Null players in weighted voting

Szilvia Keszthelyi

Applied Mathematics MSc

Thesis

Supervisors:

Ágnes Cseh
University of Bayreuth,
Fakultät für Mathematik,
Physik und Informatik,
Mathematisches Institut

Tamás Király
ELTE Institute of
Mathematics,
Department of Operations
Research

Eötvös Loránd University
Faculty of Science
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Introduction

Voting and election rules affect us in various ways, from proportionate parliamentary representation to family holiday planning. In some scenarios, voters carry different weights: in a workplace, for example, the vote of senior staff may count for more, while in an EU committee, the vote of countries might count in proportion to their population size. In such cases, a vote from a lower-weighted party may not count at all, i.e. the outcome of the vote is the same, with no regard to the vote of the so-called null player. A practical example often cited is the electoral system of the Council of Ministers of the European Union in 1958, where Luxembourg acted as a null player.

The definition of null players was first introduced by von Neumann and Morgenstern [17] in game theory, and then studied by Shapley [19] and Barthelemy [5] collected several properties of null players in weighted voting games. Null players can also be defined for weighted committee games—when weights are introduced for voting rules known from social choice theory. Weighted committee games were studied by Kurz [14], however, much less is known about the properties of null players in weighted voting rules.

An intuitive characterization of a null player is the so-called null player out property, which states that a player is a null player exactly if for all possible voting profiles the voting outcome will be exactly the same with and without the null player. This property is slightly different from the definition of a null player: there, the outcome does not depend on what the null player votes. In our work, we formally prove that the two characterizations are the same in virtually all practical voting systems, and give examples of somewhat contrived systems where they are not.

Chapter 1 provides an overview of the voting rules and related definitions. Firstly, the input and output of the voting rules are discussed. Afterwards, the characterisation of voting rules follows where their most important properties are introduced. Lastly, famous theorems are mentioned to summarise the impossibility results of voting rules satisfying certain axioms.

Chapter 2 focuses on the most well-known voting rules, their properties and real-life applications. Six voting rules will be introduced in the first part of the chapter. Most of them are applied in parliamentary or presidential elections, as detailed examples show. The second part gives a more general categorisation of voting rules by introducing the family of scoring rules and Condorcet extensions. Afterwards, strategic abstention and manipulation are discussed.

The aim of Chapter 3 is to introduce weighted voting and to formally define null players. Firstly, simple and weighted voting games from game theory are detailed and then the weighted committee games, i.e. the weighted versions of voting rules from social choice theory are introduced. The rest of the chapter provides an overview of null players, their properties, and the related literature. One can see that null players are well-studied in game theory, however, the chapter shows that they can be analysed in weighted committee games as well.

Chapter 4 introduces the so-called null player out property and studies whether it holds for certain voting rules. The property was briefly researched for weighted voting games in game theory, but not for weighted committee games, therefore our main focus in the chapter is to extend it to weighted voting rules, i.e. to find examples, counterexamples, and sufficient conditions. The results in Chapter 4 are novel.

For thousands of years, elections and votes have determined the decisions, large and small, of mankind. Recognising that a voter has little or no influence on the outcome of an election is important for proportional representation. We have demonstrated how rigorous mathematical methods can be used to filter out the occurrence of null players not only in current electoral systems, but also in those that are planned.

Chapter 1

Voting rules

This chapter provides a general overview of the voting rules. First, we introduce the input in Section 1.1 starting from the sets of players and alternatives, building up the preferences voters can have over the alternatives. Section 1.2 continues with the output, i.e. the voting rule winners. Section 1.3 details making rational choices and the properties that fair voting rules can have. Lastly, in Section 1.4, one can find a summary of the well-known results on whether voting rules exist with certain desirable properties.

1.1 The input

This section introduces the input that is needed for the voting rules. First, the set of players and alternatives will be defined, and then the voters' possible preferences over the alternatives will be analysed. The notations and basics of social choice follow the lecture notes of Brandt [7] and the books of Brandt, Conitzer, and Endriss [9, 10].

Consider the set of n players $N = \{1, \dots, n\}$ who have to vote on m alternatives with $m \geq 2$, and let U be the set of alternatives. The set of all non-empty subsets of U is denoted by $\mathcal{F}(U) = 2^U \setminus \{\emptyset\}$.

3	2	1
a	b	c
b	c	a
c	a	b

Table 1.1: Example for a preference profile

Let R be a *binary preference relation* on U and $a, b \in U$. If alternative a is at least as good as b, then it can be written as aRb using the preference relation. We assume that each player i has a complete, transitive, strict preference relation P_i on the alternatives, which means the following.

- Preference relation R is *complete* if for all $a, b \in U$ it holds that aRb or bRa.
- A relation R is *transitive* if aRb and bRc imply aRc.
- Lastly, we denote *strict preferences* with P and strictness means that aPb holds if and only if aRb and not bRa. If the latter two relations hold simultaneously, it is called *indifference* and is denoted by aIb.

Given a preference relation R and a set $A \in \mathcal{F}(U)$, the *maximal set* is $\text{Max}(R, A) = \{x \in A \colon \nexists y \in A \text{ with } yPx\}$. In other words, $\text{Max}(R, A)$ contains the best elements of A according to R.

We denote the *preference profile* by $R_N = (P_1, \ldots, P_n)$. Table 1.1 is an example of a preference profile when there are six players and three alternatives. The first line denotes the number of players choosing the preference order.The rest of the table shows the ordering of the alternatives in the columns — in each column, the higher alternative is more preferred than a lower one. For example, the first column means that three players prefer a to b to c, i.e. if P_i is the preference relation of these players, then aP_ibP_ic.

1.2 The output

Having the preference profile, the next step is to determine the output—the winner alternatives—by a collective decision. A function that maps the preferences of individual voters to a choice set of socially preferred alternatives is called a *social choice function*.

Definition 1.2.1. A *social choice function* is a function $f\colon \mathcal{R}(U)^n \times \mathcal{F}(U) \to \mathcal{F}(U)$ such that $f(R_N, A) \subseteq A$ for all R_N and A.

In other words, a social choice function is a function that chooses a set of alternatives—the winners—for a given preference profile $R_N \in \mathcal{R}(U)^n$ and feasible alternatives A. Sometimes we simplify the notation to $f(R_N)$ when A is the whole set of alternatives U. In practice, it is often required to have only one winner. This property is called *resoluteness*. Formally, a voting rule is resolute if $|f(R_N)| = 1$ for all preference profiles R_N.

1.3 The characterisation of voting rules

The section provides an overview of the different properties of voting rules. In the first part, rational choice is introduced. The most important axioms that voting rules can satisfy are detailed in the second part. These axioms help to characterise the fairness of the rules.

1.3.1 Rational choice

It is desirable to make a *rational* choice. For example, let $f\colon \mathcal{F}(U) \to \mathcal{F}(U)$ be a choice function and let c, s, v be three alternatives; they can be ice cream flavours chocolate, strawberry, and vanilla. Let $f(\{c, s, v\}) = \{c\}$ and $f(\{c, s\}) = \{s\}$, with the ice cream example: if we can choose from three different flavours, we choose the chocolate, but if the shop runs out of vanilla, we will choose strawberry instead of chocolate. This choice function is not rationalisable based on the following definition.

Definition 1.3.1. A choice function f is *rationalisable* if there exists a binary relation R on U such that $f(A) = \mathrm{Max}(R, A)$ for all A.

One can construct a binary relation based on $f(\{x, y\})$ for every $\{x, y\}$ pair. This is called the *base relation* R_f where $xR_f y \Leftrightarrow x \in f(\{x, y\})$.

Claim 1.3.2. *A choice function f is rationalisable if and only if it is rationalised by its base relation R_f.*

In other words, it means that $f(A) = \mathrm{Max}(R_f, A)$, i.e. the binary relation from Definition 1.3.1 can be chosen as the base relation of f.

1.3.2 Properties of voting rules

One would like to require several other properties for a "fair" election. Firstly, it is desirable that voters are treated equally: the outcome of the voting rule should not be affected if the players are renamed.

Definition 1.3.3. Social choice function f is *anonymous* if $f(R_N, A) = f(R'_N, A)$ for all $A \in \mathcal{F}(U)$ and preference profiles $R_N, R'_N \in \mathcal{R}(U)^n$ such that there is a permutation $\pi\colon N \to N$ with $R'_i = R_{\pi(i)}$ for all $i \in N$.

Similar can be required for the alternatives as well: by renaming them, the outcome has to change, but according to the respective permutation.

Definition 1.3.4. Social choice function f is *neutral* if $\pi(f(R_N, A)) = f(R'_N, B)$ for all $A, B \in \mathcal{F}(U)$ and $R_N, R'_N \in \mathcal{R}(U)^n$ such that there is a bijection $\pi\colon A \to B$ with $(xR_i y) \Leftrightarrow (\pi(x)R'_i\pi(y))$ for all $i \in N$ and $x, y \in A$.

It will be detailed in Section 1.4 that neutrality is too strong in some cases. However, the independence of irrelevant alternatives provides a way to weaken it. It only requires that if a set of alternatives is ordered the same way by every player in both preference profiles, then the winner of that set should be the same as well.

Definition 1.3.5. Social choice function f satisfies *independence of irrelevant alternatives* if for all $A \in \mathcal{F}(U)$ and R_N, R'_N it holds that the property $(\forall i \in N\colon R_i|_A = R'_i|_A)$ implies $f(R_N, A) = f(R'_N, A)$.

There are properties related to the positions of alternatives in the preference orderings. One of them is *monotonicity.* It can be a "fairness" assumption if an alternative is among the winners and some players rank it higher in a second preference profile, then it will stay in the winner set. More precisely, we need the ranking of the rest of the alternatives remain unchanged and only the selected alternative is rising in some of the players' preferences in the second profile.

Definition 1.3.6. Let R_N and R'_N be such that there is some $i \in N$ and $a \in U$ with $\forall j \neq i, R_j = R'_j$, furthermore, $\forall x, y \in U \setminus \{a\}$, $(aR_iy \Rightarrow aR'_iy)$, and $(aP_iy \Rightarrow aP'_iy)$. A voting rule is *monotonic* if $a \in f(R_N, A)$ implies $a \in f(R'_N, A)$.

A stronger version of monotonicity is *positive responsiveness*, which requires that a chosen alternative $a \in f(R_N)$ and $R_i|_A \neq R'_i|_A$ implies $\{a\} = f(R_n)$ with R and R' as in the definition above. If postitive responsiveness is satisfied, a is the single winner when a player ranks it higher.

Pareto-optimality

In some cases, there are alternatives that we do not want to choose if there is a strictly better option. *Pareto-optimality* describes this property. Given R_N and $x, y \in U$ then x *Pareto-dominates* y if xP_iy for all $i \in N$. Alternative z is called *Pareto-optimal* in $A \subseteq U$ if not Pareto-dominated in A. A social choice function is *Pareto-optimal*, if whenever there is a Pareto-dominated alternative y in A then it will not get selected, i.e. $y \notin f(R_N, A)$. In other words, if all voters prefer alternative x to y unanimously, then y will not be a winner.

1.4 Impossibility results

It would be useful to combine the axioms listed above to get a voting rule that can determine the socially preferred winners in a fair way. Such a voting rule exists when there are no more than two alternatives, but unfortunately, in general, it is not possible to create one. In the case of two alternatives, a straightforward voting rule turns out to be the fairest: it is enough to count the winner or the majority comparisons—equivalent to plurality for two alternatives—also called the majority rule.

Theorem 1.4.1 (May)**.** *The majority rule is the only social choice function on two alternatives that satisfy anonymity, neutrality and positive responsiveness. For two alternatives and an odd number of voters, majority rule is the unique resolute, anonymous, neutral and monotonic social choice function.*

Condorcet, May, and Arrow proved important impossibility results for more than two alternatives.

Theorem 1.4.2 (Condorcet, May)**.** *No anonymous, neutral and positive responsive social choice function is rationalisable when* $m \geq 3$ *and* $n \geq 3$*.*

The rest of the section reviews the results when some of the axioms from the above theorem are weakened. Arrow showed that the nonexistence still holds if weakened versions are used but it requires the choice function to be rationalisable by a transitive relation, which is stronger than the one in the theorem of Condorcet and May.

A way to weaken anonymity is with the help of a dictatorship: if there is a dictator, their favourite alternative will be the single winner, regardless of how other players vote.

Definition 1.4.3. A social choice function f is *dictatorial* if there exists $i \in N$ such that for all A, R and $x \in A$ $(\forall y \in A \setminus \{x\} : xP_iy) \Rightarrow f(R, A) = \{x\}$.

If the social choice function is anonymous, it follows that there is no dictator. Arrow's following theorem shows that using non-dictatorship instead of anonymity is still a too strong requirement.

Theorem 1.4.4 (Arrow)**.** *No social choice function satisfies independence of irrelevant alternatives, Pareto-optimality, non-dictatorship and transitive rationalisability when* $m \geq 3$*.*

Chapter 2

Well-known voting rules and their properties

The main focus of this chapter is on specific voting rules and their properties. Firstly, in Section 2.1, six well-known voting rules are introduced; these will be referred to in later chapters. Real-life examples show that most of these voting rules are used in politics. However, the voting rules can output multiple winners which is not fitting for many applications, therefore tie-breaking is mentioned in Section 2.2. Sections 2.3 and 2.4 present broader classes of voting rules, namely the scoring rules and Condorcet extensions and their properties. Finally, Section 2.5 considers the possibility of manipulating the voting: players might not vote according to their real preferences if they can achieve a better result by giving fake preferences or not voting. Results about voting rules being prone to manipulation are summarised in the section.

2.1 Well-known voting rules

This section will introduce some of the most important voting rules. The resulting collective decisions can differ depending on which rules we apply

4	3	2
a	c	b
b	b	c
c	a	a

Table 2.1: Example of different outcomes for a given preference profile

in each case for a given preference profile. One of the famous examples is the profile shown with the help of Table 2.1, which was detailed in the book Computational Social Choice by F. Brandt, V. Conitzer and U. Endriss [9]. Similarly to the earlier example of a preference profile, the first row of the table shows the number of players. In this case, nine voters are participating, four of whom rank the three possible alternatives in the order of preferring a to b to c.

In the example, three different ways are shown how to determine winners and one can argue in favour of any of these voting rules, depending on the situation in which one would like to apply them. Firstly, if we consider only the first-ranked alternatives, let the winner be the one with the most votes. In this case, alternative a wins with four players ranking it first. However, one can say that this rule is not the fairest, since for more voters—for five of them—a is not only not the first-ranked alternative but it is in the last place and any other alternative would be more preferred by the majority of the players.

As another approach, we can assign a value to each ranking, namely, the first-ranked alternatives get two points from each voter who chose them, the second-ranked ones get one point and the last ones none. Applying this rule (Borda's rule), alternative a gets 8 points, b gets $4+3+2\cdot 2=11$ and lastly, c gets 8 points, which leads to a different result with b being the new winner.

Alternatively, we can start by deleting the alternatives that we do not want to declare as a winner. These would be the ones that are ranked first by the least number of players. In this case, alternative b is ranked first by only two players, therefore we delete it first. After removing b from the game, five voters have c ranked first and the four other players still have a as their most preferred option. This means that the next alternative to eliminate would be a, resulting in c winning the voting.

In the following part, the above-mentioned and some new voting rules will be detailed. These are the most commonly used voting rules in the literature. The following are structured based on the lecture notes of Brandt [7].

2.1.1 Plurality

The first voting rule mentioned in the above example is called the plurality rule.

Plurality

Plurality rule considers only the first-ranked alternatives of each voter and chooses the ones that are ranked first by most voters.

In plurality (and most of the other voting rules) there can be a tie if the same number of players rank two or more alternatives first and those win.

One can check that plurality satisfies some of the above axioms, including anonymity, neutrality and monotonicity. It is also Pareto-optimal: if all voters prefer a to b then b will not be among the first-ranked alternatives, therefore, it cannot be chosen.

This voting rule is often used in real elections, for example, it is part of the German parliamentary elections. The German Bundestag (the Parliament of the Federal Republic of Germany) consists of at least 598 members, half of which, 299 members, are elected directly by Germany's constituencies and the other half by party lists in Germany's sixteen Länder (states) [3].

The two-stage elections are the following. In the first stage, voters have to vote for one person from the list of candidates in their constituency and the one who obtains the most votes gets elected to the Bundestag as the local representative of the constituency. Then voters have a second vote to cast for a party from the party list to allocate the seats proportionate to the votes each party get. The second half of the seats are given out based on these votes and additional "balance seats" are created to ensure the right ratio of the parties in the Bundestag. The first stage is a real-life application of the plurality rule. The only difference is an additional rule setting a lower limit on the number of seat parties can get: they have to gain at least three constituency seats in the first stage or more than five percent of the second votes to participate in the distribution of the seats.

2.1.2 Borda's rule

The second voting rule in the example was Borda's rule.

Borda's rule

Borda's rule has a pointing system: the first-ranked alternative out of the m gets $m-1$ points, the second-ranked alternative gets $m-2$ points, and so on, until the last but one alternative gets 1 point, while the last one gets 0. The points are calculated for all players' preferences and the alternative with the highest total score wins.

Borda's rule also satisfies anonymity, neutrality, monotonicity and Pareto-optimality.

Borda's rule or similar *scoring rules* are used in real life as well. Scoring rules are a more general version of Borda's rule. They allow a broader range of scores and will be detailed later on in Section 2.3. The Eurovision Song Contest applies a scoring rule very similar to Borda's rule: the jury of five music industry professionals and the audience of each country allocates the scores $(12, 10, 8, 7, 6, 5, 4, 3, 2, 1)$ to the ten performers that are the most popular according to the audience.

2.1.3 Sequential majority comparisons

Sequential majority comparison provides a different approach to choosing a winner alternative.

Sequential majority comparisons

Sequential majority comparisons consist of a fixed sequence of pairwise majority comparisons of the alternatives where the winner gets to the next round. The output is the winner of the last majority comparison.

These comparisons have a tree structure where the leaves are given in advance and the rest of the nodes are filled with the winner of the respective pairwise comparison. The pairwise winners are determined by the majority rule: the alternative preferred by the majority of the voters wins. Figure 2.1 is an example of the visualisation of the structure: first, alternative a and b will be compared and the majority winner will be written in the empty node and compared to c and then the next winner to d. Figure 2.2 shows another example tree where the leaves are given and the rest of the nodes are filled out with the majority comparison winners based on the preference profile from Table 2.2.

Sequential majority comparison satisfies anonymity and monotonicity, just like the previous rules, however, it fails to satisfy neutrality and Pareto-optimality.

Preference profile 2.2 from Lecture notes [7] shows an example of failing both properties. Let this profile be R_N and π be such that it leaves a and b unchanged but maps c to d and d to c. When using the tree from Figure

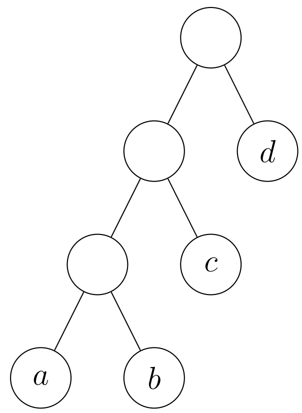

Figure 2.1: Tree for the sequential majority comparison

2.1, we compare a and b first, then compare the winner with c and then with d. In this case, the winner will be alternative d and by applying π, we get $\pi(f(R_N)) = \pi(d) = c$. If we want to calculate the other side from the definition of neutrality, we have to use the same tree but with R'_N, where c and d are swapped. This leads to $f(R'_N) = d$ and therefore a counter-example for neutrality.

Moreover, in the above preference profile R_N, alternative c is Pareto-dominated by b. By using the tree from Figure 2.2, where the order of comparing c and d is swapped—hence it is equivalent to R'_N and the first tree from the previous example—alternative c turns out to be the winner. This shows that Sequential Majority Comparison is not Pareto-optimal.

2.1.4 Instant runoff

Instant runoff applies the method of repeatedly deleting “unpopular” alternatives.

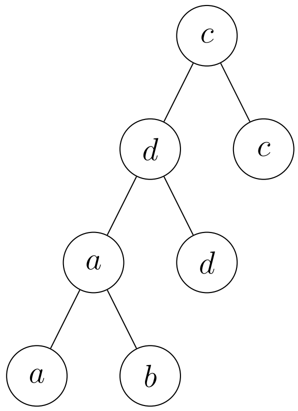

Figure 2.2: Sequential Majority Comparison—leaves are given in advance and the rest of the nodes are filled out with the pairwise majority winner.

1	1	1
a	d	b
b	a	c
c	b	d
d	c	a

Table 2.2: Sequential majority comparison does not satisfy neurtality.

Instant runoff

Alternatives ranked first by the lowest number of players get deleted in every round. The rounds are repeated until no more alternatives can be deleted.

The properties satisfied by instant runoff will be detailed in the next section, together with plurality with runoff.

Instant runoff may seem more complicated to apply in real life than other voting rules, however, it is used in parliamentary elections in Australia [4]. The House of Representatives is one of the two houses of the Australian Federal Parliament, the other being the Senate. Each Member of the House of Representatives represents an electoral division of Australia. At the elections, voters of each constituency get a ballot paper with the list of the candidates and have to write numbers from 1 to m next to each of the m alternatives to indicate their order of preference, one being the highest-ranked alternative. When counting the votes, first only the first-ranked alternatives are taken into account. If an alternative reaches the absolute majority, it is the winner. Otherwise, the candidate with the lowest first votes gets eliminated and some lower-ranked alternatives are considered too. These steps continue until an alternative with an absolute majority is found, which is equivalent to the instant runoff.

2.1.5 Plurality with runoff

Plurality with runoff has only two rounds, which are the following.

> **Plurality with runoff**
>
> In the first round, the two alternatives ranked first by most voters are selected for the second round, which is a majority comparison.

Instant runoff and plurality with runoff satisfy anonymity, neutrality and Pareto-optimality, but not monotonicity, for which Table 2.3 provides an example. In the first preference profile, b and c are ranked first by the lowest number of players—five players rank them first while a has 7 first ranks—and will be deleted according to the rules of instant runoff. After this round only a will be left, and therefore is the winner. We get the second preference profile by a rising in the preference of the player in the second column. This way, even more voters ranked a first, however, a will no longer win. In the first round, b will be deleted, therefore c gets four "extra" votes in the second round, getting a total of nine, while a has only eight votes.

Plurality with runoff is often used in presidential elections of democratic countries, for example, Austrian and French presidential elections are real-life examples, moreover, France uses a similar method at their legislative elections as well to determine the members of the parliament [2, 1]. In the

7	1	4	5
a	b	b	c
b	a	c	a
c	c	a	b

7	1	4	5
a	a	b	c
b	b	c	a
c	c	a	b

Table 2.3: Instant runoff and plurality with runoff are not monotonic

first round of the presidential elections, voters have a single vote to cast for their most preferred candidate of the list of all candidates. If none reaches an absolute majority, i.e. more than half of the votes, a second round is held four weeks after the first in Austria, while in France there are only two weeks between the two rounds. The two candidates with the two highest number of votes are competing in the second round and voters have one vote again to cast to their most preferred alternative.

2.1.6 Copeland's rule

Copeland's rule returns the alternatives that win the pairwise majority comparison against the most alternatives.

Copeland's rule

Copeland's rule $f(R_N, A) = \text{argmax}_{x \in A} |\{y \in A : xP_m y\}|$, where $xP_m y$ means that y is majority dominated by x.

It is easy to see that Copeland's rule satisfies anonymity, neutrality, monotonicity and Pareto-optimality as well. For example, monotonicity is satisfied because when a (winning) alternative is moved higher in a voter's preferences, the number of alternatives majority dominated by it will not decrease and the majority relations between the rest of the alternatives do not change.

	Anonymity	Neutrality	Monotonicity	Pareto-optimality
Plurality	✓	✓	✓	✓
Borda's rule	✓	✓	✓	✓
SMC	✓	✗	✓	✗
Instant runoff	✓	✓	✗	✓
Plurality w. runoff	✓	✓	✗	✓
Copeland's rule	✓	✓	✓	✓

Table 2.4: Properties satisfied by the voting rules.

2.1.7 Summary of the properties

Table 2.4 summarises the properties satisfied by the above voting rules.

2.2 Tie-breaking

The above voting rules exemplify how different methods can be designed to determine winners. It depends on the given situation which rule is the most suitable. These six rules also show how most of the axioms can fail. However, most of the above rules satisfy neutrality. In the cases mentioned above, it is possible to have multiple winners. If one has to guarantee a single winner, then *tie-breaking* rules can be applied. A commonly used tie-breaker is the *alphabetical tie-breaking* when the alphabetically first alternative wins out of the ones in the tie. This rule takes into account the names of the alternatives, thus it makes the voting rules fail neutrality.

2.3 Scoring rules

Some of the above voting rules can be generalised using a family of voting rules, namely the scoring rules. Given a feasible set of alternatives A, a *score vector* is a vector $s = (s_1, \ldots, s_{|A|})$ of real numbers. If a player ranks an alternative at position i, then the alternative gets s_i points. The scoring rule chooses the alternative with the highest total score. The following notation is needed for the formal definition: s^i denotes a score vector of dimension i. One gets the chosen alternatives with the following sum:

$$f(R_N, A) = \operatorname*{argmax}_{x \in A} \sum_{i \in N} s^{|A|}_{|y \in A : y R_i x|}.$$

In other words, each voter's scores for each alternative are summed up and then the one with the maximum value is chosen.

Some of the voting rules mentioned above belong to the family of scoring rules. It is easy to see that Borda's rule is a scoring rule with the score vector $s^{|A|} = (|A| - 1, |A| - 2, \ldots, 0)$. Plurality has an even simpler score vector $s^{|A|} = (1, 0, \ldots, 0)$. Another voting rule that is easy to describe as a scoring rule is *anti-plurality* with score vector $s^{|A|} = (1, \ldots, 1, 0)$. It can be interpreted as voters voting against their least favourite alternative and the one with the least votes against wins, which is equivalent to the score vector above where every alternative but the least preferred one is assigned scores.

Equivalent formulations of the same voting rule lead to an important theorem about scoring rules. The following claim characterises the properties that have to be satisfied for score vectors to describe the same voting rule.

Claim 2.3.1. *Scoring rules are invariant under component-wise positive affine transformations of the score vectors, in other words, score vectors s and t represent the same social choice function if for all i it holds that $t_i = \alpha s_i + \beta$ with $\alpha > 0$.*

Some of the axioms can be proven generally for all scoring rules. We saw that both Borda's rule and plurality satisfy monotonicity and there is a simple characterisation of monotonic scoring rules shown in the following claim.

Claim 2.3.2. *A scoring rule with score vector s is monotonic if and only if $s_1 \geq s_2 \geq \ldots \geq s_{|A|}$.*

A short proof can be given using contraposition.

2.4 Condorcet extensions

Another way of categorising voting rules is through the family of Condorcet extensions. One can look at all the pairwise majority comparisons between the alternatives and if there is one that wins against all the other options, then it is a good candidate for the winner. This alternative is called the Condorcet winner. For the formal definition, let R_m be the majority relation.

Definition 2.4.1. Alternative x is a *Condorcet winner* in A if xP_my for all $y \in A \setminus \{x\}$.

If there exists a Condorcet winner, then it is unique. A famous example of not having a Condorcet winner is shown in Table 2.5. In this preference profile, one can look at the pairwise majority comparisons and see that the roles of all alternatives are symmetric and no one wins against all the others: a wins when compared to b by one more player voting for the former. Similarly, when b and c are compared, b is more preferred, and from c and a, c ranked first the most. These pairwise comparisons can be illustrated on a directed graph by drawing a directed edge from the nodes representing one alternative to another if the first alternative wins the pairwise majority comparison over the second. This is called the majority graph. For the above example, Figure

1	1	1
a	c	b
b	a	c
c	b	a

Table 2.5: Preference profile where there is no Condorcet winner

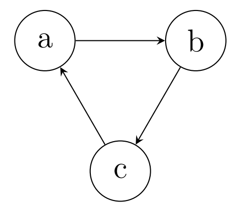

Figure 2.3: Directed graph representation of pairwise majority comparisons

2.3 shows the directed graph representing the pairwise majority comparisons of the preference profile from Table 2.5.

A social choice function f is a *Condorcet extension* if it chooses the Condorcet winner as the unique winner of f whenever there is one, i.e. $f(R_N, A) = \{x\}$ where x is the Condorcet winner in A according to R_N.

A frequently used example of a Condorcet extension is Copeland's rule. As mentioned in Section 2.1.6, this rule returns the alternatives that win the pairwise majority comparison against the most alternatives i.e. $f(R_N, A) = \text{argmax}_{x \in A} |\{y \in A : xP_m y\}|$. If there is a Condorcet winner, it will have the highest number of wins and therefore will be the unique winner.

Condorcet studied the relations between Condorcet extensions and Borda's rule which led to an interesting observation about the scoring rules failing to choose the Condorcet winner in certain cases.

Theorem 2.4.2. *No scoring rule is a Condorcet extension when $m \geq 3$.*

Lecture notes [7] provide a short proof with the help of three preference profiles that show that the Condorcet winner is not the scoring rule winner. *Black's rule* tries to provide a solution for combining scoring rules and Condorcet extensions by the rule of choosing the Condorcet winner whenever it exists and the Borda winner when there is none.

2.5 Strategic manipulation and abstention

In some cases, players can get a better result for themselves if they do not vote according to their real preferences but lie about them or not participate at all. This section summarises strategic manipulation and abstention.

If a player misrepresents their preferences and this way can make an alternative better for them to win, then the voting is not *strategyproof*. It can be formulated more precisely as follows. If there exists R_N, R'_N, A and i such that $R'_N = (R_1, \ldots, R_{i-1}, R'_i, R_{i+1}, \ldots, R_n)$, $R'_i \neq R_i$ and $f(R'_N, A) P_i f(R_N, A)$, then f is prone to strategic manipulation.

The other possibility to manipulate a voting rule is by not participating. If a voter can achieve a strictly better result by staying away from the voting, then *participation* is not satisfied. Formally, f suffers from *strategic abstention* if there exist R_N, A and i such that $f(R_{N_{-i}}, A) P_i f(R_N, A)$, where $R_{N_{-i}} = (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$, in other words the preference profile without i participating. In both cases, it is required to have a single winner to be able to compare the chosen alternatives before and after misrepresenting the preferences.

3	2	1
b	a	c
c	b	a
a	c	b

((a)) Real preferences of the voters.

3	2	1
b	a	a
c	b	c
a	c	b

((b)) Misrepresented preferences.

Table 2.6: Plurality does not satisfy strategyproofness.

As an example, plurality is not strategyproof but resistant to strategic abstention. Preference profile 2.6 shows that plurality is not strategyproof, with the real preferences in part (a) and the misrepresented ones in part (b).

3	2	3
b	a	c
c	b	a
a	c	b

Table 2.7: Plurality with runoff does not satisfy participation.

Suppose we use alphabetical tie-breaking: if two alternatives get the same number of votes and the voting would result in a tie, then the winner will be the alternative appearing earlier in the alphabet. By voting according to the players' real preferences, alternative b is declared the winner. However, b is the worst possible option for the last player and they can make a better alternative win if they misrepresent their preferences. They are the only one placing alternative c at first place, therefore it cannot win, but by swapping c and a, alternative a will receive three votes and with the help of the tie-breaking rule, will win, which is better for the player than the original winner b.

Plurality with runoff (with alphabetical tie-breaking) is an example of a voting rule that does not satisfy participation. Let preference profile 2.7 be the preferences of the voters before a player abstains. If all players vote according to their real preferences, alternative a gets eliminated in the first round, therefore b gets two more votes for the second, which leads to it winning. However, suppose two voters from the third column abstain because b is their least preferred option. In that case, a will be the winner by eliminating c in the first round and then in the second round, a receives one more vote, which puts the remaining two alternatives in a tie and results in a winning the tie-breaking.

2.5.1 Strategyproofness

The following theorem provides a short overview of the cases where voting rules are strategyproof. It turns out that most of the "reasonable" resolute social choice functions can be manipulated. By that, we mean that every alternative can be a winner with an appropriate preference profile, there are no irrelevant alternatives that can never win. A social choice function f with this property is called *non-imposing*, i.e. if for all $x \in A$ there exists R_N such that $f(R_N) = \{x\}$. Gibbard and Satterthwaite [13, 18] studied the relationship between non-imposing resolute social choice functions and strategyproofness and proved the following theorem independently of each other.

Theorem 2.5.1 (Gibbard, Satterthwaite)**.** *Every non-imposing, strategyproof, resolute social choice function is dictatorial when there are at least* 3 *alternatives.*

The above theorem is a generalization of the negative result concerning the Condorcet extensions, namely that no resolute Condorcet extension is strategyproof when $n, m \geq 3$.

2.5.2 Participation

For some voting rules, voters may be better off not participating if "too many" others are already participating. This is called the No-Show paradox. Condorcet extensions under the following conditions are an example of this.

Theorem 2.5.2 (Brandt et al. [11])**.** *No resolute Condorcet extension satisfies participation when* $n \geq 12$ *and* $m \geq 4$*.*

When $n \leq 11$ or $m \leq 3$, there exist social choice functions that satisfy participation.

	Strategyproof	Participation
Plurality	✗	✓
Borda's rule	✗	✓
SMC	✗	✗
Instant runoff	✗	✗
Plurality w. runoff	✗	✗
Copeland's rule	✗	✗

Table 2.8: Properties satisfied by the voting rules.

2.5.3 Summary of the strategic manipulation

Table 2.8 summarises which of the voting rules mentioned above satisfy strategyproofness or participation for all n, m, using lexicographic tie-breaking.

Chapter 3

Weighted voting

This chapter introduces the so-called null players and different voting game variants where null players can appear. Firstly, Section 3.1 provides an overview of special cases of cooperative games—simple and weighted voting games to see the similarities to the weighted committee games which are discussed in Section 3.2. Lastly, Section 3.3 introduces the null players and studies the connected literature.

3.1 Simple and weighted voting games

In Section 2.1, the main focus of the voting rules was to choose one alternative out of multiple possible options. In politics and other real-life cases, a much simpler version of voting is often used: there is a single option—for example, a new proposition—and the players have to vote about accepting it or not, therefore there are only two options. To model this, one can use *simple games* which constitute a special case of cooperative games. In this section, cooperative games and simple games will be introduced and afterwards, generalised for a weighted version.

3.1.1 Cooperative games

In cooperative games, players can form coalitions and make binding agreements to maximise their utility and then share it among the members of the coalition. Cooperative games are usually given in a characteristic function form (N, v), where N is the set of players and $v : 2^N \to \mathbb{R}$ is the worth of the possible coalitions, the value they can obtain by working together. Cooperative games can be divided into classes, as detailed in the lecture notes [8], which was used for writing this paragraph. When coalitions are formed, one can consider the *stability* and the *fairness* of distributing the payoff between the members. The stability is measured by the so-called *core* and the *Shapley value* defines a fair payoff.

3.1.2 Simple games

A *simple game* is a special case of a cooperative game (N, v), where N is the set of players as earlier and $v : 2^N \to \{0, 1\}$ tells us if a coalition $S \subseteq N$ is winning or losing. If $v(S) = 1$, then S is called a *winning coalition* and if $v(S) = 0$, S is *losing*. It can be interpreted as if the members of the coalition S vote for the new proposition, the rest of the players vote against it, and if it gets accepted, then S is a winning coalition, otherwise not. Simple games were characterised by von Neumann and Morgenstern in 1953 [17]. In general, it is required that (i) $v(\emptyset) = 0$, i.e. the empty coalition is losing, (ii) $v(N) = 1$, in other words, the grand coalition is winning and (iii) v is monotonic, which means that if $T \subseteq S$, then $v(T) \leq v(S)$. To put it differently, if no one votes in favour of the proposition, it will not be accepted and similarly, if everyone supports it, then the new laws will be adopted. The last condition ensures that if a coalition can make a proposition be accepted by only them voting for it, then if more players join to support them, it will not result in declining the proposition—as it is usually expected at elections. A coalition S is called a *minimal winning coalition* if S is winning but every proper subset of S is a losing coalition. A player i is called a *vetoer* if they are part of every winning coalition, formally, $v(S) = 0 \ \forall S \subseteq N \setminus \{i\}$.

3.1.3 Weighted voting games

Simple games can be extended to *weighted voting games* where the voters have non-negative weights and a coalition has to reach a given quota to win. It can be seen as the players having multiple votes: for example, at an international event, countries can have votes proportional to their population. Formally, it can be represented by the quota q and weights $w = (w_1, \dots, w_n) \in \mathbb{N}^n$. We assume that $w_1 \geq w_2 \geq \dots \geq w_n \geq 0$. A coalition is winning, i.e. $v(S) = 1$ if and only if $\sum_{i \in S} w_i \geq q$. A special case is the majority game, when a simple majority is needed for winning, formally, $q_{maj} = \frac{\sum_{i \in N} w_i}{2} + 1$ if the sum of the weights is even and $q_{maj} = \frac{\sum_{i \in N} w_i + 1}{2}$ if it is odd. It is also assumed that the quota $q \geq q_{maj}$, otherwise there could be multiple coalitions reaching the quota. Another special case is the unanimity rule where $q = \sum_{i \in N} w_i$, therefore every player has to agree on the votes and everyone has a veto power. To avoid extremities, one can consider the *admissible* games where every weight $1 \leq w_i \leq q - 1$, in other words, no player has zero votes and there is no dictator.

3.2 Weighted committee games

The voting rules for Section 2.1 can be extended to weighted versions, analogously to the above-mentioned simple voting games. Players can hold multiple votes in case of the previously described voting rules. We can model these with the *weighted committee games* which Kurz et al. studied [14]. They focused on the equivalence classes of weight distributions for plurality, Borda's rule, antiplurality and Copeland's rule for $m \geq 3$ alternatives.

In the following sections, we will consider anonymous and resolute (with appropriate tie-breaking) voting rules f. Let $w = \{w_1, \dots, w_n\} \in \mathbb{N}^n$ be the weights that voters $i = 1, \dots, n$ have. This means that the input of a voting rule f is the weighted preference profile—taking each preference ordering P_i the corresponding weight w_i times. We denote the weighted committee game with voting rule f and weight vector w with $f \circ w$.

If weight vector w is chosen to be the uniform weights $\mathbf{1}$ then $f \circ w$ is reduced to the original voting rule w. This shows that if f violates a property, for example, monotonicity, then $f \circ w$ is violating too.

Participation is also determined for the weighted committee game by the underlying social choice function as stated in [14]. Voter i can be better off with a lower weight $w'_i \leq w_i$, i.e. $f \circ w$ is worse for i than $f \circ w'$, where $w'_i < w_i$ and $w'_j = w_j$ if $j \neq i$, if and only if f suffers from the No-Show paradox.

3.3 Null players

In the Council of Ministers of the European Economic Community, the quota for approving a proposition was 12 votes out of 17 between 1958 and 1972. The number of votes was allocated proportionate to the population and therefore Luxembourg held only one vote while all the other countries had an even number of votes: Belgium and the Netherlands held 2 votes each, while Germany, Italy and France had 4 votes each. This meant that Luxembourg was not able to make any difference in the voting process [5]. This real-life example shows that a positive weight at the elections does not necessarily mean a positive share of power. When a player can vote but there is no setting when their vote can change the election result, they are called a null player or dummy player. This section focuses on the literature about null players.

The above story is the textbook example of a real-life null player, however, according to Mayer [16], there were ways out of Luxembourg being a null player in the mentioned scenario. It will be detailed in Section 3.3.3 that the real voting rule might have been slightly different, resulting in Luxembourg not being a null player anymore.

Definition 3.3.1 defines the null player formally in the more general case than the example of Luxembourg in the EEC. Null players can occur when using weighted voting rules, not only in the case of weighted voting games. However, most of the literature on null players analyses the weighted voting games only which will be detailed in Section 3.3.2.

Definition 3.3.1. For an anonymous voting rule f and non-negative weight vector $w \in \mathbb{R}^n_{\geq 0}$, player i is a *null player* if $f \circ w(P) = f \circ w(P')$ for all P and P' preference profiles that differ only in coordinate i, i.e. $P_j = P'_j$ if $j \neq i$.

It means that it does not matter how player i orders their preference list, they cannot change the winner of voting rule f with the given weights w.

3.3.1 The case of weighted voting games

In the case of the weighted voting games introduced in Section 3.1 there is a simpler way to define a null player.

Definition 3.3.2. A player $i \in N$ in game (N, v) is called a *null player* if for all $S \subseteq N$ it holds that $v(S) = v(S \cup \{i\})$.

This definition means that the null player i does not contribute to any coalition—there is no scenario where a coalition cannot make a proposition be accepted but with player i joining them, they can make a change. It follows that the null player cannot be part of any minimal winning coalition.

Authors often make no difference between the terms null player and dummy player, however, originally they were slightly different.

Definition 3.3.3. A player $i \in N$ in game (N, v) is called a *dummy player* if for all $S \subseteq N \setminus \{i\}$ it holds that $v(S \cup \{i\}) = v(S) + v(\{i\})$.

This definition of a dummy player includes null players but not only them: dummy players can either be null players or dictators. Null players were first used by von Neumann and Morgenstern [17] but the name 'null player' was created later. Dummy players were defined by Shapley [19].

Since it is possible to have positive weight but no power over changing the result of the voting, one can be interested in how much voting power a given player has in influencing the elections. There are different power indices to measure voting power, the most prominent ones and mainly used in the papers cited in the rest of the chapter are the *Shapley-Shubik index* (SSI) and the *Penrose-Banzhaf index* (PBI).

A power index is a family of functions which map a game (N, v) to a vector $f(N, v) = (f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^n$ where $f_i(N, v)$ indicates the voting power of player i. The Shapley-Shubik and the Penrose-Banzhaf indices measure the players' *marginal contributions* to all possible coalitions while considering the probability of the occurrence of different coalitions. The difference between the two is the assumption made about the probabilities. The below function f_i is the formal definition of the two power indices.

$$f_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} p_S^i (v(S \cup \{i\}) - v(S))$$

For the Shapley-Shubik index, the probability is defined as $p_S^i = \frac{s!(n-s-1)!}{n!}$, assuming that all coalitions size s are equally likely to occur and any coalition of the given size are equally likely. The Penrose-Banzhaf index chose p_S^i to be $\frac{1}{2^{n-1}}$, which assumes that every coalition occurs with the same probability. The term $v(S \cup \{i\}) - v(S)$ is called the *marginal contribution* of player i to coalition S. In the case of voting games, it can be either 1 or 0. If the contribution of player i is 1, it means that S is losing, but when i joins, $S \cup \{i\}$

is winning, therefore i is in a minimal winning coalition. If $v(S \cup \{i\}) - v(S) = 0$ then player i does not contribute to the coalition—it stays winning or losing with i too—and if the value is 0 for all coalitions, i is a null player. It follows that f_i is 0 if and only if i is a null player, since if i is part of a minimal winning coalition, at least one marginal contribution will be positive and the probabilities are positive as well.

Properties of null players

In this section, some observations about null players in weighted simple games are collected from the work of Barthelemy [5].

Suppose that player j is a null player. Then player $k > j$ is also a null player as we assumed that $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$. If there is no dictator, i.e. $w_1 \leq q - 1$, then the first and second players are not null players, furthermore, if $q = q_{maj}$, then the third player cannot be a null player as well.

3.3.2 How common is it to have a null player?

We have seen a real-life example when there was a null player at elections, and the question arises of how common it is to have them and how to find the null players. The literature focuses on finding null players in weighted voting games but not in weighted committee games.

Barthelemy et al. studied the first question in the case of small weighted voting games [5] of 3 to 5 players and then extended it to larger ones [6], up to 15 players. They analysed the probability of having a null player as the function of the quota under the assumption that every admissible weighted voting game is equally likely to occur. They found that this probability is sensitive to the choice of the quota and can be very high.

Within the case of the small number of players, having only three players is considerably easier than having four or five voters. Barthelemy et al. characterised the probability of having a null player as a function of the quota and the sum of the weight and established a lemma about the existence of null players.

Lemma 3.3.4 (Barthelemy et al.)**.** *In a* 3*-player weighted voting game, there is a null player if and only if* $w_1 + w_3 \leq q - 1$ *and* $w_1 + w_2 \geq q$*.*

Since the weights are in non-decreasing order if $w_1 + w_3 \leq q - 1$, then the third player will not be able to win by being in a coalition with only the second player either. The second condition means that the first and second voters can form a winning coalition, therefore the coalition of the three players is not a minimal winning coalition and player 3 is a null player. The other direction can be proven by starting with the observation that only the third player can be a null player and formulating the conditions needed for it.

Since the four- and five-player cases are more complex, the authors only studied the probabilities of having at least one null player when the sum of the weights tends to infinity.

Zick et al. [20] mostly studied algorithmic issues related to quota manipulation. They listed several hard questions, one of which is deciding whether a quota maximizes or minimizes an agent's Shapley-Shubik power index which is coNP-hard. However, the authors gave quotas to maximise or minimise the voting power of given players as a function of their weights. They also provided a heuristic algorithm for minimising a player's power, which takes into account the overall weight distribution. Another focus of the paper was the special case of the above-mentioned hard problem, which is checking if the quota maximises the power of the player with the smallest weight or minimises the power of the largest player. Both questions can be reduced to checking if all players have equal voting power, which they found polynomial-time solvable.

Zuckerman et al. [21] also studied how the voting power of players changes if the quota is modified in weighted voting games. They analysed both the

Shapley-Shubik and Banzhaf indices. Suppose that a manipulator wants to minimise a specific player's power, in other words, making them a null player. In this case, it is useful to know if the manipulator can succeed or how much they can change a player's voting power. The authors provided tight worst-case bounds on the difference between the powers before and after changing the quota. They also focused on the algorithmic aspects of the problem of manipulating the quota and gave an efficient algorithm to compute if there is a value of the quota when a given player becomes a null player.

However, there are some negative results regarding finding the null players. Matsui and Matsui [15] showed that calculating the Banzhaf power index and the Shapley–Shubik power index for weighted majority games are NP-complete and checking if a given player is a null player for a fixed quota is coNP-complete.

3.3.3 Ways out of being a null player

As mentioned in Section 3.3, there was a real-life example of a null player, namely, Luxembourg formally being one in the European Economic Community. Mayer [16] analysed the scenario from an economic point of view since it seemed unlikely that a country had no power in the Council of Ministers for nearly fifteen years without anyone noticing it.

He identified two possibilities which might have allowed Luxembourg to have positive voting power.

The Council's rules for voting Mayer studied the different voting rules that were used in the Council and were defined by the EEC Treaty. He found that the weighted voting with the quota of 12 votes was not the only rule but, depending on the type of the proposal, the Council made most of the decisions by simple majority voting and for decisions in policy areas that did not require an initial proposal, the so-called *double majority rule* was applied. In the latter two rules, Luxembourg has a positive voting power. Mayer claims that the voting rule when Luxembourg was a null player was not used often.

The double majority rule could be a way out of having no power, since with a small additional rule to the original one, it ensures that there are no null players in the new voting game. The first version of the weighted voting rule was $[12; 4, 4, 4, 2, 2, 1]$ which means that the quota was 12 and the larger counties (Germany, Italy and France) had 4 votes, the Netherlands and Belgium 2 and Luxembourg had 1 vote. The double majority rule keeps the above-mentioned part but adds another criterion: the proposition gets accepted if the votes reach the quota of 12 and at least 4 counties vote in favour of it. With this addition, Luxembourg is not a null player anymore: the three big counties can reach the quota together but they would still miss a vote from a fourth country which can be Luxembourg. This shows a minimal winning coalition with Luxembourg participating in it.

Formally, this new voting rule is an intersection of two weighted voting games, one of them being the original rule $[12; 4, 4, 4, 2, 2, 1]$ and the second is the game with the quota of 4 and equal weights $[4; 1, 1, 1, 1, 1, 1]$. In the intersection of two games, a coalition S wins if and only if it is winning in both games. However, the double majority has a simpler form, it is equivalent to the game $[10; 3, 3, 3, 2, 2, 1]$. This example shows that a null player can have positive power in the new voting game by slightly changing the quota and the weights.

The Benelux Union For the second escape route, Mayer considered the historical background of the Benelux countries. In the twentieth century, there were numerous agreements signed by the countries to strengthen a good economic relationship such as the Union Economique Belgo-Luxembourgeoise which founded an economic union in 1922, the Benelux Customs Union

Treaty in 1944 and then the 1958 Treaty establishing the Benelux Economic Union for the next 50 years. Mayer assumes that the three countries worked closely together in the Council of Ministers as well and agreed to act as a block and vote together.

He analysed the game where the Benelux countries made a prior decision and then acted as a single player in the EEC Council. This escape route can be modelled by the *composite games*. The composite games consist of two stages, in the first one, blocks of players decide on a common standpoint and then in the second phase where every player takes part, they are bound to vote according to the first decision. Formally, $N_1, \ldots, N_k$ are non-empty and disjoint subsets of N, moreover, $N = \bigcup_{j=1}^{k} N_j$. Let $(N_1, v_1), \ldots, (N_k, v_k)$ describe simple games with the respective player sets and (K, v) is a simple game over set $K = 1, \ldots, k$. Mayer defined the characteristic function of the composite game as $u(S) = v(j : v_j(S \cap N_j) = 1)$ where the so-called v-composition is denoted by $u = v[v_1, \ldots, v_k]$.

In the case of the EEC and Benelux Union, the blocks besides the Benelux Union are the three big counties as separate blocks where the only voters are Germany, France and Italy. In the first stage, one can consider the weights of the Netherlands, Belgium and Luxembourg, however, it can be reduced to a simple majority voting, since any two countries form a winning coalition both in the weighted and unweighted cases. The second phase is a four-player simple game with the Benelux counties, Germany, France, and Italy as players. The weighted version $[12; 5, 4, 4, 4]$ is again equivalent to the simple game with four players in this specific case: the three big counties still form a winning coalition, and since the small ones are in a block, the Benelux Union and two of the big counties are also winning.

Consider the coalition consisting of Germany, Italy, the Netherlands and Luxembourg. This is a minimal winning coalition in the composite game since the Netherlands and Luxembourg are the majority in the block stage, therefore Belgium will also support them in the second phase. By adding the block system to the original voting game, Luxembourg is not a null player in the composite game—it is part of minimal winning coalitions.

Chapter 4

Null Player Out

Null players were introduced in Section 3.3 in Definition 3.3.1: player i is a null player if $f \circ w(P) = f \circ w(P')$ for voting rule f, weight vector w and preference profiles P and P' that differ only in coordinate i. It means that player i cannot influence the winner with their votes. The question arises if they cannot change the results by voting, should null players even participate? This chapter focuses on whether null players can influence the elections by staying away from voting or if they can be left out without any changes. Section 4.1 gives a brief overview of the literature about the null player out property in cooperative games. Section 4.2 is concerned with leaving out null players from weighted committee games. So far, very little attention has been paid to this area, therefore, the main focus of the section is to analyse the different weighted voting rules from the perspective of leaving out the null player. Firstly, some examples are shown where the null player out property is unsatisfied. Afterwards, some positive results will follow about the well-known voting rules.

4.1 NPO in cooperative games

If eliminating a null player from a cooperative game does not affect the payoff of the other players then the game satisfies the *Null Player Out (NPO)* property.

Derks and Haller [12] investigated the null player out property in game theory. In cooperative games, the null player axiom assigns zero utility to null players, however, the rest of society can still be affected by the absence of the null player. They gave an example when that is the case. In their paper, Derks and Haller studied the necessary and sufficient conditions for the NPO property. They found that removing null players from cooperative games with transferable utilities does not affect the payoff of the other players.

4.2 NPO in weighted committee games

The null player out property for weighted committee games means that for null player i in voting rule f and weights w it holds that $f \circ w(P) = f \circ w_{-i}(P_{-i})$ for all P where P_{-i} is the preference profile obtained by leaving out the preferences of player i and w_{-i} is the new weight vector.

4.2.1 Counterexamples

Most weighted committee games satisfy the NPO property, however, not all of them, as the following two voting rules demonstrate.

Voting rule I.

Suppose that n players participate and let $k \in \mathbb{N}$ be s.t. $1 < k < n$. Let player $n \mod k$ be the dictator, i.e. that player's first choice will be the winner.

In this case, everyone except for the dictator is a null player. Yet if player $l > k$ stays away from voting, the dictator will change in the new scenario. The two dictators can have different alternatives ranked first, if that is the case, the winner will change after leaving out a null player.

Voting rule II.
Suppose that there are n players. Let the winner be the n'th alternative modulo m.

These voting rules do not consider the weights of the players but similar rules can be constructed using the weight vector w instead of just the number of voters. Although these rules most likely will not have any real-life application, they prove that we have to be careful when deleting null players, as it might change the winner.

4.2.2 Sufficient conditions

In many cases, null players can be eliminated from the game without any changes. For instance, every weighted scoring rule satisfies NPO.

Claim 4.2.1. *Let f be a non-trivial weighted scoring rule with score vector s and minimum score $s_{min} = 0$. Furthermore, let $w \in \mathbb{N}^n$ be a weight vector. Player i is a null player if and only if $f \circ w(P) = f \circ w_{-i}(P_{-i})$ for all preference profiles P.*

Proof. First, suppose that $f \circ w(P) = f \circ w_{-i}(P_{-i})$ holds for some player i and all profiles P. We need to show that i is a null player, i.e. $f \circ w(P) = f \circ w(P')$ for all profiles P and P' that differ only in the preferences of i.

We use the assumption $f \circ w(P) = f \circ w_{-i}(P_{-i})$ for P and P, besides the observation that $P_{-i} = P'_{-i}$, therefore $f \circ w_{-i}(P_{-i}) = f \circ w_{-i}(P'_{-i})$. We get

$$f \circ w(P) = f \circ w_{-i}(P_{-i}) = f \circ w_{-i}(P'_{-i}) = f \circ w(P'), \tag{4.1}$$

which holds for all P, P', therefore i is a null player.

For the other direction, suppose that player i is a null player. Let $s'(x)$ denote the sum of the weighted scores of alternative x given by all players except for i, formally

$$s'(x) = \sum_{j \in N, j \neq i} w_j s^{|A|}_{|y \in A : y R_j x|}. \tag{4.2}$$

The total score of alternative x is

$$s(x) = \sum_{j \in N} w_j s^{|A|}_{|y \in A : y R_j x|} = s'(x) + w_i s_{|y \in A : y R_i x|}. \tag{4.3}$$

Suppose that $x = f \circ w(P) = f \circ w(P')$ where P and P' differ only in the preferences of null player i. Alternative $x = \text{argmax}_{y \in A}\, s(y)$ is the scoring rule winner and i is a null player, equivalently,

$$s'(x) + w_i s_k \geq s'(y) + w_i s_\ell \quad \forall y \neq x,\, \forall s_k, s_\ell \text{ pair.} \tag{4.4}$$

The inequality holds in the special case when i gives the lowest score to x and it still wins.

$$s'(x) + 0 w_i \geq s'(y) + w_i s_{max} \quad \forall y \neq x, \tag{4.5}$$

where s_{max} is the highest possible score. If player i does not vote, then x has score $s'(x)$. By inequality 4.5, $s'(x) > s'(y)$ for all alternatives y since the scoring rule is non-trivial, i.e. $s_{max} > 0$. Therefore x is the scoring rule winner of $f \circ w_{-i}$. □

After seeing that scoring rules have the NPO property, one can study the more general case of the voting rules that satisfy participation.

Claim 4.2.2. *If a voting rule is resistant to strategic abstention, then it also satisfies NPO.*

Proof. Let player i be a null player and $f \circ w(P) = f \circ w(P') = x$ for all P, P' preference profiles which differ only in i. Suppose that leaving out i, alternative y wins instead, i.e. $f \circ w_{-i}(P_{-i}) = y$. Since the preferences of the null player do not affect the winner x, we can assume that yP_ix. In this case, the player can achieve a better result by strategically staying away from the voting, therefore the voting rule would not satisfy participation. □

However, the opposite direction does not hold: a voting rule that satisfies NPO is not necessarily strategyproof, therefore the two definitions are not equivalent. The proof for this statement will follow in the next section.

4.2.3 Voting rules satisfying the NPO property

This section shows that most of the voting rules mentioned in Chapter 2 satisfy the null player out property. The following claim shows that plurality with runoff, a voting rule that suffers from strategic abstention, satisfies NPO and thus proves the above statement from the previous section.

Claim 4.2.3. *Weighted plurality with runoff satisfies NPO.*

Proof. Suppose that player i is a null player and alternative $a = f \circ w(P)$ is the winner independently of how i votes. Since i is a null player, a still wins if player i ranks it as the least preferred alternative. Suppose there is a preference profile P such that if null player i does not vote, then alternative a is no longer the winner but b would win instead. We can assume that there are at least three alternatives, otherwise, the voting would be equivalent to a scoring rule, and that bP_ia and xP_ib for all $x \neq a, b$ alternatives.

In all cases when i votes, a is selected for the second round and wins it. If i stays away, then alternatives a and b get the same number of first rankings as with P as well as the other alternatives except for $\max(P_i, A)$ which gets w_i times less votes. It follows that if b wins, then a and b must be selected for the second round both in $f \circ w(P)$ and $f \circ w_{-i}(P_{-i})$. However, the second round is a majority comparison which a wins even if player i votes for b, therefore if the other players do not change their votes and i stays away and b gets less votes and cannot win. This leads to a contradiction, hence plurality with runoff satisfies NPO. □

Similarly, null players can stay away from voting without any changes in instant runoff too.

Claim 4.2.4. *Weighted instant runoff satisfies NPO.*

Proof. Suppose that i is a null player of weighted instant runoff $f \circ w$ and alternative b wins if the null player does not participate. Let P be such that bP_ix for all alternatives $x \neq b$, which can be chosen this way since i is a null player and it does not affect the winner. Our goal is to show that the same alternatives get eliminated in each round when we apply the weighted instant runoff to P and P_{-i}, therefore the winner has to be the same as well.

Alternative b is the winner of the smaller game, therefore it cannot be the one with the lowest number of first ranks in either of the rounds. Suppose that it is alternative x_1 in the first round. Since all alternatives get the same number of votes in $f \circ w$ and $f \circ w_{-i}$ except for b which gets w_i extra votes when i is playing, x_1 has to have the lowest score in the original game too and be the one eliminated in the first round. The alternatives and the rankings in the next round stay the same in both cases except for b having w_i votes more in one of them. Similarly, suppose the same alternatives are eliminated in the first $k-1$ rounds. In that case, the number of first rankings stays the same in $f \circ w$ and $f \circ w_{-i}$ except for alternative b which gets more if i plays, hence if alternative x_k is the one to be deleted in round k of the game without player i then it will be the same with them as well. The same alternatives being deleted in every round leads to the same winner, namely $b = f \circ w(P)$. □

Other well-known voting rules that do not satisfy participation are the resolute Condorcet extensions when $n \geq 12$ and $m \geq 4$ according to Theorem 2.5.2: the No-Show paradox. They can be candidates for further rules where null players cannot be left out. However, this is not the case for well-known Condorcet extensions like Copeland's rule's weighted version.

Claim 4.2.5. *The weighted Copeland's rule satisfies NPO.*

Proof. Let $f \circ w$ be the weighted Copeland's rule. The output is calculated as $f \circ w(P_N, A) = \text{argmax}_{x \in A} |\{y \in A : xP_m y\}|$. Suppose, the winner alternative is x when the input preference profile is P_N and player i is a null player. It follows, that $f \circ w(P_N) = x$ in the special case too when the null player i's preferences are $yP_i x$ for all $y \neq x$. By leaving out the null player, the number of alternatives that are majority-dominated by x does not change since x was ranked to the last place by i, i.e. $|\{y \in A : xP_m y\}| = |\{y \in A : x(P_{-i})_m y\}|$. The values of other alternatives do not increase, which means that x is the argument of the maxima and therefore the winner. □

However, as the following counter-example shows, not all Condorcet extensions satisfy the null player out property.

> **Voting rule III.**
>
> If a Condorcet winner exists, then it is selected as a winner, otherwise, apply voting rule I.

Voting rule III. is a Condorcet extension, but similarly to voting rule I., it does not satisfy the NPO property.

4.2.4 Strategyproofness

Strategyproofness and the NPO property do not imply each other in any direction, for which the following two voting rules serve as examples.

The NPO property does not necessarily hold if a voting rule is strategyproof. An example of this is voting rule I, which is strategyproof: there is a dictator whose first alternative will win, therefore they do not need to misrepresent their preferences, while the other voters cannot change the result of the voting by any ranking of the alternatives. However, it was shown that voting rule I. does not satisfy the null player out property.

Similarly, plurality is a counter-example for the other direction: NPO property does not imply strategyproofness since the null player can be left out without affecting the results, but plurality is not strategyproof.

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