, i_{n+2} = 2, \dots, i_k = n$ we have

$$\int_\Omega f(\omega)\, t^\ell_{i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega) \geq \int_\Omega \varphi(\omega)\, t^\ell_{i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega) \geq K_0 - \varepsilon \quad \forall\, \ell \in \mathbb{N}.$$

We can choose K_0 as large and ε as small as we like, so we are done. ■

3.3 Asset pricing

Let's establish the pricing of an asset inside the type space setting.

Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$.

Let $f\colon \Omega \to [0,\infty)$ be a random variable that represents the asset the players are thinking about. We define the valuation variables $V_1, V_2, \dots, V_n\colon \Omega \to [0,\infty)$ as follows:

$$V_i(\omega^*) \triangleq \int_\Omega f(\omega)\, t_i(\omega^*)(\mathrm{d}\omega) \quad \forall\, \omega^* \in \Omega.$$

Suppose Romeo and Juliet are two of the players. They are theorizing about each other's valuation. A logical assumption to be made is that there exists some notion of distance for which Romeo's beliefs about the distance of Juliet's valuation from his own don't depend on his valuation and vice versa. In other words, there should be some form of homogeneity in their beliefs.

We define the distance as the difference of some transform of the asset valuations, as follows:

$$\xi_i \triangleq h(V_i) \quad \forall\, i \in N,$$

$$\eta_{i,j} \triangleq \xi_i - \xi_j \quad \forall\, i, j \in N,$$

where $h\colon [0,\infty) \to \mathbb{R}$ is some $\mathcal{B}(\mathbb{R})$-measurable function.

V_i is constant under $t_i(\omega^*)$ $\forall\, \omega^* \in \Omega$, $\forall\, i \in N$, therefore ξ_i is also constant under the distribution $t_i(\omega^*)$ $\forall\, \omega^* \in \Omega$, $\forall\, i \in N$ (the players know their own valuation).

Take for example the case where $\eta_{1,2}$ is the difference between the asset's log returns that Romeo and Juliet predict for this year. That would mean f is the (proportional) value of a security next year and $h(\cdot) = \log(\cdot)$.

Suppose that if Romeo predicts the log return to be $\xi_{Romeo} = 5\%$ then he believes that with probability $\frac{1}{3}$ Juliet predicts the log return to be $\xi_{Juliet} = 6\%$. Assuming homogeneity would mean that if Romeo predicted the log return to be 6% then he would believe that with probability $\frac{1}{3}$ Juliet predicts the log return to be 7%.

In general, this would mean that the distribution of $\eta_{i,j}$ under $t_j(\omega)$ doesn't depend on ω, in which case calculations can be significantly simplified by the following lemma:

Lemma 3.13. *Let $\xi_i\colon \Omega \to \mathbb{R}\colon i \in N$ be random variables, let $\omega^* \in \Omega$ and suppose ξ_i is constant under $t_i(\omega^*)$ $\forall\, i \in N$. Let $\eta_{i,j} = \xi_i - \xi_j$ $\forall\, i, j \in N$.*
Suppose that for any $i, j \in N$, , the distribution of $\eta_{i,j}$ under $t_j(\omega)$ is $\nu_{i,j}$ for all $\omega \in \Omega$. Then for any $i_0, i_1, i_2, \ldots, i_k \in N$:

$$t_{i_1,i_2,\ldots,i_k}(\omega^*)(\xi_{i_0} \in B) = (\nu_{i_0,i_1} * \nu_{i_1,i_2} * \ldots * \nu_{i_{k-1},i_k})(B - \xi_{i_k}(\omega^*))$$

for any $B \in \mathcal{B}(\mathbb{R})$, where $*$ denotes the convolution of probability measures.

Proof.
By induction on k:
For $k = 1$:

$$t_{i_1}(\omega^*)(\xi_{i_0} \in B) = t_{i_1}(\omega_1^*)(\xi_{i_1} + \eta_{i_0,i_1} \in B) = (\nu_{i_0,i_1})(B - \xi_{i_1}(\omega^*))$$

by the definition of $\eta_{i,j}$ and because ξ_i is constant under $t_i(\omega^*)$ $\forall\, \omega^* \in \Omega$, $\forall\, i \in N$.

Now suppose the statement is true for $k-1$. Then

$$
\begin{aligned}
& t_{i_1,i_2,\dots,i_k}(\omega_1^*)(\xi_{i_0} \in B) \\
&= \int_\Omega \underbrace{\left[\dots \left[\int_\Omega \left[\int_\Omega t_{i_1}(\omega_1)(\xi_{i_0} \in B)\, t_{i_2}(\omega_2)(\mathrm{d}\omega_1) \right] t_{i_3}(\omega_3)(\mathrm{d}\omega_2) \right] \dots \right]}_{t_{i_1,\dots,i_{k-1}}(\omega_{k-1})(\xi_{i_0} \in B)} t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1}) \\
&= \int_\Omega (\nu_{i_0,i_1} * \nu_{i_1,i_2} * \dots * \nu_{i_{k-2},i_{k-1}})(B - \xi_{i_{k-1}}(\omega_{k-1}))\, t_{i_k}(\omega^*)(\mathrm{d}\omega_{k-1}) \\
&= (\nu_{i_0,i_1} * \nu_{i_1,i_2} * \dots * \nu_{i_{k-2},i_{k-1}} * (t_{i_k}(\omega^*) \circ \xi_{i_{k-1}}^{-1}))(B) \\
&= (\nu_{i_0,i_1} * \nu_{i_1,i_2} * \dots * \nu_{i_{k-2},i_{k-1}} * (t_{i_k}(\omega^*) \circ (\xi_{i_k} + \eta_{i_{k-1},i_k})^{-1}))(B) \\
&= (\nu_{i_0,i_1} * \nu_{i_1,i_2} * \dots * \nu_{i_{k-2},i_{k-1}} * (t_{i_k}(\omega^*) \circ \eta_{i_{k-1},i_k}^{-1}) * (t_{i_k}(\omega^*) \circ \xi_{i_k}^{-1}))(B) \\
&= (\nu_{i_0,i_1} * \nu_{i_1,i_2} * \dots * \nu_{i_{k-2},i_{k-1}} * (t_{i_k}(\omega^*) \circ \eta_{i_{k-1},i_k}^{-1}))(B - \xi_{i_k}(\omega^*)) \\
&= (\nu_{i_0,i_1} * \nu_{i_1,i_2} * \dots * \nu_{i_{k-1},i_k})(B - \xi_{i_k}(\omega^*)).
\end{aligned}
$$

■

Remark 3.14. Note that Lemma 3.13 also applies to Section 2.2 with ξ_1, ξ_2 as the liquidity values and $\eta_{1,2} \sim U(-1,0,1)$ under $t_2(\omega)$ $\forall \omega \in \Omega$ and $\eta_{2,1} \sim U(-1,0,1)$ under $t_1(\omega)$ $\forall \omega \in \Omega$.

Example 3.15. Suppose $h(\cdot) = \log(\cdot)$ and

$$\eta_{i,j} \sim U(-1,0,1) \;\text{ under } t_j(\omega) \;\; \forall \omega \in \Omega, \;\; \forall i,j \in N, \; i \neq j.$$

By Lemma 3.13, we get the convolution of discrete uniform variables, so for $\omega^* \in \Omega$ and $i_1, i_2, \dots, i_k \in N$ with $i_r \neq i_{r+1}$ $\forall r$:

$$t_{i_1,i_2,\dots,i_k}(\omega^*)(\xi_{i_0} = m) = \begin{cases} \frac{1}{3^k} \binom{k}{m - \xi_{i_k}(\omega^*)}_2 & \text{if } \xi_{i_k}(\omega^*) - k \leq m \leq \xi_{i_k}(\omega^*) + k, \\ 0 & \text{otherwise,} \end{cases}$$

where $m \in \mathbb{Z}$ and $\binom{a}{b}_2$ is the trinomial coefficient, that is, the coefficient of x^{a+b} in the expansion of $(1 + x + x^2)^a$.

Let $\omega^* \in \Omega$ and $i_0, i_1, i_2, \dots \in N$ be a sequence of players such that $i_r \neq i_{r+1}$ $\forall r$. Let $\mathbb{P} \in \Delta(\Omega, \mathcal{A})$ be a probability measure and let $U_1, U_2, \dots \colon \Omega \to \mathbb{R}$ be a sequence of random variables independent, identically distributed under $\mathbb{P}$, such that

$$U_s \sim U(-1,0,1) \;\text{ under } \mathbb{P} \;\; \forall s \in \mathbb{N}.$$

Then by Lemma 3.13:

$$\int_\Omega f(\omega)\, t_{i_0,i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega) = \int_\Omega \exp\{V_{i_0}(\omega)\}\, t_{i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega)$$

$$= \mathbb{E}_\mathbb{P}\left[\exp\left(\sum_{s=1}^k U_s\right)\right] = \prod_{s=1}^k \mathbb{E}_\mathbb{P}\left[\exp(U_s)\right] = \mathbb{E}_\mathbb{P}\left[\exp(U_1)\right]^k = \left[\frac{\frac{1}{e}+1+e}{3}\right]^k \xrightarrow{k\to\infty} \infty.$$

Also, note that the sequence is increasing as k increases. The only restriction we had on $i_0, i_1, i_2, \ldots \in N$ was that $i_r \neq i_{r+1}\ \forall\, r$, therefore the above implies that $\forall\, K > 0$ $\exists\, i_1, i_2, \ldots, i_k \in N$ with $\bigcup_{j=1}^k \{i_j\} = N$ such that

$$\int_\Omega f(\omega)\, t^\ell_{i_1,i_2,\dots,i_k}(\omega^*)(\mathrm{d}\omega) > K \quad \forall\, \ell \in \mathbb{N}.$$

Hence, f is a hyperbolic financial bubble in any state $\omega^* \in \Omega$.

Remark 3.16. By the proof we can see that f is a hyperbolic financial bubble in any state if $h(\cdot) = \log(\cdot)$ and

$$\eta_{i,j} \sim \nu \ \text{ under } t_j(\omega) \ \ \forall\, \omega \in \Omega, \quad \forall i, j \in N, \ i \neq j.$$

where ν is any distribution for which the expectation of its exponential is greater than 1.

Chapter 4

Conclusions

In this paper, we rigorously explored the often debated concept of financial bubbles within a mathematical framework, specifically leveraging belief hierarchies represented as Markov chains. By applying Bayesian game theoretical approaches, we effectively modeled unusual market behaviors. Our contributions include three distinct definitions: illusions, defined as events possible only in higher-order beliefs; financial bubbles, characterized as random variables valued more highly by higher-order beliefs than by any player's individual beliefs; and hyperbolic financial bubbles, which are random variables assigned unbounded values by higher-order beliefs.

Our analysis revealed significant interconnections among these concepts and their relation to common priors. Specifically, Corollary 3.7 demonstrated how illusions can imply the existence of financial bubbles. Conversely, Propositions 3.8 and 3.11 established the conditions under which financial and hyperbolic financial bubbles respectively imply the presence of illusions. Furthermore, Proposition 3.3 and Theorems 3.5 and 3.12 provided necessary and sufficient conditions for the existence of illusions, financial bubbles, and hyperbolic financial bubbles, based on the properties of the common prior.

This research contributes robust tools for the quantitative analysis of speculative bubbles, enhancing our understanding of complex market dynamics within knowledge-belief spaces.

Összefoglaló

A nem-teljes információs játékok területén a több szereplő közötti vélekedések modellezése jelentős kihívásokat jelent, különösen a vélekedési hierarchiák figyelembevételével — azaz az ügynökök vélekedései mások vélekedéseiről, és így tovább. Harsányi (1967) típusainak bevezetése egy módszert kínál ezen bonyodalmak kezelésére, a játékosok vélekedéseit egy típusfüggvénnyel reprezentálva. A kutatásunk célja az volt, hogy a típustér modellt adaptáljuk a pénzügyi buborékok jelenségére, mely meglepő módon nem rendelkezik egyetemesen elfogadott, szigorú matematikai definícióval a korábbi irodalomban. Törekedtünk a pénzügyi buborékok formális meghatározására és egy olyan közös prior felállítására, amely megengedi egy ilyen típustér kialakulását. Tanulmányunk betekintést kívánt nyújtani a spekulatív buborékok mögött rejlő folyamatokba a pénzügyi buborékok kialakulásához szükséges és elégséges feltételek elemzésével, mindezt a típusterek kontextusában.

A pénzügyi buborékok gyakran vitatott fogalmát szigorú matematikai keretben, különösen Markov-láncként ábrázolt vélekedési hierarchiákat alkalmazva vizsgáltuk meg. Bayesi játékelméleti megközelítésben hatékonyan modelleztük a szokatlan piaci viselkedéseket. Hozzájárulásaink három különálló definíciót tartalmaznak: illúziók, amelyeket csak magasabb rendű vélekedések szerint lehetséges eseményekként definiálunk; pénzügyi buborékok, amelyeket olyan valószínűségi változóként jellemzünk, amelyek magasabb rendű vélekedések szerint nagyobb értékkel bírnak, mint bármely játékos egyéni vélekedései szerint; és hiperbolikus pénzügyi buborékok, amelyek olyan valószínűségi változók, amelyekhez a magasabb rendű vélekedések korlátlan értéket rendelnek.

Elemzésünk jelentős összefüggéseket tárt fel ezen fogalmak között és azok közös priorokkal való kapcsolatában. A 3.7 Következmény megmutatta, hogyan következnek pénzügyi buborékok illúziókból. Fordítva, a 3.8 és 3.11 Állítások meghatározták azokat a feltételeket, amelyek mellett a pénzügyi és hiperbolikus pénzügyi buborékok illúziók létezését jelentik. Továbbá a 3.3 Állítás és a 3.5 és 3.12 Tételek megadták az illúziók, pénzügyi buborékok és hiperbolikus pénzügyi buborékok létezésének szükséges és elégséges feltételeit a közös priorok tulajdonságai alapján.

Szószedet

English	**Magyar**
aperiodic	aperiodikus
common prior	közös prior
convex hull	konvex burok
expectation/expected value	várható érték
field of events	eseménytér
financial bubble	pénzügyi buborék
higher-order belief function	magasabb rendű vélekedési függvény
hyperbolic financial bubble	hiperbolikus pénzügyi buborék
illusion	illúzió
invariant/stationary distribution	stacionárius eloszlás
irreducible communicating class	kapcsolatos osztály
knowledge partition	tudáspartíció
liquidity	likviditás
Markov chain	Markov lánc
Markov transition matrix	Markov átmenetmátrix
measurable	mérhető
meet	hálómetszet
prior	prior
probability measure	valószínűségi mérték
random variable	valószínűségi változó
set of the states	állapottér
type function	típusfüggvény
type space	típustér

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