Minimally rigid tensegrity frameworks

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1 Introduction

Rigidity theory usually deals with the examination of static structures consisting of joints and some rigid bars connecting pairs of them. The structure is typically described by a graph, where the vertices represent point-like joints that can rotate freely in any direction, and the edges represent inextendible, incompressible bars. This model is relatively simple, mathematically tractable, and also very helpful for understanding a wide range of practical structures; however, for the sake of manageability, we need to neglect several aspects that may be significant in certain real-world applications. For example, in practice it is common for a structure to be designed in such a way that certain bars only experience tensile forces during their use, and they do not need to withstand compression at all. In such cases, it seems wasteful to use a rigid bar that can bear forces in all directions; instead, we can replace it with a cable that can withstand tensile forces very well but immediately deforms under compressive forces. For example, consider the bar structure modeling a hook mounted on the wall as shown in the left side of Figure 1. It is clear that here, instead of the upper bar, we could use a cable, and instead of the lower bar, a strut (which is the opposite of a cable: it can withstand any compressive force, but deforms under any tensile force), while still retaining the weight hung on it, thereby potentially reducing the overall weight and material cost of the structure.

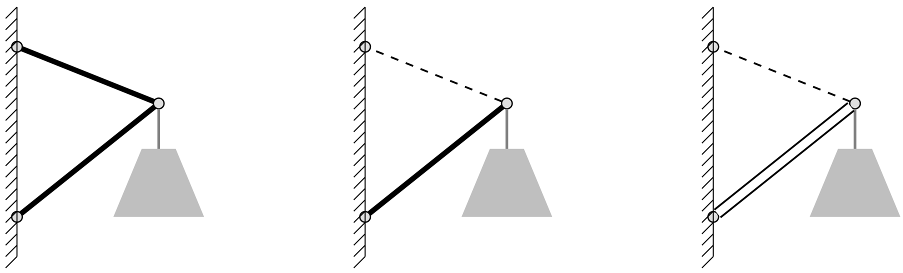


Figure 1: Simple models of a hook mounted on the wall. Bars are denoted by solid lines, cables by dashed lines and struts by double lines.

This approach naturally motivates a generalization of bar-and-joint frameworks, which we call tensegrity frameworks. A tensegrity graph is a graph where the edges are labeled according to whether they represent cables or struts. By assigning positions to the vertices of a tensegrity graph, we obtain a tensegrity framework. In the literature, tensegrity graphs are sometimes defined to allow three types of edge labels: cable, strut, and bar. If a cable and a strut both run parallel between two of the joints, then this joint pair will behave exactly as if they are connected by a bar in the model (the cable prevents their distance from increasing, while the strut prevents their distance from decreasing). Therefore, here we allow such parallel cable-strut pairs, but we only use these two types (cable and strut) of labels. We give precise definitions in the next section.

The interest in tensegrity frameworks was significantly raised by the works of the sculptor Kenneth Snelson in the 1940s. His creations, consisting of bars and cables, are visually striking as they stably stand with the bars not touching each other, creating an illusion of them floating in the air, see for example Figure 2. The term "tensegrity" was coined by Snelson's advisor, Richard Buckminster Fuller, and it originates from the combination of "tension" and "structural integrity".

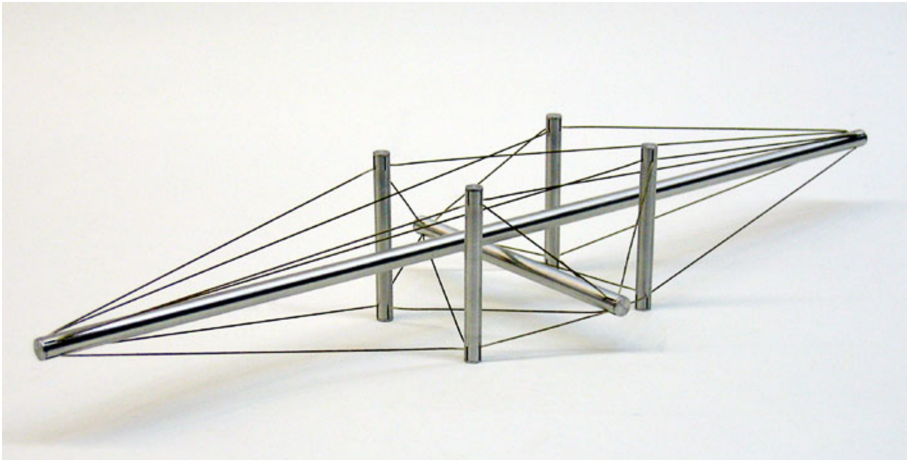
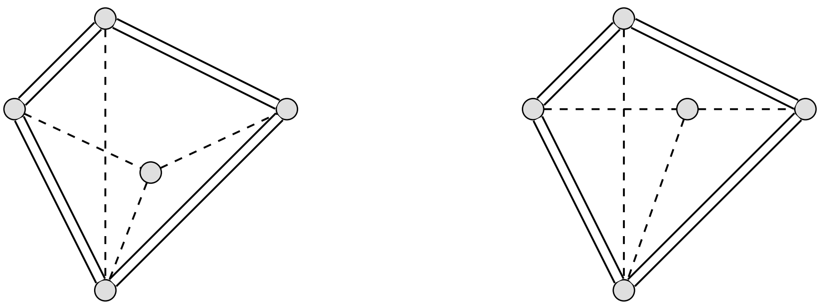


Figure 2: One of Kenneth Snelson's small sculptures, called *X-tend* [19].

Similar to bar-and-joint frameworks, the fundamental question about tensegrity frameworks is whether they are stable (rigid) or not. This area motivates numerous interesting mathematical questions, problems, and conjectures, and has obvious practical applications. Although the fundamental questions are similar, it is not surprising that there are significant differences between the practical and theoretical perspectives. For example, in the mathematical approach that we use in this thesis, we assume that the cables resist any attempt to pull them apart with any force: they do not stretch at all and do not snap. Similarly, we assume that struts do not break, and their length does not decrease under any compressive force. Moreover, we also assume that struts can be stretched to any length. It is clear that in a physical realization we cannot rely on these simplifications and must consider the load-bearing capacity and flexibility of the elements of the structure and compare it with the forces acting on them. Additionally, mathematicians do not usually consider the extension of joints; they simply regard them as point-like, nor do we consider in the mathematical model whether two members (cables or struts) intersect each other. Another significant difference between the two perspectives is that in engineering applications, frameworks do not typically "float in the air"; there is always some support or suspension holding the structure. However, in mathematics, there is a rich literature in studying frameworks only represented by a graph without any pinned points, and this thesis also deals with such frameworks. Furthermore, mathematicians often study frameworks in high dimensional spaces, while understandably, engineers are more interested in 1, 2 and 3-dimensions.

But what do we mean by saying that a structure is stable? A d-dimensional tensegrity framework is an edge-labeled graph alongside with a representation of its vertices in the d-dimensional space. Thus, the length of each cable and strut is defined by the Euclidean distance between its endpoints. A deformation of a framework is an operation which is not an isometry, that is, the distance between some two vertices changes. We say that a tensegrity framework is rigid if it cannot be continuously deformed in such a way that the lengths of the cables do not increase and the lengths of the struts do not decrease. For example, the frameworks in Figure 3a and 3b are both rigid in 2-dimensions. However, many engineers would be unhappy with calling the 3b framework rigid, because if we do not use ideal cables (i.e. they can stretch a bit), then we can move the central point of the framework slightly downward with much less force in Figure 3b than the central point of the framework in Figure 3a, as the cables only hold it horizontally. This occurrence motivates a stronger rigidity definition, called infinitesimal rigidity.



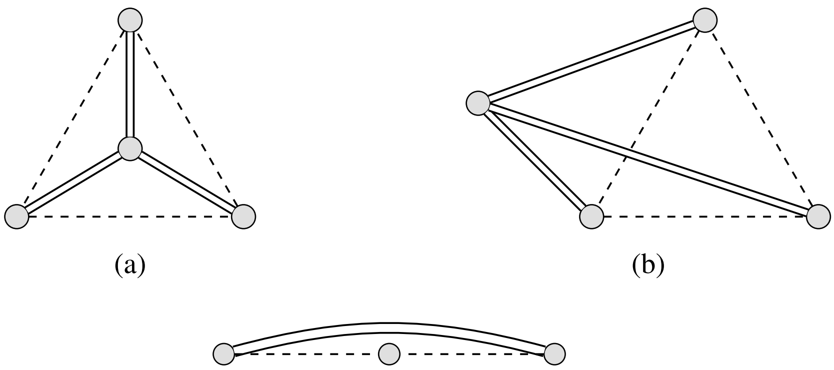
(a) (b)

Figure 3: Tensegrity frameworks in $\mathbb{R}^2$.

An infinitesimal motion of a tensegrity framework can be seen as a function that assigns a vector to each vertex, satisfying that if the vertices move with velocities equal to the given vectors, then the first derivative of the cable's length is at most zero and the first derivative of the strut's length is at least zero. Clearly, applying a translation or rotation to the entire framework and assigning the initial velocity of this motion to the points results in an infinitesimal motion, which we call a trivial infinitesimal motion. The framework is infinitesimally rigid if it has no infinitesimal motion other than the trivial ones. Therefore, the structure in Figure 3b is not infinitesimally rigid, since by assigning a downward-pointing vector to the central vertex and zero vectors to the other vertices, we obtain a non-trivial infinitesimal motion.

Another stronger definition of rigidity is global rigidity. If a d-dimensional tensegrity framework that satisfies that all d-dimensional representations in which the lengths of the cables are not greater and the lengths of the struts are not smaller are congruent to it, then we say that the structure is globally rigid. That is, the framework does not even have a non-continuous deformation. If this also holds for every D-dimensional representation where $D \geq d$, then the structure is universally rigid.

There are frameworks that are infinitesimally rigid but not globally rigid. For example, the framework in Figure 4a is infinitesimally rigid and has a non-continuous deformation as shown in Figure 4b. There are also examples where the converse holds: the framework in Figure 4c is globally rigid (and also universally rigid) but not infinitesimally rigid in $\mathbb{R}^2$.



(a) (b) (c)

Figure 4: Tensegrity frameworks in $\mathbb{R}^2$.

Studying the rigidity of frameworks, it turns out that it causes many difficulties when some vertices are in "special" positions in the representation, as is the case with Figures 3a and 3b, where the difference in their rigidity is caused by three vertices lying on a straight line in Figure 3b. In higher dimensions and larger structures, such geometric coincidences might significantly increase the difficulty of addressing rigidity questions. Therefore, in rigidity theory, many statements are formulated for frameworks where we assume that there are no algebraic relationships among the coordinates of their vertices. These representations of vertices are called generic configurations. On one hand, this is not a very strong assumption, as randomly choosing the coordinates with a natural continuous distribution, the chance of hitting a non-generic configuration is zero. However, note that the majority of real-life structures are not generic.

The rest of this thesis is organized as follows. In Section 2, after introducing the basic definitions, we summarize some previous results related to the rigidity of tensegrities. In Section 3, we present some definitions and results from the field of matroid theory, especially the concepts related to the rigidity matroid. We use these in Section 4 to prove the main results of this thesis. In Section 4.1 we show sharp upper bounds in one, two, and three dimensions on the number of edges of minimally infinitesimally rigid tensegrity frameworks containing no parallel members. These bounds turn out to be significantly better than the also sharp upper bound for the general case, which we prove for all dimensions in Section 4.2. The results presented in Section 4 come from our joint research with Adam Clay and Tibor Jordán, and will be published in an upcoming paper. Finally, in Section 5, we list some related open problems.

For more details about tensegrity frameworks we recommend [1].

2 Tensegrity frameworks

2.1 Definitions

A *tensegrity graph* $T = (V, C \cup S)$ is a graph on vertex set V, in which each edge e is labelled as a *cable* or a *strut*. Accordingly, the edge set of T is partitioned into two sets, C and S. We call the elements of $C \cup S$ the *members* of T.

A *d-dimensional tensegrity framework* (T, p) is a pair, where $T = (V, C \cup S)$ is a tensegrity graph and $p : V \to \mathbb{R}^d$ is a map, satisfying $p(u) \neq p(v)$ for each $uv \in C \cup S$. We also say that (T, p) is a *d-dimensional realization* of T.

The *underlying graph* of $T = (V, C \cup S)$, denoted by $\overline{T} = (V, E)$, is a simple graph on vertex set V in which $uv \in E$ if and only if $uv \in C \cup S$ holds. The bar-and-joint framework $(\overline{T}, p)$ is obtained from the tensegrity framework (T, p) by replacing the cables and struts between each of the adjacent pair of vertices with a single bar.

Let (T, p) and (T, q) be two d-dimensional realizations of the tensegrity graph $T = (V, C \cup S)$. The framework (T, p) *dominates* (T, q) if we have

$$\begin{aligned} ||p(u) - p(v)|| &\geq ||q(u) - q(v)|| \text{ for each cable } uv \in C, \\ ||p(u) - p(v)|| &\leq ||q(u) - q(v)|| \text{ for each strut } uv \in S. \end{aligned}$$

In this case we also use the term (T, q) satisfies the *member constraints* of (T, p).

The frameworks (T, p) and (T, q) are *congruent* if we have

$$||p(u) - p(v)|| = ||q(u) - q(v)|| \text{ for each vertex } u, v \in V.$$

Here $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$.

A tensegrity framework (T, p) is *rigid* if there is some $\varepsilon > 0$ such that any other realization (T, q) with $||p(v) - q(v)|| \leq \varepsilon$ for all $v \in V$ that satisfies the member constraints of (T, p) is, in fact, congruent to it.

An *infinitesimal motion* of a tensegrity framework (T, p) is an assignment $m : V \longrightarrow \mathbb{R}^d$ which satisfy

$$(p(u) - p(v)) \cdot (m(u) - m(v)) \leq 0 \text{ for each cable } uv \in C,$$
$$(p(u) - p(v)) \cdot (m(u) - m(v)) \geq 0 \text{ for each strut } uv \in S.$$

A d-dimensional tensegrity framework (T, p) is *infinitesimally rigid* if every infinitesimal motion of (T, p) is an infinitesimal isometry of $\mathbb{R}^d$.

The *rigidity matrix* $R(T, p)$ of a tensegrity framework (T, p) is a matrix of size $|C \cup S| \times d|V|$, where, for each edge $e = uv \in C \cup S$, in the row corresponding to e, the entries in the two columns corresponding to vertices u and v contain the d coordinates of $(p(u) - p(v))$ and $(p(v) - p(u))$, respectively, and the remaining entries are zeros.

For a subset A of edges of T we use $R_A(T, p)$ to denote the submatrix of the rigidity matrix $R(T, p)$ induced by the rows of A. An infinitesimal motion of a tensegrity framework (T, p) is often considered as a vector $m \in \mathbb{R}^{d|V|}$ such that $R_C(T, p) \cdot m \leq 0$ and $R_S(T, p) \cdot m \geq 0$.

A d-dimensional tensegrity framework (T, p) is *globally rigid* if the only d-dimensional realizations satisfying its member constraints are the ones congruent to it.

A *stress* of a tensegrity framework (T, p) is a function $\omega : C \cup S \longrightarrow \mathbb{R}$, which assigns a scalar to each edge of T such that

$$\omega(e) \leq 0 \text{ for each cable } e \in C,$$
$$\omega(e) \geq 0 \text{ for each strut } e \in S,$$

and

$$\sum_{uv \in C \cup S} \omega(uv)(p(u) - p(v)) = 0 \text{ for each vertex } v \in V.$$

A stress is often considered as a vector $\omega \in \mathbb{R}^{C \cup S}$ such that $\omega(e) \leq 0$ for each $e \in C$, $\omega(e) \geq 0$ for each $e \in S$, and $\omega \cdot R(T, p) = 0$.

The *support* of a stress ω of (T, p) is the set of edges with non-zero stress, i.e.

$$\text{supp}(\omega) = \{e \in C \cup S : \omega(e) \neq 0\}.$$

A *proper stress* ω of a tensegrity framework (T, p) is a stress of (T, p) with every edge in its support, i.e.

$$\omega(e) < 0 \text{ for each cable } e \in C,$$
$$\omega(e) > 0 \text{ for each strut } e \in S,$$

and

$$\sum_{uv \in C \cup S} \omega(uv)(p(u) - p(v)) = 0 \text{ for each vertex } v \in V.$$

A d-dimensional realization (T, p) of the tensegrity graph $T = (V, C \cup S)$ is called *generic realization* if the set of the $d|V|$ coordinates of the points $p(v)$, $v \in V$, is algebraically independent over the rationals (i. e. the only polynomial with integer coefficients satisfied by the coordinates of all the vertices is the zero polynomial).

The d-dimensional framework (T, p) is in *general position* if

$$\text{rank} R_A(T, p) = \max\{\text{rank} R_A(T, q) : (T, q) \text{ is a } d\text{-dimensional realization of } T\}$$

for every non-empty $A \subseteq C \cup S$. We also use the term that (T, p) is a *general realization*.

A realization p is *injective* if $p(u) \neq p(u)$ for all pairs of distinct vertices $u, v \in V$.

A *d-dimensional bar-and-joint framework* (G, p) is a pair, where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map, satisfying $p(u) \neq p(v)$ for each $uv \in E$.

The *rigidity (infinitesimal rigidity, global rigidity resp.)* of a bar-and-joint framework (G, p) with $G = (V, E)$ is equivalent to the rigidity (infinitesimal rigidity, global rigidity) of the tensegrity framework (T, p), where T is constructed from G by replacing each bar with a parallel cable and strut, i.e. $T = (V, C \cup S)$ where $C = S = E$.

For a bar-and-joint framework (G, p) with $G = (V, E)$, the rigidity matrix $R(G, p)$ is the same as the rigidity matrix of a tensegrity framework (T, p) with $T = (V, C \cup S)$ and $C \cup S = E$.

2.2 Previous results

It will be convenient to use the following notation:

$$S(|V|, d) = \begin{cases} d|V| - \binom{d+1}{2} & \text{if } |V| \geq d + 2 \\ \binom{|V|}{2} & \text{if } |V| \leq d + 1 \end{cases}$$

A fundamental result about infinitesimally rigid bar-and-joint frameworks is the following.

Lemma 2.1. [[9], Lemma 1.2.1.] Let (G, p) be a bar-and-joint framework in $\mathbb{R}^d$, where $G = (V, E)$. Then

$$\mathrm{rank} R(G, p) \leq S(|V|, d),$$

and (G, p) is infinitesimally rigid if and only if

$$\mathrm{rank} R(G, p) = S(|V|, d).$$

It is proved in [2], that infinitesimal rigidity and rigidity are equivalent for tensegrity frameworks in general position. Note that if (T, p) is generic then it is general.

The rigidity and global rigidity of bar-and-joint frameworks is a generic property, that is, for any fixed dimension d, either all generic realizations in $\mathbb{R}^d$ are rigid (globally rigid, respectively), or none of them are [3, 4]. So the rigidity (global rigidity) of a framework depends only on its graph and not the particular realization, if it is assumed to be generic. We say that a graph is rigid (globally rigid) in d-dimensions if every (or equivalently, if some) of its generic realization in $\mathbb{R}^d$ is rigid (globally rigid). Both of the problems of characterizing when a graph is rigid and when a graph is globally rigid have been solved for $d = 1, 2$ and are major open problems for $d \geq 3$.

For tensegrity graphs, the situation is different: it may happen that some generic realizations are rigid, while others are not. For example, consider the graph on Figure 4a and 4b. It has rigid as well as non-rigid generic realizations in $\mathbb{R}^2$. This makes some of the questions concerning tensegrities more difficult than the corresponding questions for bar-and-joint frameworks. However, we can still explore the connection between a tensegrity graph and the rigidity of its generic realizations.

We may ask which tensegrity graphs have a rigid (globally rigid, respectively) generic realization in d-dimensions. We call these tensegrity graphs *weakly rigid* (*weakly globally rigid*) in $\mathbb{R}^d$. We may also require every generic realization of a tensegrity graph in d-dimensions to be rigid (globally rigid, respectively). Tensegrity graphs with this stronger property are called *strongly rigid* (*strongly globally rigid*) in $\mathbb{R}^d$. Characterizing which graphs are weakly and strongly rigid is an interesting but as yet not very well understood area. In the following, we summarize some of the important previous results related to generic rigidity of tensegrity graphs.

Regarding weak rigidity, the only case we can handle is the 1-dimensional case, for which the following polynomial-time checkable characterization is known, attributed to Recski and Shai.

Theorem 2.1 (Recski, Shai [5])**.** A tensegrity graph $T = (V, C \cup S)$ is weakly rigid in $\mathbb{R}^1$ if and only if

- $\overline{T}$ is 2-edge-connected,

- every 2-vertex-connected component of T contains at least one cable and one strut.

Characterizing the weakly rigid tensegrity graphs is open for $d \geq 2$.

In the case of strong rigidity, Jackson, Jordán and Király gave a combinatorial characterization of strongly rigid graphs in $\mathbb{R}^1$, called alternating cycle property.

Let $T = (V, C \cup S)$ be a tensegrity graph. A cycle in T is *alternating* if no two incident edges along the cycle have the same label. We say that T has the *alternating cycle property* if for all proper bipartitions $(U, V - U)$ of V there is an alternating cycle in the bipartite subgraph $H = (V, E(U, V - U))$ of T induced by the bipartition.

Theorem 2.2 (Jackson, Jordán, Király [6]). Let $T = (V, C \cup S)$ be a tensegrity graph. Then T is strongly rigid in $\mathbb{R}^1$ if and only if T has the alternating cycle property.

They also showed that if there exists a polynomial-time algorithm that decides whether a 2-edge-labeled graph has the alternating cycle property, then we could solve the 3-SAT problem in polynomial time, leading to the following theorem.

Theorem 2.3 (Jackson, Jordán, Király [6]). Recognizing strongly rigid tensegrity graphs in $\mathbb{R}^1$ is co-NP-hard.

It is an open question, whether this hardness result concerning strong rigidity extends to higher dimensions.

Regarding strong and weak global rigidity, quite similar results are known. Garamvölgyi proved the following necessary condition for d-dimensional weak global rigidity.

Theorem 2.4 (Garamvölgyi [7]). Let $T = (V, C \cup S)$ be a weakly globally rigid tensegrity graph. Then either it is a complete graph with only parallel cable-strut members, or it has at least $d + 2$ vertices and satisfies the following conditions:

- $\overline{T}$ is globally rigid in $\mathbb{R}^d$
- T contains at least $\frac{d+1}{2}$ struts
- The graph (V, C) is connected.

He also showed that for $d = 1$, the condition of the theorem is sufficient, however, for $d \geq 2$ it is not. Thus, similarly to weak rigidity, we have a polynomial-time checkable characterization for weakly globally rigid graphs in $\mathbb{R}^1$, and for $d \geq 2$ the problem of characterizing globally rigid graphs is still open.

Theorem 2.5 (Garamvölgyi [7]). A tensegrity graph $T = (V, C \cup S)$ with $|V| \geq d + 2$ is weakly globally rigid in $\mathbb{R}^1$ if and only if

- $\overline{T}$ is 2-connected,
- T contains at least one strut,
- the graph (V, C) is connected.

The proof of this theorem also implies that a tensegrity graph has a generic globally rigid realization in $\mathbb{R}^1$ if and only if it has a generic universally rigid realization in $\mathbb{R}^1$.

In this paper Garamvölgyi also proved a necessary condition for strong global rigidity of tensegrity graphs, called the odd cycle property, and showed that it is sufficient in some special cases. Using this, he proved the following hardness result about d-dimensional strongly rigid tensegrity graphs.

Theorem 2.6 (Garamvölgyi [7])**.** For any $d \geq 1$ recognizing strongly globally rigid tensegrity graphs in $\mathbb{R}^d$ is co-NP-hard.

Another approach to the topic of tensegrities is studying the rigidity properties of tensegrity frameworks instead of tensegrity graphs. So, given a tensegrity graph T and a d-dimensional realization p of its vertices, the question is whether the resulting framework (T, p) is rigid (resp. infinitesimally rigid, globally rigid).

The following fundamental theorem of Roth and Whiteley gives a nice characterization of infinitesimally rigid tensegrity frameworks.

Theorem 2.7 (Roth, Whiteley [8])**.** Let (T, p) be a tensegrity framework in $\mathbb{R}^d$. Then (T, p) is infinitesimally rigid if and only if

- $(\overline{T}, p)$ is infinitesimally rigid,
- there exists a proper stress of (T, p).

Another interesting lemma from the same paper, which we will use later is the following.

Lemma 2.2 (Roth, Whiteley [8])**.** Let (T, p) be a realization of the tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^d$ and let $e \in C \cup S$. If there exists a stress of (T, p) with e in its support, then there exists a stress ω of (T, p) with e in its support and such that $\text{rank} R_A(T, p) = |A|$ for every $A \subset \text{supp}(\omega)$.

3 Rigidity matroid

The aim of this section is to define concepts and prove some statements from the field of matroid theory so that they can be used in Section 4 to prove the main results of this thesis. For more details, we recommend [3, 9, 10], which are also the sources we used for this section.

3.1 Matroids

A set-system $\mathcal{M} = (E, \mathcal{I})$ is called a *matroid* if it satisfies the following properties, called *independence axioms*.

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $X \subseteq Y \in \mathcal{I}$, then $X \in \mathcal{I}$,

(I3) for every subset $X \subseteq E$, the maximal subsets of X which are in $\mathcal{I}$ have the same cardinality.

The members of $\mathcal{I}$ are *called independent sets*; all other subsets of E are dependent. Axiom (I1) requires the empty set to be independent, (I2) means that the subset of an independent set is independent, while (I3) is another way of saying that the maximal independent subsets of each subset of E are of the same size. This maximum number is called the *rank* of X and is denoted by $r(X)$, where r is the *rank function* of the matroid and $r(E)$ is the *rank of the matroid*. A subset $B \subseteq E$ is called a *basis* of the matroid if B is a maximal independent subset of E. A subset $C \subseteq E$ is called a *circuit* of the matroid if C is dependent but every proper subset of C is independent.

Theorem 3.1. [[10], Theorem 5.2.1] Let C_1 and C_2 be two distinct circuits of the matroid $\mathcal{M}$ and let $e \in C_1 \cap C_2$. Then there is a circuit C for which $C \subseteq C_1 \cup C_2 - e$.

Proof. Suppose indirectly that two circuits C_1 and C_2 are violating the statement. Then $C_1 \cup C_2 - e$ is independent, while $C_1 \cup C_2$ is not, thus $r(C_1 \cup C_2) = |C_1 \cup C_2| - 1$. On the other hand $C_1 \cap C_2$ is independent, so it can be extended to a maximal independent subset F of $C_1 \cup C_2$, for which $|F| = |C_1 \cup C_2| - 1$. But F includes neither C_1 nor C_2, so its cardinality is at most $|C_1 \cup C_2| - 2$, a contradiction. □

Theorem 3.2. [[10], Theorem 5.2.3] Let C_1 and C_2 be two distinct circuits of the matroid $\mathcal{M}$ and let $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$. Then there is a circuit C for which $f \in C \subseteq C_1 \cup C_2 - e$.

Proof. Suppose indirectly that two circuits C_1 and C_2 are violating the statement, select C_1, C_2 such that $|C_1 \cup C_2|$ is minimal. By Theorem 3.1 there is a circuit C_3 for which $C_3 \subseteq C_1 \cup C_2 - e$. Then $f \notin C_3$.

Since C_3 is not a subset of C_1, there is an element $g \in C_3 - C_1$ which is in C_2. By the minimality of $|C_1 \cup C_2|$, the statement of the theorem holds for circuits C_2 and C_3. Thus, there exists a circuit $C_4 \subseteq C_2 \cup C_3 - g$ for which $e \in C_4$. Since $f \in C_1$ and $f \notin C_4$, $C_1 \neq C_4$.

Since $C_1 \cup C_4 \subset C_1 \cup C_2$, the statement of the theorem holds for C_1 and C_4, and hence there is a circuit $C \subseteq C_1 \cup C_4 - e \subseteq C_1 \cup C_2 - e$ for which $f \in C$, a contradiction. □

An element $e \in E$ is a *bridge* if $r(E - e) = r(E) - 1$ holds. This definition is equivalent to requiring that e is the element of every basis of $\mathcal{M}$, or equivalently: e is not included in any circuit of $\mathcal{M}$. A matroid $\mathcal{M}$ is *bridgeless*, if it contains no bridges.

3.2 Circuit decomposition of matroids

In the following we present the definition of a new concept called matroid circuit decomposition, and prove some properties related to it. The results demonstrated here are used in Section 4 for the new results about minimal tensegrities.

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $(C_1, \ldots, C_t)$ be a non-empty sequence of circuits of $\mathcal{M}$. Let $D_0 = \emptyset$, and $D_i = C_1 \cup \ldots \cup C_i$ for $1 \le i \le t$. We say that $(C_1, \ldots, C_t)$ is a *partial circuit decomposition* of $\mathcal{M}$ if for any $2 \le i \le t$ the following properties hold:

(E1) $C_i - D_{i-1} \neq \emptyset$

(E2) no circuit C'_i satisfying (E1) has $C'_i - D_{i-1}$ properly contained in $C_i - D_{i-1}$.

A *circuit decomposition* of $\mathcal{M}$ is a partial circuit decomposition with $D_t = E$. The set $C_i - D_{i-1}$ is denoted by $\tilde{C}_i$ for $1 \le i \le t$.

Note that if we replace (E1) with the property

(E1') $C_i - D_{i-1} \neq \emptyset$ and $C_i \cap D_{i-1} \neq \emptyset$

in the definition of partial circuit decomposition, then we obtain the definition of partial ear decomposition, which is a well-known concept in matroid theory [11].

Lemma 3.1. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then $\mathcal{M}$ has a circuit decomposition if and only if $\mathcal{M}$ is bridgeless. Furthermore, if $\mathcal{M}$ is bridgeless, then every partial circuit decomposition of $\mathcal{M}$ can be extended to a circuit decomposition.

Proof. By definition, $\mathcal{M}$ is bridgeless if and only if each $e \in E$ is in a circuit C_e of $\mathcal{M}$, so the "only if" direction is immediate.

Suppose that $\mathcal{M}$ is bridgeless and we have a partial circuit decomposition $(C_1, \ldots, C_j)$ of $\mathcal{M}$ with $E - D_j \neq \emptyset$. Let $e \in E - D_j$. Since $\mathcal{M}$ is bridgeless, there exists a circuit C_e with $e \in C_e$. The circuit C_e satisfies (E1), so there exists a circuit C' satisfying both (E1) and (E2). Then $(C_1, \ldots, C_j, C')$ is a partial circuit decomposition of $\mathcal{M}$. Clearly, we can choose a first circuit C_1 and iterating this method we get a circuit decomposition of $\mathcal{M}$. □

Lemma 3.2. Let $(C_1, \ldots, C_t)$ be a circuit decomposition of $\mathcal{M}$. Then

$$r(D_j) - r(D_{j-1}) = |\tilde{C_j}| - 1$$

for all $1 \leq j \leq t$.

Proof. Let $e \in \tilde{C_j}$. By (E2), there is no circuit C' in $\mathcal{M}$ with $C' - D_{j-1} \neq \emptyset$ and $C'_i - D_{i-1}$ properly contained in $C_i - D_{i-1}$, so each $f \in \tilde{C_j} - e$ is a bridge in $D_j - e$. Thus, $r(D_j) \geq r(D_{j-1}) + |\tilde{C_j}| - 1$. As the inequality cannot be strict, the lemma follows. □

3.3 Rigidity matroid

The *rigidity matroid* of a d-dimensional framework (G, p) is defined on the edge set E of G, where $F \subseteq E$ is independent if and only if the corresponding rows of the rigidity matrix $R(G, p)$ are linearly independent. Any two general d-dimensional frameworks (G, p) and (G, q) have the same rigidity matroid, because for a subset A of the edges of G, if the rows of $R_A(G, p)$ are linearly independent then the rows of $R_A(G, q)$ must also be linearly independent and vice versa.

It is not difficult to see that $\mathcal{R}_1(G)$ is the circuit matroid of G. According to Laman's theorem, $\mathcal{R}_2(G)$ is also well-characterized, since it is equivalent to the sparsity matroid of G. More precisely, the edge set of a subgraph H of G is independent in the rigidity matroid if and only if H is sparse, i.e. for every subset X of at least 2 vertices in H, the number of edges spanned by X is at most $2|X| - 3$. It is an open problem to find good characterizations for the d-dimensional rigidity matroid of a graph when $d \geq 3$ [9].

A simple graph $G = (V, E)$ is said to be an $\mathcal{R}_d$-circuit if E is a circuit (i.e. a minimal dependent set) in $\mathcal{R}_d(G)$. The following property is proved in [12].

Theorem 3.3 ([12], Corollary 2.6)**.** Let G be an $\mathcal{R}_d$-circuit. Then G is $(d + 1)$-edge-connected.

For a graph $G = (V, E)$ with $|V| \geq d + 2$ let

$$k_d(G) = d|V| - \binom{d+1}{2} - r_d(G)$$

denote the *degrees of freedom* of G. By Lemma 2.1 we have $k_d(G) \geq 0$.

The following lemma gives an upper bound on the number of edges of an $\mathcal{R}_d$-circuit for $d \geq 2$, which we need for the proofs of the new results about minimal tensegrities presented in Section 4.

Lemma 3.3. Let $G = (V, E)$ be an $\mathcal{R}_d$-circuit for some $d \geq 2$. Then

$$|E| \leq (d+1)|V| - \binom{d+2}{2} - \frac{d+1}{d-1} k_d(G).$$

Proof. By the definition of $k_d(G)$ and from the fact that if G is an $\mathcal{R}_d$-circuit then it has $r_d(G) + 1$ edges:

$$|E| = d|V| - \binom{d+1}{2} - k_d(G) + 1. \tag{1}$$

It follows from Theorem 3.3 that the minimum degree of an $\mathcal{R}_d$-circuit is at least $d + 1$. Therefore, from (1) we have

$$k_d(G) = -|E| + d|V| - \binom{d+1}{2} + 1 \leq$$
$$\leq -\frac{d+1}{2}|V| + d|V| - \binom{d+1}{2} + 1,$$

which leads to

$$k_d(G) \leq \frac{d-1}{2}|V| - \binom{d+1}{2} + 1. \tag{2}$$

Multiplying both sides of the inequality (2) by $-\frac{2}{d-1}$, adding $|V|$ to both sides and then performing transformations on the right-hand side:

$$|V| - \frac{2}{d-1} k_d(G) \geq \frac{2}{d-1}\binom{d+1}{2} - \frac{2}{d-1} =$$
$$= \frac{d(d+1)}{d-1} - \frac{2}{d-1} = \frac{d^2+d-2}{d-1} = d + 2 =$$
$$= d + 1 + 1 = \binom{d+2}{2} - \binom{d+1}{2} + 1,$$

where in the last step we use the well-known equality: $\binom{d+2}{2} = \binom{d+1}{2} + \binom{d+1}{1}$. Since $\frac{2}{d-1} k_d(G) = \frac{d+1}{d-1} k_d(G) - k_d(G)$, we get

$$|V| + \binom{d+1}{2} + k_d(G) - 1 \geq \binom{d+2}{2} + \frac{d+1}{d-1} k_d(G).$$

Multiplying by -1 and adding $(d+1)|V|$ to both sides:

$$d|V| - \binom{d+1}{2} - k_d(G) + 1 \leq (d+1)|V| - \binom{d+2}{2} - \frac{d+1}{d-1} k_d(G),$$

where the left-hand side is equal to $|E|$ by (1), resulting the inequality that we required.

□

4 Minimally rigid tensegrities

An infinitesimally rigid tensegrity framework (T, p) in $\mathbb{R}^d$ is called *minimally infinitesimally rigid*, if $(T - e, p)$ is not infinitesimally rigid in $\mathbb{R}^d$ for every edge e of T.

Studying the minimal elements of a graph family is often a key step in finding a constructive characterization. The fundamental question of this section is that how many edges a minimally infinitesimally rigid tensegrity framework can have.

An easy consequence of Lemma 2.1 is the following.

Corollary 4.1. Let (G, p) be a minimally infinitesimally rigid bar-and-joint framework in $\mathbb{R}^d$, where $G = (V, E)$. Then

$$|E| = S(|V|, d).$$

Note that if a d-dimensional tensegrity framework (T, p) has at most $d + 1$ vertices, then, according to Theorem 2.7 and Corollary 4.1, if (T, p) is minimally infinitesimally rigid, then T can only be the cable-strut complete graph (i.e., $(V, C) = (V, S) = K_{|V|}$). In this case, we understand the minimal instances well, and they are not very interesting. Therefore, in the following, we will only consider cases where the number of vertices is at least $d + 2$.

By Theorem 2.7, there is a linear dependence on the rows of the rigidity matrix of an infinitesimally rigid tensegrity framework, thus, it is immediate from Lemma 2.1 that a d-dimensional minimally infinitesimally rigid tensegrity framework on at least $d + 2$ vertices must have at least $d|V| - \binom{d+1}{2} + 1$ edges.

This trivial lower bound is also sharp: let $G = (V, E)$ be an $\mathcal{R}_d$-circuit with $|E| = d|V| - \binom{d+1}{2} + 1$ and (G, p) be a rigid d-dimensional realization of G as a bar-and-joint framework. Note that there exists such a framework for any d and $|V| \geq d + 2$, since any generic d-dimensional realization of the $\mathcal{R}_d$-circuits obtained by 1-extensions from K_{d+2} are rigid [3]. Since G is an $\mathcal{R}_d$-circuit, there is a linear dependence λ of the rows of $R(G, p)$ with $\mathrm{supp}(\lambda) = E$. Let $C = \{e \in E : \lambda(e) < 0\}$ and $S = \{e \in E : \lambda(e) > 0\}$ and (T, p) be the tensegrity framework, where $T = (V, C \cup S)$. Then λ is a proper stress of (T, p), the bar-and-joint framework $(\overline{T}, p)$ is infinitesimally rigid, and there is no proper stress of $(T - e, p)$ for any edge $e \in C \cup S$, because the rows of $R(T - e, p)$ are linearly independent. Therefore, (T, p) is a d-dimensional minimally infinitesimally rigid tensegrity framework with exactly $d|V| - \binom{d+1}{2} + 1$ edges.

However, there are minimally infinitesimally rigid tensegrity frameworks with more edges than this, see Figure 6 for example. The question that we investigate in the rest of this section is that how many edges can a minimally infinitesimally rigid tensegrity framework have at most.

4.1 Upper bound without parallel members

In this section, we consider tensegrity graphs, in which a cable is not allowed to be parallel to a strut.

We show tight upper bounds for $1 \leq d \leq 3$ on the number of edges of d-dimensional minimally infinitesimally rigid tensegrity frameworks in general position with no parallel members. Interestingly, with this natural restriction, the upper bounds become significantly better than the (also tight) generalized upper bounds presented in Section 4.2 bellow.

We need the concepts and results related to matroid theory introduced in Section 3. Using them, we can formulate Lemma 2.2 as follows.

Corollary 4.2. Let (T, p) be a general realization of the tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^d$ and let $e \in C \cup S$. If there exists a stress of (T, p) with e in its support, then there exists a stress ω_e of (T, p) with e in its support and such that $\text{supp}(\omega_e)$ is a circuit of $\mathcal{R}_d(T)$.

.

4.1.1 One dimension

First we prove the 1-dimensional case, Theorem 4.3. We will see later that this result is an easy consequence of the generalized case, however the proof presented here is easily understandable and instructive; it can be helpful in illustrating the fundamental idea behind the higher-dimensional proofs.

Lemma 4.1. Let (T, p) be a minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^d$. Then every infinitesimally rigid subframework of (T, p) is minimally infinitesimally rigid.

Proof. Let (T, p) be a minimally infinitesimally rigid tensegrity framework, and let (T', p') be an infinitesimally rigid subframework of T, where p' denotes the restriction of p to the vertices of T'. Suppose for a contradiction that T' has an edge $uv = e$ such that $(T' - e, p')$ remains infinitesimally rigid. Consider the framework $(T - e, p)$, where the minimality of (T, p) implies the existence of a non-trivial infinitesimal motion m satisfying $(p(u) - p(v))(m(u) - m(v)) \neq 0$. Restricting m to the vertices of T' yields a non-trivial infinitesimal motion of $(T' - e, p')$, a contradiction. □

Theorem 4.3. Let (T, p) be a 1-dimensional general realization of the tensegrity graph $T = (V, C \cup S)$ with no parallel members. Suppose that (T, p) is minimally infinitesimally rigid. Then

$$|C \cup S| \leq 2|V| - 3.$$

Proof. By Theorem 2.7, there exists a proper stress ω of (T, p). Thus, by Corollary 4.2, for each edge $e \in C \cup S$, there is a stress ω_e of (T, p), such that $\mathrm{supp}(\omega_e)$ is a cycle H_e containing e.

Since T is connected, there exists a permutation $(H_1, \ldots, H_t)$ of the cycles in $\{H_e : e \in C \cup S\}$ such that

$$V(H_i) \cap V(H_1 \cup \ldots \cup H_{i-1}) \neq \emptyset$$

for $i = 2, \ldots, t$.

We prove the inequality by induction on t. For $t = 1$, T is a cycle on at least 3 vertices, thus $|C \cup S| = |V| \leq 2|V| - 3$ holds. Suppose that $t \geq 2$. Let (T', p') be the subframework on $H_1 \cup \ldots \cup H_{t-1}$ with $T' = (V', E')$. Since T' is connected, $(\overline{T'}, p')$ is infinitesimally rigid in $\mathbb{R}^1$, and $\sum_{e \in E'} \omega_e$ is a proper stress of (T', p'), therefore, (T', p') is infinitesimally rigid. Moreover, by Lemma 4.1, (T', p') is minimally infinitesimally rigid.

Let $E^+ = H_t - E'$ and $V^+ = V(H_t) - V'$. Since (T', p') is minimally infinitesimal rigid, the vertex set V' induces no other members of T but the edges of T', so if $V^+ = \emptyset$, then $E^+ = \emptyset$ and the statement follows from the induction hypothesis. If $V^+ \neq \emptyset$ then the degree of any vertex in V^+ is at most 2, so $|E^+| \leq 2|V^+|$.

Then, by the induction hypothesis

$$|C \cup S| = |E'| + |E^+| \leq 2|V'| - 3 + 2|V^+| = 2|V| - 3,$$

as required.

□

The upper bound in Theorem 4.3 is the best possible, since the framework in Figure 5 (with arbitrarily many vertices in the middle) has $2|V| - 3$ edges and a minimally infinitesimally rigid realization on the line: every representation is valid in which one endpoint of the strut is the first, and the other endpoint is the last.

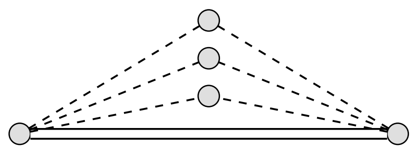


Figure 5: Tensegrity graph with $2|V| - 3$ edges that has a minimally infinitesimally rigid realization in $\mathbb{R}^1$.

4.1.2 Two and three dimensions

Let (T, p) be a realization of the tensegrity graph $T = (V, C \cup S)$ and let $F \subseteq C \cup S$. We say that a linear dependence ω of the rows of $R(T, p)$ is a *semi-stress of* (T, p) *with* *respect to* F, if $\text{supp}(\omega) - F \neq \emptyset$ and it satisfies the sign constraints on $(C \cup S) - F$, i. e. $\omega(c) \leq 0$ for each $c \in C - F$ and $\omega(s) \geq 0$ for each $s \in S - F$. We call ω a *minimal semi-stress of* (T, p) *with respect to* F, if ω is a semi-stress of (T, p) with respect to F, for which $\text{supp}(\omega) - F$ is minimal and under this condition, $\text{supp}(\omega) \cap F$ is minimal.

Lemma 4.2. Let (T, p) be a general d-dimensional realization of the tensegrity graph $T = (V, C \cup S)$ and let $F \subseteq C \cup S$. Suppose that ω is a minimal semi-stress with respect to F, and let H be a circuit of $\mathcal{R}_d(T)$ with $H \subseteq F \cup \text{supp}(\omega)$ and $H - F \neq \emptyset$. Let λ_H be a dependence with $\text{supp}(\lambda_H) = H$. Then λ_H or $-\lambda_H$ is a semi-stress with respect to F and $\text{supp}(\lambda_H) - F = \text{supp}(\omega) - F$.

Proof. Since H is a circuit in $\mathcal{R}_d(T)$, λ_H exists.

By replacing λ_H with $-\lambda_H$, if necessary, we may assume that for at least one member f of $H - F$ the sign of $\lambda_H(f)$ and the sign of $\omega(f)$ agrees.

If the two sign patterns agree on all members of $H - F$, then, since ω is a semi-stress of (T, p) with respect to F, λ_H is also a semi-stress of (T, p) with respect to F. The minimality of ω implies that $\text{supp}(\lambda_H) - F = \text{supp}(\omega) - F$.

If the two sign patterns do not agree on all members of $H - F$ then let

$$t = \min\left\{\frac{\omega(e)}{\lambda_H(e)} : e \in H - F, \text{sign}(\lambda_H(e)) = \text{sign}(\omega(e))\right\}.$$

Then $t > 0$ and $\mu = \omega - t\lambda_H$ is a semi-stress of (T, p) with respect to F, for which $\emptyset \neq \text{supp}(\mu) - F \subset \text{supp}(\omega) - F$, contradicting the minimality of ω.

□

Corollary 4.4. Let (T, p) be a general d-dimensional realization of the tensegrity graph $T = (V, C \cup S)$, $F \subseteq C \cup S$ and ω be a minimal semi-stress of (T, p) with respect to F. Then $\text{supp}(\omega)$ is a circuit of $\mathcal{R}_d(T)$.

Proof. Let H be a circuit of $\mathcal{R}_d(T)$ with $H \subset \text{supp}(\omega)$ and λ_H a dependence with $\text{supp}(\lambda_H) = H$.

If $H - F \neq \emptyset$, then by Lemma 4.2, λ_H or $-\lambda_H$ is a semi-stress of (T, p) with respect to F, contradicting the minimality of ω.

Suppose that $H - F = \emptyset$. Let $e \in H$ fixed and

$$\mu = \omega - \frac{\omega(e)}{\lambda_H(e)}\lambda_H.$$

Then μ is a semi-stress of (T, p) for which $\text{supp}(\mu) - F = \text{supp}(\omega) - F$ and $\text{supp}(\mu) \cap F \subset \text{supp}(\omega) \cap F$, contradicting the minimality of ω.

Therefore, there is no circuit H of $\mathcal{R}_d(T)$ with $H \subset \text{supp}(\omega)$, thus $\text{supp}(\omega)$ is a circuit of $\mathcal{R}_d(T)$. □

For a partial circuit decomposition $(H_1, \ldots, H_t)$ of $\mathcal{R}_d(T)$, denote $D_j = H_1 \cup \ldots \cup H_j$ for $j = 1, \ldots, t$.

Corollary 4.5. Let (T, p) be a tensegrity framework in general position in $\mathbb{R}^d$ and $(H_1, \ldots, H_j)$ a partial circuit decomposition of $\mathcal{R}_d(T)$. Let ω be a minimal semi-stress of (T, p) with respect to D_j and $\text{supp}(\omega) = H$. Then $(H_1, \ldots, H_j, H)$ is a partial circuit decomposition of $\mathcal{R}_d(T)$.

Proof. By Corollary 4.4, H is a circuit of $\mathcal{R}_d(T)$. The definition of a semi-stress and Lemma 4.2 implies that $(H_1, \ldots, H_j, H)$ satisfies (E1) and (E2). □

Lemma 4.3. Let (T, p) be a general realization of the tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^d$, $F \subseteq C \cup S$ and assume that (T, p) has a proper stress. Then for any $f \in (C \cup S) - F$ there exists a minimal semi-stress ω_f of (T, p) with respect to F with $f \in \text{supp}(\omega_f)$.

Proof. Let ω be a semi-stress of (T, p) with respect to F, such that $f \in \text{supp}(\omega)$, under this condition, $\text{supp}(\omega) - F$ is minimal and among all these, $\text{supp}(\omega) \cap F$ is minimal. Suppose that ω is not a minimal semi-stress of (T, p) with respect to F. Then there exists a minimal semi-stress ω' of (T, p) with respect to F such that $f \notin \text{supp}(\omega')$ and $\text{supp}(\omega') - F \subset \text{supp}(\omega) - F$. Let

$$t = \min \left\{ \frac{\omega(e)}{\omega'(e)} : e \in \text{supp}(\omega') - F \right\}.$$

Then $t > 0$ and $\mu = \omega - t\omega'$ is a semi-stress of (T, p) with respect to F, for which $f \in \text{supp}(\mu)$ and $\text{supp}(\mu) - F \subset \text{supp}(\omega) - F$, contradicting the minimality of ω. □

A *properly stressed circuit decomposition* of a d-dimensional tensegrity framework (T, p) is a circuit decomposition $(H_1, \ldots, H_t)$ of $\mathcal{R}_d(T)$, for which the subframework on D_j has a proper stress for all $1 \leq j \leq t$.

Corollary 4.6. Let (T, p) be a general realization of the tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^d$, assume that (T, p) has a proper stress. Then (T, p) has a properly stressed circuit decomposition $(H_1, \ldots, H_t)$. Moreover, if T is connected then it can be chosen such that $V(D_{j-1}) \cap V(H_j) \neq \emptyset$ for $j = 1, \ldots, t$.

Proof. For each edge $e \in C \cup S$ there is a properly stressed circuit containing e by Lemma 2.2, so we can choose a first circuit H_1.

Suppose that $(H_1, \ldots, H_j)$ is a partial circuit decomposition of $\mathcal{R}_d(T)$ and there is a proper stress ω of the subframework on D_j. Let ω' be a minimal semi-stress of (T, p) with respect to D_j with $H = \text{supp}(\omega')$. Since (T, p) has a proper stress, such ω' exists. By the connectivity of T and Lemma 4.3, it can be chosen such that $V(H) \cap V(D_j) \neq \emptyset$. By Corollary 4.5 $(H_1, \dots, H_j, H)$ is a partial circuit decomposition of $\mathcal{R}_d(T)$. Then $\omega + \varepsilon\omega'$ is a proper stress on $(H_1, \dots, H_j, H)$ for a sufficiently small positive ε. Repeating this method, we get a properly stressed circuit decomposition of (T, p). □

A tensegrity framework (T, p) is *minimally properly stressed* if (T, p) has a proper stress, but $(T - e, p)$ has no proper stress for every edge e of T.

Lemma 4.4. Let (T, p) be an infinitesimally rigid tensegrity framework in $\mathbb{R}^d$. Then (T, p) is minimally infinitesimally rigid if and only if (T, p) is minimally properly stressed.

Proof. The "if" direction immediately follows from Theorem 2.7.

For the "only if" direction suppose that (T, p) is minimally infinitesimally rigid, but $(T - e, p)$ is properly stressed for some member e of T. Since by Theorem 2.7, $(\overline{T}, p)$ is infinitesimally rigid and (T, p) has a proper stress, $(\overline{T - e}, p)$ is infinitesimally rigid. Thus $(T - e, p)$ is also infinitesimally rigid by Theorem 2.7. □

Lemma 4.5. Suppose that (T, p) is a minimally properly stressed tensegrity framework in general position in $\mathbb{R}^d$ and let $(H_1, \dots, H_t)$ be a properly stressed circuit decomposition of (T, p). Then the subframework on D_j is minimally properly stressed for all $1 \leq j \leq t$.

Proof. Since the subframework on H_1 is clearly minimally properly stressed, we assume that $t \geq 2$. We also assume that $j = t - 1$. Suppose for a contradiction that the subframework (T', p') on $D_{t-1} - e$ has a proper stress ω' for some $e \in D_{t-1}$.

We show that there exists a circuit H in $\mathcal{R}_d(T)$ with $e \notin H$ and $H - D_{t-1} \neq \emptyset$. If $e \notin H_t$, then the statement clearly holds with $H = H_t$. Assume that $e \in H_t$. Let $H' \subseteq D_{t-1}$ be a circuit of $\mathcal{R}_d(T)$ with $e \in H'$ and let $f \in H_t - D_{t-1}$. By the strong circuit exchange axiom (Theorem 3.2) there is a circuit H of $\mathcal{R}_d(T)$ with $f \in H$, $e \notin H$, and $H \subset H' \cup H_t$, thus the statement holds for H.

Let ω be a minimal semi-stress of (T, p) with respect to D_{t-1}. By Corollary 4.4, $\mathrm{supp}(\omega)$ is a circuit of $\mathcal{R}_d(T)$. Since $(H_1, \dots, H_t)$ is a circuit decomposition, $H_t - D_{t-1} = \mathrm{supp}(\omega) - D_{t-1} = H - D_{t-1}$. Then by Lemma 4.2, λ_H or $-\lambda_H$ is a semi-stress of (T, p) with respect to D_{t-1}. So $\omega' + \varepsilon\lambda_H$ or $\omega' - \varepsilon\lambda_H$ is a proper stress of $(T - e, p)$ for a sufficiently small positive ε, contradicting the minimality of (T, p).

□

Theorem 4.7. Let (T, p) be a general realization of the connected tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^2$ with no parallel members. Suppose that (T, p) is minimally properly stressed. Then

$$|C \cup S| \leq 3|V| - 6 - 3k_2(T).$$

Proof. By Corollary 4.6, there is a properly stressed circuit decomposition $(H_1, \ldots, H_t)$ of (T, p) such that $V(H_j) \cap V(D_{j-1}) \neq \emptyset$ for $j = 1, \ldots, t$. We prove the inequality by induction on t. For $t = 1$, the graph T is a $\mathcal{R}_2$-circuit and the bound follows from Lemma 3.3. Assume that $t \geq 2$. Let $T' = (V(D_{t-1}), D_{t-1})$ be the subgraph of T induced by D_{t-1}, let $E^+ = H_t - D_{t-1}$ and $V^+ = V(H_t) - V(D_{t-1})$.

By Lemma 3.2 and the definition of $k_2(T)$, we have

$$|E^+| = r_2(T) - r_2(T') + 1 = 2|V^+| + 1 + k_2(T') - k_2(T). \tag{3}$$

If $V^+ = \emptyset$ then the minimality of (T, p) implies that $|E^+| \geq 2$. If $V^+ \neq \emptyset$ then by Theorem 3.3, there are at least 3 edges between V^+ and $V - V^+$ and the degree of any vertex in V^+ is at least 3. So in both cases, when $V^+ = \emptyset$ and when $V^+ \neq \emptyset$, the inequality

$$|E^+| \geq \frac{3}{2}(|V^+| + 1)$$

holds. Combining this with (3) we get

$$1 \leq |V^+| + 2k_2(T') - 2k_2(T).$$

Plugging this into the right hand side of (3) we have

$$|E^+| \leq 3|V^+| + 3k_2(T') - 3k_2(T).$$

By Lemma 4.5, the subframework on D_{t-1} is minimally properly stressed. Then, by the induction hypothesis

$$|C \cup S| = |D_{t-1}| + |E^+| \leq 3|V(D_{t-1})| - 6 - 3k_2(T') + 3|V^+| + 3k_2(T') - 3k_2(T) =$$

$$= 3|V| - 6 - 3k_2(T),$$

as required. □

We can prove a quite similar result in the 3-dimensional case.

Theorem 4.8. Let (T, p) be a general realization of the connected tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^3$ with no parallel members. Suppose that (T, p) is minimally properly stressed. Then

$$|C \cup S| \leq 4|V| - 10 - 2k_3(T).$$

Proof. By Corollary 4.6, there is a properly stressed circuit decomposition $(H_1, \ldots, H_t)$ of (T, p) such that $V(H_j) \cap V(D_{j-1}) \neq \emptyset$ for $j = 1, \ldots, t$. We prove the inequality by induction on t. For $t = 1$, the graph T is a $\mathcal{R}_3$-circuit and the bound follows from

Lemma 3.3. Assume that $t \geq 2$. Let $T' = (V(D_{t-1}), D_{t-1})$ be the subgraph of T induced by D_{t-1}, let $E^+ = H_t - D_{t-1}$ and $V^+ = V(H_t) - V(D_{t-1})$.

By Lemma 3.2 and the definition of $k_3(T)$, we have

$$|E^+| = r_3(T) - r_3(T') + 1 = 3|V^+| + 1 + k_3(T') - k_3(T). \quad (4)$$

If $V^+ = \emptyset$ then the minimality of (T, p) implies that $|E^+| \geq 2$. If $V^+ \neq \emptyset$ then by Theorem 3.3, there are at least 4 edges between V^+ and $V - V^+$ and the degree of any vertex in V^+ is at least 4. So in both cases, when $V^+ = \emptyset$ and when $V^+ \neq \emptyset$, the inequality

$$|E^+| \geq 2(|V^+| + 1)$$

holds. Combining this with (4) we get

$$1 \leq |V^+| + k_3(T') - k_3(T).$$

Plugging this into the right hand side of (4) we have

$$|E^+| \leq 4|V^+| + 2k_3(T') - 2k_3(T).$$

By Lemma 4.5, the subframework on D_{t-1} is minimally properly stressed. Then, by the induction hypothesis

$$|C \cup S| = |D_{t-1}| + |E^+| \leq 4|V(D_{t-1})| - 10 - 2k_3(T') + 4|V^+| + 2k_3(T') - 2k_3(T) =$$

$$= 4|V| - 10 - 2k_3(T),$$

as required. □

By Lemma 4.4, and Theorem 4.7 and 4.8 combined with the fact that $k_d(T) = 0$, if (T, p) is an infinitesimally rigid tensegrity framework without parallel edges, we obtain the main results.

Theorem 4.9. Let (T, p) be a 2-dimensional general realization of the tensegrity graph $T = (V, C \cup S)$ with no parallel members. Suppose that (T, p) is minimally infinitesimally rigid. Then

$$|C \cup S| \leq 3|V| - 6.$$

Theorem 4.10. Let (T, p) be a 3-dimensional general realization of the tensegrity graph $T = (V, C \cup S)$ with no parallel members. Suppose that (T, p) is minimally infinitesimally rigid. Then

$$|C \cup S| \leq 4|V| - 10.$$

The upper bounds in Theorem 4.9 and Theorem 4.10 are the best possible: consider the tensegrity framework in $\mathbb{R}^2$ obtained from a triangle of struts, by adding $|V| - 3$ vertices inside and connecting each of them to every vertex of the triangle with cables, see for example the left framework in Figure 6. It is infinitesimally rigid and has $3|V| - 6$ edges. Similarly, the framework obtained from a tetrahedron of struts in $\mathbb{R}^3$ with $|V| - 4$ vertices inside, each of them connected to every vertex of the tetrahedron, is infinitesimally rigid and has $4|V| - 10$ edges, see Figure 7. Note that there exists other type of extremal examples, see for example the right framework in Figure 6.

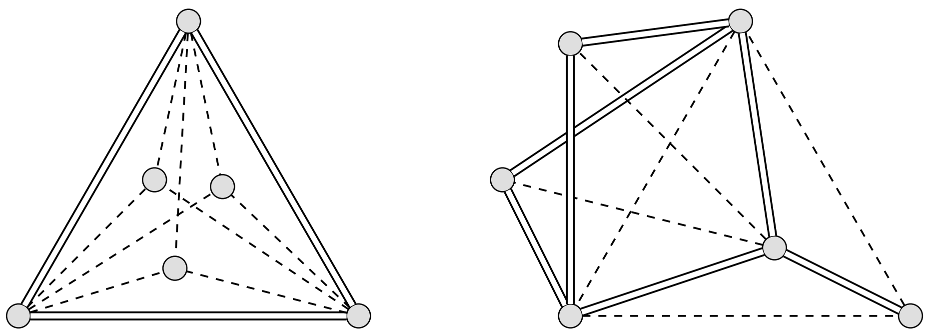


Figure 6: Minimally infinitesimally rigid tensegrity frameworks in $\mathbb{R}^2$ with $3|V| - 6$ edges.

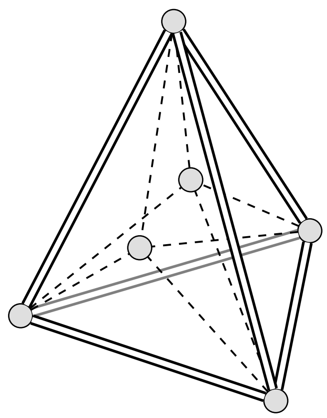
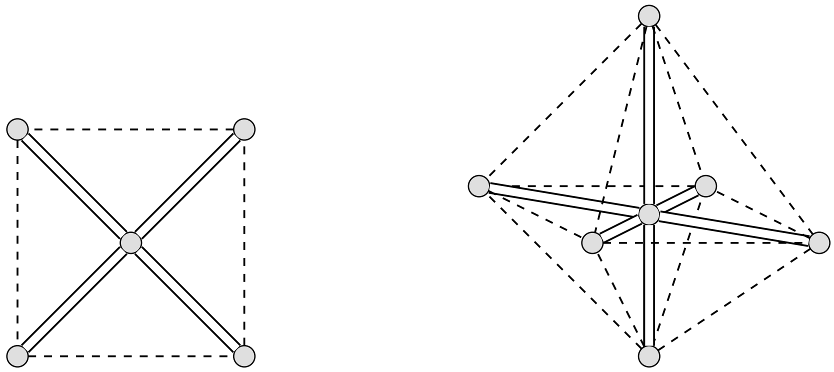


Figure 7: Minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^3$ with $4|V| - 10$ edges.

Consider the tensegrity framework in Figure 8a and assume that there are multiple copies of the central vertex at the same position and each of the copies are connected to every vertex of the square with struts. It can be easily verified that this framework is minimally infinitesimally rigid in $\mathbb{R}^2$. Similarly: the framework in Figure 8b with multiple copies of the central vertex and the attached struts is minimally infinitesimally rigid in $\mathbb{R}^3$. Note that these frameworks are not in general position (in fact, they are not even injective). The 2-dimensional frameworks have $4(|V| - 4) + 4 = 4|V| - 12$ edges and the 3-dimensional frameworks have $6(|V|-6)+12 = 6|V|-24$ edges, showing that the bounds of Theorem 4.9 and 4.10 do not hold without any restrictions on p.



(a) (b)

Figure 8: Non-generic minimally infinitesimally rigid tensegrity frameworks.

It is important to mention that the condition in Theorem 4.9 and Theorem 4.10, which forbids any parallel edges in the framework, can be weakened. Let (T, p) be a minimally infinitesimally rigid tensegrity framework in general position in $\mathbb{R}^d$, and ω be a proper stress of T, which exists by Theorem 2.7. If a cable c and a strut s are parallel in T, then $\omega(s) = -\omega(c)$ must hold, since otherwise $(T-s, p)$ or $(T-c, p)$ has a proper stress, thus it is infinitesimally rigid by Theorem 2.7. Then the framework $(T-c-s, p)$ has a proper stress, hence the bar-and-joint framework $(\overline{T-e}, p)$, where e is the edge of $\overline{T}$ corresponding to s and c, cannot be infinitesimally rigid. Therefore, $\operatorname{rank} R(\overline{T-e}, p) = \operatorname{rank} R(\overline{T}, p) - 1$, so e is a bridge in $\mathcal{R}_d(T)$. Hence, in Theorem 4.9 and Theorem 4.10, instead of forbidding any parallel edges, it is enough to assume that for any parallel cable-strut pair of T, the corresponding edge of $\overline{T}$ is not a bridge in $\mathcal{R}_d(T)$.

4.2 The general upper bound

In this section, we consider tensegrity graphs, in which a cable is allowed to be parallel to a strut.

4.2.1 Without bars

Consider a minimally infinitesimally rigid bar-and-joint framework (G, p) in $\mathbb{R}^d$ with $G = (V, E)$. That is, for any edge $e \in E$ there exists a non-trivial infinitesimal motion

$m \in \mathbb{R}^{d|V|}$ of $(G-e,p)$ for which $R_{E-\{e\}}(T,p)\cdot m = 0$ and $R_{\{e\}}(T,p)\cdot m \neq 0$ hold. Clearly, the negative of m is also an infinitesimal motion of $(G-e,p)$ and the sign of $R_{\{e\}}(T,p)\cdot(-m)$ is the opposite of the sign of $R_{\{e\}}(T,p)\cdot m$. Therefore, replacing each bar in (G,p) with a parallel cable-strut pair leads to an infinitesimally rigid tensegrity framework (it is intuitively clear and also easy to verify using the conditions in Theorem 2.7) and removing any member from this framework allows an infinitesimal motion. Thus, tensegrity frameworks constructed this way are minimally infinitesimally rigid. By Corollary 4.1, every d-dimensional minimally infinitesimally rigid bar-and-joint framework with $|V| \geq d+2$ has exactly $d|V| - \binom{d+1}{2}$ edges, implying the existence of minimally infinitesimally rigid tensegrity frameworks in $\mathbb{R}^d$ with $2\cdot(d|V| - \binom{d+1}{2})$ edges for any d and $|V| \geq d+2$.

This means that there exist a minimally infinitesimally rigid tensegrity framework with exactly $2|V|-2$ edges in 1-dimension, $4|V|-6$ edges in 2-dimensions and $6|V|-12$ edges in 3-dimensions. Therefore, the upper bounds of Theorem 4.3, 4.9 and 4.10 do not hold, if we allow parallel cable-strut pairs to appear in the framework.

We show that a minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^d$ cannot have more edges than $2\cdot(d|V| - \binom{d+1}{2})$. The proof employs a variant of Carathéodory's theorem from convex geometry, attributed to Ernst Steinitz [13], see also [14, Theorem 4.22.].

Let $X = \{x_1,\ldots,x_m\} \subset \mathbb{R}^n$ be a finite set of points. The *(linear) span* of X is the subspace $\operatorname{span}(X) = \{\sum_{i=1}^m \lambda_i x_i : \lambda_1,\ldots,\lambda_m \in \mathbb{R}\}$. The *convex hull* of X is the set $\operatorname{conv}(X) = \{\sum_{i=1}^m \lambda_i x_i : \sum_{i=1}^m \lambda_i = 1,\ 0 \leq \lambda_1,\ldots,\lambda_m \in \mathbb{R}\}$. The polytope $\operatorname{conv}(X)$ is *k-dimensional*, if there exists a k-dimensional affine subspace in $\mathbb{R}^n$ containing $\operatorname{conv}(X)$ and there is no $(k-1)$-dimensional affine subspace containing $\operatorname{conv}(X)$. A point x is in the *relative interior* of $\operatorname{conv}(X)$ if $\operatorname{conv}(X)$ is k-dimensional and there exists a k-dimensional ball centered at x contained in $\operatorname{conv}(X)$.

Lemma 4.6. Let $X \subset \mathbb{R}^n$ be a finite set of points. A point x is in the relative interior of $\operatorname{conv}(X)$ if and only if there is a strictly positive convex combination of the elements of X resulting in x.

Proof. Let $X = \{x_1,\ldots,x_m\}$, and suppose that $\operatorname{conv}(X)$ is k-dimensional. First, assume that x is the origin. If the origin is in the relative interior of $\operatorname{conv}(X)$ then $-\varepsilon\cdot\sum_{i=1}^m x_i$ is in $\operatorname{conv}(X)$ for a sufficiently small positive ε, thus

$$-\varepsilon \sum_{i=1}^m x_i = \sum_{i=1}^m \lambda_i x_i,$$

where $\lambda_1, \ldots, \lambda_m \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. Then

$$0 = \sum_{i=1}^{m} \frac{\lambda_i + \varepsilon}{1 + m \cdot \varepsilon} x_i$$

is a strictly positive convex combination of the elements of X resulting in zero.

Conversely, let $\sum_{i=1}^m \alpha_i x_i = 0$ be a strictly positive convex combination of the elements of X resulting in the origin. It is enough to show that for any $h \in \operatorname{span}(X)$ there exists a positive ε such that εh is in $\operatorname{conv}(X)$. Let $h \in \operatorname{span}(X)$ and $h = \sum_{i=1}^m \lambda_i x_i$. Denote $\lambda = \sum_{i=1}^m \lambda_i$. If $\lambda \neq 0$, then there exists $\mu \in \mathbb{R}$ with the same sign as λ and $|\mu|$ being sufficiently small such that

$$\mu \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} x_i + (1 - \mu) \sum_{i=1}^{m} \alpha_i x_i = \frac{\mu}{\lambda} h$$

is a convex combination of the elements of X resulting in εh, where $\varepsilon = \frac{\mu}{\lambda} > 0$. If $\lambda = 0$ then for a sufficiently small positive ε,

$$\varepsilon \sum_{i=1}^{m} \lambda_i x_i + \sum_{i=1}^{m} \alpha_i x_i = \varepsilon h$$

is a convex combination of the elements of X resulting in εh.

Finally, observe that if x is not the origin then x is in the relative interior of $\operatorname{conv}(X)$ if and only if the origin is in the relative interior of $\operatorname{conv}(X')$, where $X' = \{x_1 - x, \ldots, x_m - x\}$. Also, there is a strictly positive convex combination of the elements of X resulting in x if and only if there is a strictly positive convex combination of the elements of X' resulting in the origin, since if $\sum_{i=1}^m \alpha_i = 1$ then

$$0 = \sum_{i=1}^{m} \alpha_i (x_i - x)$$

is equivalent to

$$x = \sum_{i=1}^{m} \alpha_i (x_i - x) + x = \sum_{i=1}^{m} \alpha_i x_i - x \sum_{i=1}^{m} \alpha_i + x = \sum_{i=1}^{m} \alpha_i x_i.$$

□

Lemma 4.7. Let $X \subset \mathbb{R}^n$ be a finite set of points. If there exists a strictly positive combination of the elements of X resulting in zero then

- the origin is in the relative interior of conv(X) and
- the dimension of the subspace span(X) is equal to the dimension of the polytope conv(X).

Proof. Let $X = \{x_1, \ldots, x_m\}$ and $\sum_{i=1}^m \lambda_i x_i$ be a strictly positive combination of the elements of X resulting in zero. Denote $\lambda = \sum_{i=1}^m \lambda_i$. Then $\sum_{i=1}^m \frac{\lambda_i}{\lambda} x_i$ is a strictly positive convex combination resulting in zero, thus the origin is in the relative interior of conv(X) by Lemma 4.6.

Assume that span(X) is k-dimensional. Since conv(X) is contained in span(X), conv(X) is at most k-dimensional. Assume that conv(X) is contained in a $(k-1)$-dimensional affine subspace $H \subset \text{span}(X)$. The origin is in conv(X), thus it is in H. Therefore, H is a subspace. Since H contains all elements of X, it contains span(X), contradicting the fact that H is $(k-1)$-dimensional.

□

The theorem of Steinitz is the following.

Theorem 4.11 (Steinitz [13])**.** Let $X \subset \mathbb{R}^n$ be a finite set of points, conv(X) k-dimensional and a point x in the relative interior of conv(X). Then there is a subset $Y \subseteq X$ of at most $2k$ points such that conv(Y) is k-dimensional and x is in the relative interior of conv(Y).

Using this theorem, we prove a sharp upper bound on the edge count of minimally infinitesimally rigid tensegrity frameworks, depending on the number of vertices and dimensions.

Theorem 4.12. Let (T, p) be a minimally infinitesimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^d$ with $|V| \geq d + 2$. Then

$$|C \cup S| \leq 2 \cdot \left(d|V| - \binom{d+1}{2} \right).$$

Proof. Let $N = d|V| - \binom{d+1}{2}$. For a d-dimensional tensegrity framework (T, p) let $R'(T, p)$ be the matrix obtained by replacing the rows corresponding to cables in the rigidity matrix $R(T, p)$ with their negatives, and let X denote the set of points in $\mathbb{R}^{d|V|}$ whose coordinates are the rows of $R'(T, p)$.

According to Theorem 2.7, the tensegrity framework (T, p) with $|V| \geq d+2$ is infinitesimally rigid if and only if

(1) $\overline{T}$ is infinitesimally rigid, or equivalently, the rank of $R'(T, p)$ is N, or equivalently, the subspace span(X) is N-dimensional, and

(2) there exists a proper stress of (T, p), or equivalently, there exists $\omega \in \mathbb{R}^E$ such that $\omega > 0$ and $\omega \cdot R'(T, p) = 0$, or equivalently, there is a strictly positive combination of the elements of X resulting in in zero.

Assume for a contradiction that (T, p) is a minimally infinitesimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^d$ with $|C \cup S| \geq 2N + 1$ and $|V| \geq d + 2$.

Since (T, p) is infinitesimally rigid, (1) and (2) hold. Therefore, by Lemma 4.7, $\mathrm{conv}(X)$ forms an N-dimensional polytope containing the origin in its relative interior.

By Theorem 4.11 of Steinitz, one can select a set $Y \subset X$ of at most $2N$ elements such that $\mathrm{conv}(Y)$ forms an N-dimensional polytope with the origin in its relative interior. By Lemma 4.7, $\mathrm{span}(Y)$ is N-dimensional, thus both (1) and (2) hold for the submatrix of $R'(T, p)$ only containing the rows corresponding to Y. So the tensegrity framework obtained from (T, p) by deleting the edges corresponding to $X - Y$ is infinitesimally rigid. Notice that adding edges to an infinitesimally rigid tensegrity, it remains infinitesimally rigid.

Therefore, if $|C \cup S| \geq 2N + 1$ then there exists an edge such that by removing it (1) and (2) still hold, thus the framework remains infinitesimally rigid, contradicting the minimality of (T, p).

□

It is also proved in Steinitz's paper that in Theorem 4.11 if $\mathrm{conv}(X)$ is k-dimensional and Y is a minimal subset of X such that $\mathrm{conv}(Y)$ is also k-dimensional and x is in the relative interior of $\mathrm{conv}(Y)$, then either $|Y| \leq 2k - 1$ or Y consists of $2k$ points collinear in pairs with x [14]. Therefore, a minimally infinitesimally rigid tensegrity framework (T, p) has $2 \cdot \left(d|V| - \binom{d+1}{2}\right)$ edges if and only if the points in $\mathbb{R}^{d|V|}$ whose coordinates are the rows of $R'(T, p)$ are collinear in pairs with the origin. By the definition of the rigidity matrix, two different rows of $R'(T, p)$ only can be collinear with the origin if they are the negatives of each other, which means that they correspond to a parallel cable-strut pair in T.

So, we can characterize the minimally infinitesimally rigid tensegrity frameworks for which the number of edges equals the upper bound provided by Theorem 4.12. These extremal frameworks are exactly the minimally infinitesimally rigid bar-and-joint frameworks with parallel cable-strut pairs instead of bars.

Notice, that this characterization gives a different proof for the upper bound $2|V| - 3$ proved in Theorem 4.3 in the 1-dimensional non-parallel case. If (T, p) is a minimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^1$, and there are no parallel cable-strut pairs in T, then (T, p) is not an extremal example of Theorem 4.12, thus $|C \cup S| \leq 2|V| - 3$. And this is sharp, because there exist 1-dimensional minimally infinitesimally rigid tensegrity frameworks with exactly $2|V| - 3$ edges, see Figure 5. Note that we do not need to assume here that p is in general position.

Now we prove a lemma, which is a consequence of Theorem 4.11 of Steinitz, and then use it to show a better upper bound for frameworks that are not extremal examples of Theorem 4.12. This lemma will also be useful to prove a generalization of Theorem 4.12 in Section 4.2.2.

For a subspace $H \subseteq \mathbb{R}^n$ the *orthogonal complement* of H is the subspace containing the elements of $\mathbb{R}^n$ orthogonal to each element of H.

Lemma 4.8. Let $X \subset \mathbb{R}^n$ be a finite set of points, $\mathrm{conv}(X)$ k-dimensional and a point x in the relative interior of $\mathrm{conv}(X)$. Let $Z \subseteq X$ such that $\mathrm{conv}(Z)$ is k'-dimensional and x is in the relative interior of $\mathrm{conv}(Z)$. Then there is a subset $Y \subseteq X$ of at most $2(k-k') + |Z|$ points such that $\mathrm{conv}(Y)$ is k-dimensional, x is in the relative interior of $\mathrm{conv}(Y)$ and $Z \subseteq Y$.

Proof. We can assume without loss of generality that x is the origin.

Let U denote the subspace $\mathrm{span}(Z)$, which is k'-dimensional by Lemma 4.7, and W denote the subspace obtained by the intersection of the orthogonal complement of U and $\mathrm{span}(X)$. For a point $h \in \mathrm{span}(X)$, denote the component of h in U by h^U, and the component in W by h^W (thus, $h = h^U + h^W$), and for a subset $Y \subseteq X$ denote $Y^W = \{y^W : y \in Y\}$ the orthogonal projection of Y to W.

By Lemma 4.7, $\mathrm{span}(X)$ is k-dimensional, $\mathrm{span}(Z)$ is k'-dimensional, hence W is $(k-k')$-dimensional. Any $h \in W$ can be expressed as a linear combination of elements of X, and orthogonally projecting both sides of this linear expression onto W, we obtain h as a linear combination of elements in X^W (since the orthogonal projection eliminates the U-directional components of the elements of X). So $\mathrm{span}(X^W) = W$, and therefore $\mathrm{span}(X^W)$ is $(k-k')$-dimensional. Moreover, if we take a strictly positive combination of the elements of X expressing the origin, then projecting this orthogonally onto W yields to a strictly positive combination of the elements of X^W resulting in the origin.

Thus, by Lemma 4.7, $\mathrm{conv}(X^W)$ is a $(k-k')$-dimensional polytope with the origin in its relative interior. Applying Theorem 4.11 of Steinitz to X^W we obtain that there exists $Y \subseteq X$ with at most $2(k-k')$ elements, such that $\mathrm{conv}(Y^W)$ is $(k-k')$-dimensional and it contains the origin in its relative interior.

Then $Y \cup Z$ has at most $2(k-k') + |Z|$ elements. Now we prove that $\mathrm{conv}(Y \cup Z)$ is k-dimensional and contains the origin in its relative interior.

Let $Z = \{z_1, \ldots, z_{|Z|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$. First we show that the elements of $Y \cup Z$ generate $\mathrm{span}(X)$. Let $h \in \mathrm{span}(X)$. Since by Lemma 4.7, the elements of Y^W generate W, we can express h^W as

$$h^W = \sum_{i=1}^{|Y|} \alpha_i y_i^W.$$

Combining the corresponding elements in Y with the same coefficients we obtain

$$\sum_{i=1}^{|Y|} \alpha_i y_i = \sum_{i=1}^{|Y|} \alpha_i (y_i^W + y_i^U) = h^W + \sum_{i=1}^{|Y|} \alpha_i y_i^U.$$

Since $\sum \alpha_i y_i^U \in \operatorname{span}(Z)$,

$$\sum_{i=1}^{|Y|} \alpha_i y_i^U = \sum_{i=1}^{|Z|} \beta_i z_i,$$

and by $h^U \in \operatorname{span}(Z)$

$$h^U = \sum_{i=1}^{|Z|} \gamma_i z_i.$$

So

$$h = h^W + h^U = \sum_{i=1}^{|Y|} \alpha_i y_i - \sum_{i=1}^{|Z|} \beta_i z_i + \sum_{i=1}^{|Z|} \gamma_i z_i.$$

Thus, h can be expressed as a linear combination of elements of $Y \cup Z$.

Similarly, we can show that there is a strictly positive combination of the elements of $Y \cup Z$ resulting in the origin. We chose the set Y such that

$$\sum_{i=1}^{|Y|} \alpha_i y_i^W = 0,$$

where $\alpha_1, \dots, \alpha_{|Y|} > 0$. Combining the corresponding elements in Y with the same coefficients, we obtain

$$\sum_{i=1}^{|Y|} \alpha_i y_i = \sum_{i=1}^{|Y|} \alpha_i (y_i^W + y_i^U) = 0 + \sum_{i=1}^{|Y|} \alpha_i y_i^U \in \operatorname{span}(Z).$$

Since the origin is in the relative interior of conv(Z), for a sufficiently small positive ε, the vector $\varepsilon \cdot \left(-\sum \alpha_i y_i^U\right)$ can be expressed as a strictly positive convex combination of elements in Z, thus

$$-\sum_{i=1}^{|Y|} \alpha_i y_i^U = \sum_{i=1}^{|Z|} \beta_i z_i,$$

where $\beta_1, \ldots, \beta_{|Z|} > 0$. Therefore, we have

$$\sum_{i=1}^{|Y|} \alpha_i y_i + \sum_{i=1}^{|Z|} \beta_i z_i = 0,$$

which is a strictly positive combination of elements in $Y \cup Z$ resulting in the origin.

Therefore, by Lemma 4.7, $\operatorname{conv}(Y \cup Z)$ is k-dimensional and it contains the origin in its relative interior. □

The following corollary is immediate from Lemma 4.8 and it also follows from the result of Bonnice and Reay proved in [15].

Corollary 4.13. Let $X \subset \mathbb{R}^n$ be a finite set of points, $\operatorname{conv}(X)$ k-dimensional and a point x in the relative interior of $\operatorname{conv}(X)$. Let k' be the dimension of the highest dimensional simplex with vertices in X and having x in its relative interior. Then there is a subset $Y \subseteq X$ of at most $2k - k' + 1$ points such that $\operatorname{conv}(Y)$ is k-dimensional and x is in the relative interior of $\operatorname{conv}(Y)$.

This implies the following modification of Theorem 4.12, which improves the upper bound, if the tensegrity framework is in general position and it contains a properly stressed $\mathcal{R}_d$-circuit.

Corollary 4.14. Let (T, p) be a general minimally infinitesimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^d$ with $|V| \geq d + 2$. Let ω be a stress of (T, p) such that $\operatorname{supp}(\omega)$ is a circuit of $\mathcal{R}_d(T)$. Let $H = \operatorname{supp}(\omega)$, then

$$|C \cup S| \leq 2 \cdot \left(d|V| - \binom{d+1}{2} \right) - |H| + 2.$$

Proof. Let $R'_H(T, p)$ be the matrix obtained by replacing the rows corresponding to cables in the matrix $R_H(T, p)$ with their negatives, and let X denote the set of points in $\mathbb{R}^{d|V|}$ whose coordinates are the rows of $R'_H(T, p)$. Since $\operatorname{supp}(\omega) = H$, there is a strictly positive combination of the elements of X resulting in zero. Moreover, H is a circuit of $\mathcal{R}_d(T)$, so the dimension of $\operatorname{span}(X)$ is $|H| - 1 = |X| - 1$. Therefore, by Lemma 4.7, $\operatorname{conv}(X)$ is $(|X| - 1)$-dimensional, thus it is a simplex, and the origin is in the relative interior of $\operatorname{conv}(X)$. So in the proof of Theorem 4.12 we can improve the upper bound on $|Y|$ by Corollary 4.13 with $|Y| \leq 2N - (|X| - 1) + 1 = 2N - |H| + 2$. □

Using this, we can improve the upper bound, when the framework is in general position and it is not an extremal example.

Corollary 4.15. Let (T, p) be a general minimally infinitesimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^d$ and $|V| \geq d + 2$. Suppose that there exist two vertices u, v of T with only a cable or only a strut connecting them. Then

$$|C \cup S| \leq 2 \cdot \left(d|V| - \binom{d+1}{2}\right) - \binom{d+2}{2} + 2.$$

Proof. By Corollary 4.2, there exists a stress ω of (T, p) with its support being an $\mathcal{R}_d$-circuit containing the single edge uv. Since $\mathcal{R}_d$-circuits have at least $\binom{d+2}{2}$ edges, see [3], Corollary 4.15 follows from Corollary 4.14. □

4.2.2 With bars

In the previous sections, we considered tensegrity graphs in which edges are labeled as cables or struts, and in certain cases, a cable is allowed to be parallel to a strut. In the literature, tensegrity graphs are often considered of the form $T = (V, B \cup C \cup S)$ in which edges are labeled as bars, cables, or struts. As we mentioned above, there is no significant difference between the two approaches in terms of (any kind of) rigidity, since using parallel cable-strut pairs instead of bars and vice versa does not change the rigidity of the framework.

However, it is not clear, how to define the minimality of infinitesimally rigid tensegrity frameworks consisting of bars.

It is convenient to say, that a tensegrity framework (T, p), where $T = (V, B \cup C \cup S)$ is minimally infinitesimally rigid, if the framework (T', p) is minimally infinitesimally rigid, where T' is the tensegrity graph that we get by replacing each of the bars in T with a parallel cable and strut. Thus, by this definition, if a tensegrity framework is minimally infinitesimally rigid, then replacing any of its bars with a single cable or a single strut, makes it no longer infinitesimally rigid. Call this definition the first definition of minimal infinitesimal rigidity. Using the first definition, it follows immediately from Theorem 4.12, that if (T, p) is a d-dimensional minimally infinitesimally rigid realization of the tensegrity graph $T = (V, B \cup C \cup S)$, then $2|B| + |C \cup S| \leq 2(d|V| - \binom{d+1}{2})$.

However, in the beginning of this section, we define minimal infinitesimal rigidity such that an infinitesimally rigid tensegrity framework (T, p) in $\mathbb{R}^d$ is called minimally infinitesimally rigid, if $(T - e, p)$ is not infinitesimally rigid in $\mathbb{R}^d$ for any edge e of T. It looks natural to use this definition for tensegrity frameworks consisting of bars. Call this the second definition. Notice, that the second definition generalizes the first definition, and there exist tensegrities, which are only minimally infinitesimally rigid by the second definition and not by the first one, see Figure 9 and Figure 10. In the following, we use the second definition and show that the inequality $2|B| + |C \cup S| \leq 2(d|V| - \binom{d+1}{2})$ still holds for d-dimensional minimally infinitesimally rigid tensegrity framework consisting bars, generalizing Theorem 4.12.

The following result is the consequence of Lemma 4.8.

Corollary 4.16. Let $X \subset \mathbb{R}^n$ be a finite set of points and conv(X) be a k-dimensional polytope with the origin in its relative interior. Let $Z \subseteq X$ such that $Z = Z^+ \dot{\cup} Z^-$ where $Z^- = \{-z : z \in Z^+\}$. Then there is a subset $Y \subseteq X$ of at most $2k$ points such that conv(Y) is k-dimensional, the origin is in the relative interior of conv(Y) and for each point $z \in Z$ either both z and $-z$ are in Y or neither of them is.

Proof. Let $Z_1 \subseteq Z^+$ be a maximal linearly independent system in Z^+, and let $Z_2 = \{-z : z \in Z_1\} \subseteq Z^-$. The sum of the elements of $Z_1 \cup Z_2$ is zero, thus, by Lemma 4.7, conv$(Z_1 \cup Z_2)$ is $|Z_1|$-dimensional. Then by Lemma 4.8, there is a subset $Y \subseteq X$ of at most $2(k - |Z_1|) + 2|Z_1| = 2k$ points such that conv(Y) is k-dimensional, the origin is in the relative interior of conv(Y) and $Z_1 \cup Z_2 \subseteq Y$, so for each point $z \in Z$ either both z and $-z$ are in Y or neither of them is. □

A d-dimensional infinitesimally rigid realization (T, p) of tensegrity graph $T = (V, B \cup C \cup S)$ is called *minimally infinitesimally rigid*, if $(T - e, p)$ is not infinitesimally rigid in $\mathbb{R}^d$ for every edge e of T.

Theorem 4.17. Let (T, p) be a minimally infinitesimally rigid realization of $T = (V, B \cup C \cup S)$ in $\mathbb{R}^d$ with $|V| \geq d + 2$. Then

$$2|B| + |C \cup S| \leq 2 \cdot \left(d|V| - \binom{d+1}{2}\right).$$

Proof. Let $N = d|V| - \binom{d+1}{2}$. For a d-dimensional tensegrity framework (T, p) let $R'(T, p)$ be the matrix obtained by replacing the rows corresponding to cables in the rigidity matrix $R(T, p)$ with their negatives, and adding the negatives of the rows of $R(T, p)$ corresponding to bars (so there are $2|B| + |C \cup S|$ rows of $R'(T, p)$). Note that the rank of $R'(T, p)$ is equal to the rank of $R(T, p)$. Let X denote the set of points in $\mathbb{R}^{d|V|}$ whose coordinates are the rows of $R'(T, p)$, and let $Z \subseteq X$ denote the $2|B|$ points corresponding to the bars.

According to Theorem 2.7 by Roth and Whiteley, the tensegrity framework (T, p) with $|V| \geq d + 2$ is infinitesimally rigid if and only if

(1) $\overline{T}$ is infinitesimally rigid, or equivalently, the rank of $R'(T, p)$ is N, or equivalently, the subspace span(X) is N-dimensional, and

(2) there exists a proper stress of (T, p), or equivalently, there exists $\omega \in \mathbb{R}^E$ such that $\omega > 0$ and $\omega \cdot R'(T, p) = 0$, or equivalently, there is a strictly positive combination of the elements of X resulting in in zero.

Assume for a contradiction that (T, p) is a minimally infinitesimally rigid realization of $T = (V, B \cup C \cup S)$ in $\mathbb{R}^d$ with $2|B| + |C \cup S| \geq 2N + 1$ and $|V| \geq d + 2$.

Since (T, p) is infinitesimally rigid, (1) and (2) hold. Therefore, by Lemma 4.7, conv(X) forms an N-dimensional polytope containing the origin in its relative interior.

By Corollary 4.16, one can select a set $Y \subset X$ of at most $2N$ elements such that conv(Y) forms an N-dimensional polytope with the origin in its relative interior and for each point $z \in Z$ either both z and $-z$ are is Y or neither of them is. Thus, each bar $b \in B$ satisfies that either both of the corresponding points are in Y or neither of them is, so deleting the edges corresponding to $X - Y$ results in a subframework of (T, p). Moreover, by Lemma 4.7, span(Y) is N-dimensional, thus both (1) and (2) hold for the submatrix of $R'(T, p)$ only containing the rows corresponding to Y. So the tensegrity framework obtained from (T, p) by deleting the edges corresponding to $X - Y$ is infinitesimally rigid. Notice that adding edges to an infinitesimally rigid tensegrity, it remains infinitesimally rigid.

Therefore, if $2|B| + |C \cup S| \geq 2N + 1$ then there exists an edge such that by removing it (1) and (2) still hold, thus the framework remains infinitesimally rigid, contradicting the minimality of (T, p).

□

Clearly, equality holds in Theorem 4.17, if (T, p) is a minimally infinitesimally rigid bar-and-joint framework with some of its bars (possibly none of them) replaced by parallel cable-strut pairs. However, there exist different types of extremal examples. It is not too difficult to see that the frameworks in Figure 9 and Figure 10 are minimally infinitesimally rigid in $\mathbb{R}^2$ and $\mathbb{R}^3$.

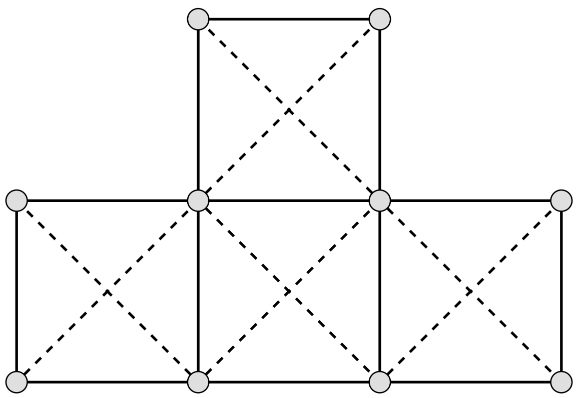


Figure 9: Minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^2$ with $2|B| + |C \cup S| = 4|V| - 6$.

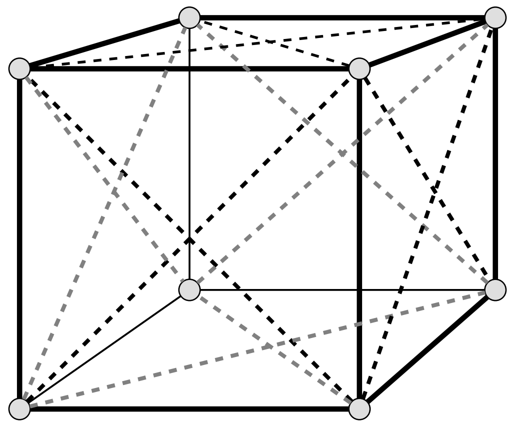


Figure 10: Minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^3$ with $2|B|+|C\cup S| = 6|V|-12$.

4.3 Consequences for weak rigidity

A tensegrity graph T is *minimally weakly rigid* in $\mathbb{R}^d$, if T is weakly rigid in $\mathbb{R}^d$ but $T-e$ is not weakly rigid in $\mathbb{R}^d$ for every edge e of T.

It is immediate that the upper bounds of Theorem 4.3, Theorem 4.9 and Theorem 4.10 hold for the number of edges of minimally weakly rigid tensegrity graphs without parallel edges. This is because a d-dimensional weakly rigid tensegrity graph T has an infinitesimally rigid representation (T,p) in $\mathbb{R}^d$ and if T is minimally weakly rigid then (T,p) must be minimally infinitesimally rigid (otherwise there would be an edge e of T such that p is a rigid realization of $T-e$ in $\mathbb{R}^d$, contradicting the minimality of T).

Thus, Theorem 4.3, Theorem 4.9 and Theorem 4.10 imply the following.

Corollary 4.18. Let $T=(V,C\cup S)$ be a tensegrity graph with no parallel members. Suppose that T is minimally weakly rigid in $\mathbb{R}^1$. Then

$$|C\cup S| \leq 2|V|-3.$$

Corollary 4.19. Let $T=(V,C\cup S)$ be a tensegrity graph with no parallel members. Suppose that T is minimally weakly rigid in $\mathbb{R}^2$. Then

$$|C\cup S| \leq 3|V|-6.$$

Corollary 4.20. Let $T=(V,C\cup S)$ be a tensegrity graph with no parallel members. Suppose that T is minimally weakly rigid in $\mathbb{R}^3$. Then

$$|C\cup S| \leq 4|V|-10.$$

Consider the tensegrity graph obtained from K_{d+1} of struts, by adding $|V| - d - 1$ vertices and connecting each of them to every vertex of the K_{d+1} with cables. Then it has $(d+1)|V| - \binom{d+2}{2}$ edges. There is an example for $d = 1$ in Figure 5, for $d = 2$ in Figure 6 and for $d = 3$ in Figure 7. Notice that if there are only cables attached to a vertex v of an infinitesimally rigid framework (T, p), then v must be in the interior of the convex hull of the vertices of T (otherwise there is no proper stress of (T, p), contradicting Theorem 2.7). Since in these examples there are only $d + 1$ vertices that have struts attached to them, their convex hull must form a d-dimensional simplex in every general d-dimensional rigid realization (rigidity and infinitesimal rigidity are equivalent for frameworks in general position). Using this observation, it is not too difficult to see that there is no general d-dimensional rigid realization of tensegrity graphs from this family, if we remove any one of their edges. Thus, the upper bounds of Corollary 4.18, Corollary 4.19 and Corollary 4.20 are tight.

Also, from Theorem 4.17 we obtain the following general upper bounds.

Corollary 4.21. Let $T = (V, B \cup C \cup S)$ be a tensegrity graph. Suppose that T is minimally weakly rigid in $\mathbb{R}^d$ and $|V| \geq d + 2$. Then

$$2|B| + |C \cup S| \leq 2 \cdot \left(d|V| - \binom{d+1}{2} \right).$$

This is also sharp, see for example minimally rigid bar-and-joint graphs.

5 Open questions

In this last section, we list some open questions related to edge counting of minimally rigid tensegrity frameworks and graphs.

The most natural related open problem is the generalization of the results in Section 4.1 for $d \geq 4$. The conjecture is the following.

Conjecture 5.1. Let (T, p) be a d-dimensional general realization of the tensegrity graph $T = (V, C \cup S)$ with no parallel members. Suppose that (T, p) is minimally infinitesimally rigid. Then

$$|C \cup S| \leq (d+1)|V| - \binom{d+2}{2}$$

As explained in Section 4.3, this would also give sharp upper bounds on the number of edges of minimally weakly rigid tensegrity graphs without parallel edges.

It is also a natural and currently open problem to give upper bounds on the number of edges of minimally infinitesimally rigid frameworks with no parallel edges and no bars, but without requiring the framework to be in general position. We saw non-injective and non-parallel examples in Section 4.1, where the number of edges is more than the upper bound given by Theorem 4.9 and Theorem 4.10. We also showed that there is no framework from this family with its edge number reaching the upper bound of Theorem 4.12 (since the extremal examples are only containing parallel members). However, we do not have a sharp upper bound on the number of edges of a framework from this family.

A tensegrity graph T is *minimally strongly rigid* in $\mathbb{R}^d$, if T is strongly rigid in $\mathbb{R}^d$ but $T - e$ is not strongly rigid in $\mathbb{R}^d$ for every edge e of T. For strongly rigid tensegrity graphs, it is an open problem to bound the number of edges of the minimal instances for any $d \geq 1$.

It is natural to ask the same questions with other rigidity definitions. The majority of these questions have not been answered yet. Problems related to global and universal rigidity of tensegrity frameworks seem more difficult than the corresponding questions for infinitesimal rigidity, because while Theorem 2.7 provides a manageable characterization for infinitesimal rigidity, no such characterization is known for global and universal rigidity of tensegrity frameworks. There is currently no bound known on the number of edges for minimally globally and universally rigid tensegrity frameworks for any $d \geq 1$.

The 1-dimensional weakly globally rigid tensegrity graphs are characterized in Theorem 2.5 by Garamvölgyi. Using this result, it can be shown that a 1-dimensional minimally weakly globally rigid tensegrity graph has at most $2|V| - 3$ edges, and this upper bound is sharp [17]. For $d \geq 2$, no upper bound on the number of edges of minimally weakly globally rigid tensegrity graphs is known. In regard to strongly globally rigid tensegrity graphs, it is also an open problem to provide upper bounds for the number of edges of the minimal instances for any $d \geq 1$.

Furthermore, all of the corresponding questions about minimally weakly and strongly universally rigid tensegrity graphs remain unanswered.

References

[1] R. Connelly, S. D. Guest. Frameworks, tensegrities, and symmetry. Cambridge University Press. 2022.

[2] L. Asimow, B. Roth. The rigidity of graphs. Transactions of the American Mathematical Society, 245, 279-289. 1978.

[3] W. Whiteley. Some matroids from discrete applied geometry. Contemporary Mathematics, 197, 171-312. 1996.

[4] Gortler, Steven J., Alexander D. Healy, Dylan P. Thurston. Characterizing generic global rigidity. American Journal of Mathematics 132(4), 897-939. 2010.

[5] A. Recski, O. Shai. Tensegrity frameworks in one-dimensional space. European Journal of Combinatorics, 31(4), 1072-1079. 2010.

[6] B. Jackson, T. Jordán, C. Király. Strongly rigid tensegrity graphs on the line. Discrete Applied Mathematics, 161(7-8), 1147-1149. 2013.

[7] D. Garamvölgyi. On the global rigidity of tensegrity graphs. Discrete Applied Mathematics, 302, 114-122. 2021.

[8] B. Roth, W. Whiteley. Tensegrity frameworks. Transactions of the American Mathematical Society, 265(2), 419-446, 1981.

[9] T. Jordán. Combinatorial rigidity: graphs and matroids in the theory of rigid frameworks. Discrete Geometric Analysis, MSJ Memoirs (Vol. 34, pp. 33-112). 2016.

[10] A. Frank. Connections in Combinatorial Optimization. Connections in combinatorial optimization (Vol. 38). Oxford: Oxford University Press. 2011.

[11] C. R. Coullard, L. Hellerstein. Independence and port oracles for matroids, with an application to computational learning theory. Combinatorica, 16(2), 189-208. 1996.

[12] B. Jackson, T. Jordán. The Dress conjectures on rank in the 3-dimensional rigidity matroid. Advances in Applied Mathematics, 35(4), 355-367. 2005.

[13] E. Steinitz. Bedingt konvergente Reihen und konvexe Systeme. i-ii-iii. J. Reine Angew. Math., 143 (1913), 128–175, 144 (1914), 1–40, 146 (1916), 1-52.

[14] V. Soltan. Lectures on convex sets. World Scientific. 2019.

[15] W. E. Bonnice, J. R. Reay. Interior points of convex hulls. Israel Journal of Mathematics, 4, 243-248. 1966.

[16] R. Connelly. The rigidity of certain cabled frameworks and the second-order rigidity of arbitrarily triangulated convex surfaces. Advances in Mathematics, 37(3), 272-299. 1980.

[17] A. Clay, T. Jordán, J. Palmer. Minimally rigid tensegrity graphs on the line, manuscript. 2023.

[18] A. Clay, T. Jordán, S. H. Tóth. Minimally rigid tensegrity frameworks, manuscript. 2024.

[19] Kenneth Snelson's small sculptures.
`http://kennethsnelson.net/category/sculptures/small-sculptures/`
2024.06.05.