Eötvös Loránd University
Institute of Mathematics
Department of Algebra and Number Theory

Families of overconvergent Galois representations

Thesis

Donát Pigler
Mathematician MSc

Supervisor:
Gergely Zábrádi
Department of Algebra and Number Theory

2024

FAMILIES OF OVERCONVERGENT GALOIS REPRESENTATIONS

DONÁT PIGLER

Contents

Introduction

One of the most significant challenges in arithmetic geometry is to comprehend the absolute Galois group of rational numbers, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, or at the very least, its action on representations derived from geometry. However, posing the question alone presents significant challenges, and it is advisable to employ a special strategy, namely the theory of p-adic representation theory. Let us fix a prime number p in the set of integers $\mathbb{Z}$ and let $\mathfrak{p}$ be a prime ideal lying above p in the ring of integers $\mathcal{O}_{\overline{\mathbb{Q}}}$. The decomposition subgroup associated with $\mathfrak{p}$, $G_{\mathfrak{p}} = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma(\mathfrak{p}) = \mathfrak{p}\}$ is canonically isomorphic to the absolute Galois group of p-adic numbers, $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. The objective of p-adic representation theory is to elucidate the action of $G_{\mathbb{Q}_p}$ on vector spaces over $\mathbb{Q}_p$. This can be regarded as the local version of the global problem previously described.

Fontaine's theory of (φ,Γ)-modules proved to be a highly valuable resource in understanding the nature of continuous representations of Galois groups of finite extensions of $\mathbb{Q}_p$ over finite $\mathbb{Q}_p$-vector spaces. One of the principal results of the theory is that the category of continuous p-adic representations of the Galois group G_K of a finite extension $K/\mathbb{Q}_p$ is equivalent to the category of the so-called étale (φ,Γ)-modules over a certain field, $\mathbf{B}_K$. The field $\mathbf{B}_K$ is the p-adic completion of the field of Laurent series in one variable. However, for certain applications in p-adic Hodge theory one has to work with Laurent-series that are convergent in a specific annulus (of outer radius 1). This refinement of Fontaine's equivalence of categories was first established by Cherbonnier and Colmez [CC98] and can be expressed in terms of the notion of so-called overconvergence, namely that all $\mathbb{Q}_p$-representations of G_K are overconvergent (Corollary 6), i.e. Fontaine's equivalence of categories remains valid if one replaces $\mathbf{B}_K$ with the subfield ${\mathbf{B}^\dagger}_K$ of overconvergent elements. Subsequently, Colmez [Col10] demonstrated the existence of a natural, almost bijective correspondence between the two-dimensional p-adic representations of $G_{\mathbb{Q}_p}$ and p-adic Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. This correspondence represents a significant special case of the p-adic Langlands programme. A natural generalisation of this approach can be made in a number of different ways. One possible avenue of enquiry is to investigate the p-adic Laglands correspondence in the case of $\mathrm{GL}_m(\mathbb{Q}_p)$. In an attempt to generalise parts of Colmez's work to $\mathrm{GL}_m(\mathbb{Q}_p)$ ($m > 2$), Zábrádi [Záb18] proved a generalisation of Fontaine's equivalence for p-adic representations of direct powers of $G_{\mathbb{Q}_p}$ (see also [ZKC21] for a different proof, which includes the generalisation to finite extensions $K/\mathbb{Q}_p$ and overconvergence). The multivariable analogue of the theorem of Cherbonnier and Colmez was first established by Pal and Zábrádi [PZ19] showing that the p-adic representations of direct powers of $G_{\mathbb{Q}_p}$ exhibit overconvergence properties. Consequently, the category of these representations is equivalent to the category of overconvergent multivariable (φ, Γ)-modules. The findings on overconvergence are of significant importance in general, as they facilitate the interconnection between the category of Galois representations and the category of representations derived from geometry.

Another tool of representation theory is to consider a family of representations collectively rather than examining them individually. Let K be a finite extension of $\mathbb{Q}_p$ and let S be a Banach $\mathbb{Q}_p$-algebra denoting its maximal spectrum by $\mathcal{X}$. A family of representations of G_K is then defined as a free S-module V of finite rank endowed with a continuous S-linear action of G_K. The objective of this thesis is to examine families of p-adic Galois representations, with a particular focus on the contributions of Berger and Colmez as outlined in their seminal paper, [BC08]. Furthermore, this study will offer a comprehensive analysis of the ring theoretical constructions that are essential for a thorough understanding of the main result of this thesis, namely that the families of p-adic representations of G_K are overconvergent (Theorem 7).

The following provides a description of the essay's structure. In the initial chapter 1, the theoretical foundations of p-adic representations and their families are established, with reference to the article [BC08] by Berger and Colmez. In chapter 2, the Tate-Sen conditions and their general consequences are presented. This section is also based on the referenced article [BC08] by Berger and Colmez. Furthermore, the proofs of the technical lemmas in subsection 2.2 are primarily derived from the notes of Brinon and Conrad [BC]. In chapter 3 and 4, the requisite ring constructions for the principal results of Sen's method are presented. These are mainly based on Berger's article [Ber04] and notes [Ber], the notes of Brinon and Conrad [BC], the article of Colmez [Col08], and for some technicalities on Wang's note [Yup]. Finally, in chapter 5, I demonstrate two consequences of Sen's method: the classical Sen theory and the principal theorems on overconvergent (φ, Γ)-modules. The structure of the cited article by Berger and Colmez [BC08] forms the foundation of this chapter. For technical proofs I used [Ber] and [Yup].

Acknowledgment. I would like to express my gratitude to Gergely Zábrádi for bringing this intriguing topic to my attention and for providing me with invaluable supervision and advice throughout the process of writing my thesis.

1. Preliminaries

1.1. p-adic representations.

This subsection offers a concise theoretical introduction to the fundamental concepts and propositions that will be required later in the thesis. In accordance with this format, the results presented here are not accompanied by proof.

A profinite group, defined as the inverse limit of finite groups, can be viewed as a topological group. This is achieved by considering the subspace topology of the product of the groups, where the discrete topology is applied to each group. This yields a compact, Hausdorff, and totally disconnected topological space.

Definition 1. *A p-adic representation of a profinite group Γ is a representation $\rho : \Gamma \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ of Γ on a finite-dimensional $\mathbb{Q}_p$-vector space V such that ρ is continuous.*[1]

The following concept constitutes a fundamental ingredient of this framework. Let K be a field with a fixed separable closure K^s and let p be a prime ($\neq \mathrm{char} K$). The group of p^nth roots of unity in $(K^s)^\times$ is designated by μ_{p^n}. Furthermore, let $\mu_{p^\infty} := \varinjlim_n \mu_{p^n}$. The action of G_K on μ_{p^∞} is given by $g(\xi) = \xi^{\chi(g)}$, where $\chi(g)$ is a unique element of $\mathbb{Z}_p^\times$. Indeed, if $\xi \in \mu_{p^n}$, then $\chi(g)$ is solely relevant $\mod p^n$, and the reduction of $\chi(g)$ mod p^n, namely $\overline{\chi} \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ describes the action of g on the finite cyclic group μ_{p^n} of order p^n. This map $\chi : G_K \to \mathbb{Z}_p^\times$ is called the *p-adic cyclotomic character* of K. Since $\ker(\overline{\chi})$ corresponds to the finite extension $K(\mu_{p^n})/K$, it is open. Consequently, χ is a continuous character and its twisting action is observed in a wide range of contexts within the theory.

Definition 2. *A p-adic field is a field K of characteristic 0 that is complete with respect to a fixed discrete valuation, with a perfect residue field k of characteristic $p > 0$.*

Let K be a p-adic field (for a fixed prime p) and let us fix an algebraic closure $\overline{K}$. The absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ will be denoted by G_K. The completion $\widehat{\overline{K}}$ of $\overline{K}$ is referred to as $\mathbb{C}_K$, which is endowed with a unique normalised ($v(p) = 1$) valuation extending the given one on K. Moreover, it can be demonstrated that $\mathbb{C}_K$ is algebraically closed. Since G_K acts on $\overline{K}$ by isometries, this action uniquely extends to the field $\mathbb{C}_K$. Furthermore, the following classical result holds.

Theorem 1 (Ax-Sen-Tate)**.** *Let H be a closed subgroup of G_K. Then $(\mathbb{C}_K)^H$ is the completion $\widehat{L}$ of $L = (\overline{K})^H$.*

In the investigation of p-adic representations of G_K (which are also known as *p-adic Galois representations*) for a finite extension $K/\mathbb{Q}_p$, it is often advantageous to consider the case, where the residue field k is algebraically closed. As the majority

[1] We consider $\mathrm{Aut}_{\mathbb{Q}_p}(V)$ to be $\mathrm{GL}_d(\mathbb{Q}_p)$ by choosing a basis, where the choice is inconsequential.

of the relevant properties of p-adic representations can be identified within the inertia subgroup $I_K = G_{K^{\mathrm{ur}}} = G_{\widehat{K^{\mathrm{ur}}}}$, in practice we are replacing K with the p-adic completion of its maximal unramified extension $\widehat{K^{\mathrm{ur}}}$ within $\overline{K}$.

Remark 1. Let F be a local field with a perfect residue field k. It can be shown that there is an equivalence of categories between the extension of k and the unramified extensions of K. Since the unique extension of $\mathbb{F}_p$ of degree n is the splitting field of $x^{p^n} - x$, it follows that the unique unramified extension of $\mathbb{Q}_p$ is also by adjoining p^nth roots of unity, namely $\mathbb{Q}_p(\mu_{p^n})$. Moreover, $\mathbb{Q}_p^{\mathrm{ur}} = \mathbb{Q}_p(\mu_{p^\infty})$. In general, if K is a local field with perfect residue field k, then K^{ur} corresponds to the separable closure k^s of k, which in this case is $\overline{k}$. In particular, if k is a finite (thus perfect) field, then $\overline{k} = k(\mu_{p^\infty})$ for some p prime to char$k$. Consequently, $K^{\mathrm{ur}} = K(\mu_{p^\infty})$.

The conventional methodology entails the construction of a kind of dictionary that relates exemplary categories of p-adic representations of G_K with assorted categories of semi-linear algebraic objects "over K". By working over these algebras, it is often more straightforward to perform operations such as deformation, descent, construction of families, and so forth, than by working solely with Galois representations. One objective of this study is to demonstrate one such equivalence of categories, which will be discussed in greater detail in section 5.

1.2. **Algebra of coefficients and completed tensor products.**

It is assumed throughout this section that S is a Banach $\mathbb{Q}_p$-algebra, that is to say, an associative $\mathbb{Q}_p$-algebra, which is also a Banach space.

Definition 3. *Denote $\mathcal{X}$ the space associated to S, that is the set of maximal ideals of S.*

The elements of $\mathcal{X}$ are considered to be points and the notation $\mathfrak{m}_x$ is employed to denote the maximal ideal of S corresponding to to the point x. If $f \in S$, then the image of f in the quotient $E_x = S/\mathfrak{m}_x$ is denoted by $f(x)$. A subspace P of $\mathcal{X}$ is said to be an *S-analytic subspace* if there exists an ideal I of S such that $P = \{x \in \mathcal{X} \mid I \subset \mathfrak{m}_x\}$ or, which is equivalent, if there exists a family of elements $\{f_\alpha\}_\alpha$ of S such that $P = \{x \in \mathcal{X} \mid f_\alpha(x) = 0 \text{ for all } \alpha\}$. Rather than working with norms, we prefer to work with valuations over S. For $f, g \in S$, these do not satisfy the usual identity $\mathrm{val}_S(fg) = \mathrm{val}_S(f) + \mathrm{val}_S(g)$, but only $\mathrm{val}_S(fg) \geq \mathrm{val}_S(f) + \mathrm{val}_S(g)$. This means that val_S fulfils the following properties:

(1) $\mathrm{val}_S(f) = \infty \Leftrightarrow f = 0$;
(2) $\mathrm{val}_S(fg) \geq \mathrm{val}_S(f) + \mathrm{val}_S(g)$;
(3) $\mathrm{val}_S(f + g) \geq \inf(\mathrm{val}_S(f), \mathrm{val}_S(g))$;

Let $\mathcal{O}_S = \{f \in S \mid \mathrm{val}_S(f) \geq 0\}$ be the ring of integers in S for val_S.

Definition 4. *S is said to be an algebra of coefficients, if*

(1) *S contains $\mathbb{Q}_p$ and the restriction of val_S from S to $\mathbb{Q}_p$ is the p-adic valuation val_p;*
(2) *for all $x \in \mathcal{X}, E_x$ is a finite extension of $\mathbb{Q}_p$;*
(3) *the Jacobson radical, $J(S) = \bigcap_{x\in\mathcal{X}} \mathfrak{m}_x = 0$. In particular S is reduced.*

Let S be an algebra of coefficients and let $\mathcal{Y}$ be a S-analytic subspace of $\mathcal{X}$ defined by an ideal I. Then $\mathcal{Y}$ is the space associated to the algebra of coefficients $S/\sqrt{I}$.

Lemma 1. *Let S be an algebra of coefficients.*

(1) *If $f \in S$ is such that for all $x \in \mathcal{X}$ $f(x) \neq 0$, then f is a unit of $\mathcal{X}$.*
(2) *If M is a flat S-module, and $y \in M$ such that for all $x \in \mathfrak{X}$, $y(x) \in M/\mathfrak{m}_x M$ is 0, then $y = 0$.*

Proof. (1) arises from the fact that f is not contained in any maximal ideal, and thus a unit. (2) results from the fact that the map $S \to \prod S/\mathfrak{m}_x$ is injective, which follows from the assumption that $J(S) = 0$, and then the injectivity of the map $M \to \prod M/\mathfrak{m}_x M$ is maintained if M is a flat module. □

If $x \in \mathcal{X}$, then the field E_x is a finite extension of $\mathbb{Q}_p$ and therefore has the p-adic valuation val_p. If $f \in S$, then *the spectral valuation* is defined by $\mathrm{val}_{sp}(f) = \inf_{x\in\mathcal{X}} \mathrm{val}_p(f(x))$. Finally, if E and F are two Banach spaces, we denote by $E\widehat{\otimes}F$ their completed tensor product over $\mathbb{Q}_p$. If E and F are two complete topological $\mathbb{Z}_p$-modules, then $E\widehat{\otimes}F$ denotes their completed tensor product over $\mathbb{Z}_p$, which is defined as follows. (see [BGR84], §2.1.7.)

Definition 5. *Let M and N be normed R-modules, and let us consider the (ordinary) tensor product $M \otimes_R N$. The function $|\cdot| : M \otimes_R N \to \mathbb{R}_{\geq 0}$ is defined as follows: $t \in M \otimes_R N$, let*

$$|t| := \inf(\max_{1\leq i \geq n} |x_i||y_i|),$$

where the infimum is taken over all possible representations $t = \sum_{i=1}^n x_i \otimes y_i$ ($x_i \in M$, $y_i \in N$). It can be readily demonstrated that with this $|\cdot|$, $M \otimes_R N$ is a semi-normed R-module. As a semi-normed group, we can construct the completion $M\widehat{\otimes}_R N$, as detailed in [BGR84], Proposition 1.1.7. This will result in a normed R-module.

1.3. **Étale descent.**

Let B be a Banach $\mathbb{Q}_p$-algebra equipped with a continuous action of a finite group G. Let $B^\natural$ denote the ring B on which G acts trivially. It is assumed that:

(1) the B^G-module B is free of finite rank and faithfully flat[2];
(2) $B^\natural \otimes_{B^G} B \cong \oplus_{g\in G} B^\natural \cdot e_g$ (where $e_g^2 = e_g, e_g e_h = 0$ if $g \neq h$ and $g(e_h) = e_{gh}$).

Proposition 1. *If S is a Banach-algebra on which G acts trivially, and if M is a locally free finitely generated $S\widehat{\otimes}B$-module with a semi-linear action of G, then*

(1) M^G is a locally free[3] finitely generated $S\widehat{\otimes}B^G$-module;
(2) $(S\widehat{\otimes}B) \otimes_{S\widehat{\otimes}B^G} M^G \to M$ is an isomorphism.

Proof. Let $\pi_G = \frac{1}{|G|}\sum_{g\in G} g \in B[G]$. If N is a $B[G]$-module, then we have a decomposition $N = \pi_G N \oplus \ker \pi_G$ and $N^G = \pi_G N$. In particular, we have that $M = M^G \oplus \ker \pi_G$, which implies that M^G is a $S\widehat{\otimes}B^G$-module. As a direct component of M, it is a locally free $S\widehat{\otimes}B^G$-module of finite type. This demonstrates (1).

We will now proceed to demonstrate item (2). Since the rank of B is finite over B^G and the tensor product is associative, we have an isomorphism

$$(S\widehat{\otimes}B) \otimes_{S\widehat{\otimes}B^G} M^G \cong B \otimes_{B^G} (S\widehat{\otimes}B) \otimes_{S\widehat{\otimes}B^G} M^G \cong B \otimes_{B^G} M^G.$$

It is therefore sufficient to show that the map $B \otimes_{B^G} M^G \to M$ is an isomorphism. As we assume that the B^G-module $B^\natural$ is faithfully flat, it is sufficient to demonstrate that the map:

$$B^\natural \otimes_{B^G} (B \otimes_{B^G} M^G) \to B^\natural \otimes_{B^G} M$$

is an isomorphism. From associativity, we have that $B^\natural \otimes_{B^G} (B \otimes_{B^G} M^G) \cong B \otimes_{B^G} (B^\natural \otimes_{B^G} M)^G$. Furhtermore, $B^\natural \otimes_{B^G} M$ is a $B^\natural \otimes_{B^G} B$-module. Since $B^\natural \otimes_{B^G} B \cong \oplus_{g\in G} B^\natural \cdot e_g$, it therefore decomposes into $B^\natural \otimes_{B^G} M \cong \oplus_{g\in G} N \cdot e_g$, where $N \cdot e_g = (B^\natural \cdot e_g) \cdot (B^\natural \otimes_{B^G} M)$. The map from $N \cdot e_1$ to $(B^\natural \otimes_{B^G} M)^G$, which associates $n \cdot e_1$ with $(g(n) \cdot e_g)_{g\in G}$ induces an isomorphism from $N \cdot e_1$ to $(B^\natural \otimes_{B^G} M)^G$. Thus, we have:

$$\begin{aligned} B \otimes_{B^G} N \cdot e_1 &= (B \otimes_{B^G} B^\natural) \otimes_{B^\natural} N \cdot e_1 = (\oplus_{g\in G} B^\natural \cdot e_g) \otimes_{B^\natural} N \cdot e_1 \\ &= \oplus_{g\in G} N \cdot e_g = B^\natural \otimes_{B^G} M, \end{aligned}$$

and the map $B^\natural \otimes_{B^G} (B \otimes_{B^G} M^G) \to B^\natural \otimes_{B^G} M$ is an isomorphism. □

Remark 2. Proposition 1 is particularly applicable in the case where B is a finite Galois extension of $\mathbb{Q}_p$ and where G is a subgroup of $\mathrm{Gal}(B/\mathbb{Q}_p)$, which ensures that B/B^G is a Galois extension. Condition (1) stated at the beginning of the paragraph is self-evident, while condition (2) is a classical result. For a second example, please refer to Lemma 21.

[2]That is to say, taking the tensor product with a sequence produces an exact sequence if and only if the original sequence is exact.
[3]For all $\mathfrak{p} \in \mathrm{Spec}$, $M_{\mathfrak{p}}$ is free.

1.4. Family of p-adic representations.

Let K be a finite extension of $\mathbb{Q}_p$ and let G_K denote the Galois group, $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. Let S be a Banach-algebra. A *family of p-adic representations* is defined as a free S-module V of finite rank d, endowed with a continuous linear action of the group G_K. It should be recalled that $\mathcal{O}_S$ denotes the ring of integers of S with respect to the valuation val_S. It is assumed that there exists a free $\mathcal{O}_S$-module T of rank d such that $V = S \otimes_{\mathcal{O}_S} T$. In the case where $S = E$ is a field, this condition is automatically satisfied, as demonstrated by the following lemma.

Lemma 2. *Let V be an E-representation of dimension d, where E is a field. Then there exists an $\mathcal{O}_E$-module T of rank d that is invariant under G_K such that $V = E \otimes_{\mathcal{O}_E} T$.*

Proof. If a basis of V is chosen, the representation corresponds to a continuous map $\rho : G_K \to GL_d(E)$. It follows that there exists an integer $n \geq 0$ such that the image of G_K is contained in $M_d(p^{-n}\mathcal{O}_E)$. If T_0 denotes the $\mathcal{O}_E$-module generated by the chosen basis, we see that if $g \in G_K$, then $g(T_0) \subset p^{-n}T_0$. Therefore, setting $T = \sum_{g \in G_K} g(T_0)$, we have $T_0 \subset T \subset p^{-n}T_0$, and thus T is free of rank d and invariant under the action of G_K. □

2. Sen's method

This section presents an explanation of Sen's method, which facilitates the calculation of specific sets of Galois cohomologies.

2.1. The Tate-Sen conditions.

In this chapter, G_0 is a profinite group endowed with a continuous character $\chi : G_0 \to \mathbb{Z}_p^\times$ having an open image $\chi(G_0)$. Let $H_0 = \ker \chi$. For any element $g \in G_0$, let $n(g)$ be the integer defined by $n(g) = \mathrm{val}_p(\chi(g) - 1)$. If G is an open subgroup of G_0 and $H = G \cap H_0$, then let $N_{G_0}(H)$ be the normaliser of H in G_0.

The objective is to ascertain the openness of the group $N_{G_0}(H)$ in G_0. Since a closed subgroup of a profinite group is profinite for the subspace topology, open subgroups of H_0 that are normal in G_0 constitute a base of open subgroups in H_0. Consequently, since H is an open subgroup in H_0, there exists a subgroup $N \subset H$ that is open in H_0 and normal in G_0. It is now necessary to consider the resulting containment of finite subgroups $H/N \subset H_0/N$ inside of G_0/N, with H_0/N normal in G_0/N. Since the action of $(G_0/N)/(H_0/N) = \chi(G_0)$ on the finite group H_0/N via conjugation is continuous, it follows that some open subgroup of G_0/N must centralize H_0/N (because the permutation group of the finite set H_0/N is finite). The preimage of this in G_0 is an open subgroup that normalizes H, as desired. In particular, $\widetilde{\Gamma}_H = N_{G_0}(H)/H$ is open in G_0/H, and therefore $\chi : \widetilde{\Gamma}_H \to \chi(G_0)$ is an open mapping. Let and C_H be the centre of $\widetilde{\Gamma}_H$.

Lemma 3. *C_H is an open subgroup of $\widetilde{\Gamma}_H$.*

Proof. The kernel of the restriction of $\chi^{2(p-1)}$ to $\widetilde{\Gamma}_H$ is a finite group A and the group $\widetilde{\Gamma}_H$ can be written as the following exact sequence: $1 \to A \to \widetilde{\Gamma}_H \to B \to 1$, where B is an open subgroup of $\mathbb{Z}_p^\times = \mathbb{Z}/(p-1)\mathbb{Z} \times (1+p\mathbb{Z}_p)$ without torsion, which is why the exponent $2(p-1)$ was included. Consequently, B is an open subgroup of $(1+p\mathbb{Z}_p) \cong \mathbb{Z}_p$ and thus is isomorphic to $\mathbb{Z}_p$. (This statement can be proven with a slight adjustment in the case where $p = 2$). The group $\widetilde{\Gamma}_H$ is therefore a semi-direct product of $\mathbb{Z}_p$ and a finite group. An element g of $\mathbb{Z}_p \subset \widetilde{\Gamma}_H$ is in the centre of $\widetilde{\Gamma}_H$ if and only if its image in $\mathrm{Aut}(A)$ (g acting by conjugation on A) is trivial. Since the group $\mathrm{Aut}(A)$ is finite, the intersection of C_H with $\mathbb{Z}_p \subset \widetilde{\Gamma}_H$ is of finite index in $\mathbb{Z}_p$ and thus also in $\widetilde{\Gamma}_H$. This allows us to conclude. □

The smallest integer $n > 1$ such that $\chi(C_H)$ contains $1 + p^n\mathbb{Z}_p$ is denoted by $n_1(H)$. The preceding lemma demonstrates that $n_1(H) \neq \infty$.

Let S be a Banach algebra and let $\widetilde{\Lambda}$ be a $\mathcal{O}_S$-algebra. The map $\mathrm{val}_\Lambda : \widetilde{\Lambda} \to \mathbb{R} \cup \{+\infty\}$ must satisfy the following weakening of the valuation axioms (in the spirit of semi-norm):

(1) $\mathrm{val}_\Lambda(x) = +\infty \Leftrightarrow x = 0$;
(2) $\mathrm{val}_\Lambda(xy) \geq \mathrm{val}_\Lambda(x) + \mathrm{val}_\Lambda(y)$;
(3) $\mathrm{val}_\Lambda(x + y) \geq \inf(\mathrm{val}_\Lambda(x), \mathrm{val}_\Lambda(y))$;
(4) $\mathrm{val}_\Lambda(p) > 0$ and $\mathrm{val}_\Lambda(px) = \mathrm{val}_\Lambda(p) + \mathrm{val}_\Lambda(x)$, if $x \in \widetilde{\Lambda}$.

It should be noted that condition (4) encompasses the case where p equal to zero in Λ. Condition (3) permits the use of val_Λ to provide $\widetilde{\Lambda}$ with a topology by using the additive subgroups $\widetilde{\Lambda}^{\geq a} := \mathrm{val}_\Lambda([a, +\infty])$ as a base of opens around 0. Condition (1) demonstrates that this topology is Hausdorff and every point has a countable base of open neighborhoods, so we can probe the topology with using sequences. Furthermore, it implies that $\mathrm{val}_\Lambda(-x) \geq \mathrm{val}_\Lambda(x)$ for all x, so $\mathrm{val}_\Lambda(x) = \mathrm{val}_\Lambda(-x)$. It is assumed that $\widetilde{\Lambda}$ is complete with respect to this topology.

Let $d \geq 1$ be an integer and $U \in M_d(\widetilde{\Lambda})$, and let

$$\mathrm{val}_\Lambda(U) = \min_{1 \leq i,j \leq d} \mathrm{val}_\Lambda(u_{i,j}), \qquad \text{where } U = (u_{i,j})_{1 \leq i,j \leq d}.$$

The following immediate result will be used extensively in what follows.

Lemma 4. *If $d \geq 1$ integer and $U \in M_d(\widetilde{\Lambda})$ verifies $\mathrm{val}_\Lambda(U - 1) > 0$, then $U \in GL_d(\widetilde{\Lambda})$ and its inverse is equal to $\sum_{n=0}^{+\infty}(1 - U)^n$.*

Now, let us suppose that $\widetilde{\Lambda}$ has a continuous $\mathcal{O}_S$-linear action of G_0 such that $\mathrm{val}_\Lambda(g(x)) = \mathrm{val}_\Lambda(x)$ ("isometry") for all $g \in G_0$ and $x \in \widetilde{\Lambda}$. Then the group G_0 acts continuously on $GL_d(\widetilde{\Lambda})$, where $d \geq 1$.

Remark 3. Given that G_0 acts continuously on $\mathrm{GL}_d(\widetilde{\Lambda})$, it is reasonable to form the pointed set of continuous cohomology $H^1(G, \mathrm{GL}_d(\widetilde{\Lambda}))$ when G is a subgroup of G_0 (with subspace topology). This, in fact, classifies the isomorphism classes of finite free $\widetilde{\Lambda}$-modules equipped with a semi-linear action of G that is continuous for the natural topology of finite free $\widetilde{\Lambda}$-modules. To illustrate, in Sen's context with $G = G_0 = G_K$, this is the problem of classifying d-dimensional continuous semi-linear representations of G_K over $\mathbb{C}_K$.

We are interested in pointed sets of continuous cohomology $H^1(G_0, GL_d(\widetilde{\Lambda}))$. Sen's method allows for a significant reduction in the apparent complexity of these sets under certain so-called Tate-Sen conditions.

Definition 6. *The Tate-Sen conditions are the following three conditions:*

(TS1) For every open subgroup $H_1 \subset H_2$ of H_0, there exists $c_1 > 0$ such that there exists an element $\alpha \in \widetilde{\Lambda}^{H_1}$ satisfying $\mathrm{val}_\Lambda(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.

(TS2) For any open subgroup H of H_0, there exists a number $c_2 > 0$ and an integer $n(H) \in \mathbb{N}$, along with an increasing sequence of closed $\mathcal{O}_S$-subalgebras of $\widetilde{\Lambda}^H$, denoted by $(\Lambda_{H,n})_{n\in\mathbb{N}}$, and for each $n \geq n(H)$, an $\mathcal{O}_S$-linear map $R_{H,n} : \widetilde{\Lambda}^H \to \Lambda_{H,n}$ satisfying the properties:

1. *if $H_1 \subset H_2$, then $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ and the restriction of $R_{H_1,n}$ to $\widetilde{\Lambda}^{H_2}$ coincides with $R_{H_2,n}$;*
2. *$R_{H,n}$ is $\Lambda_{H,n}$-linear and $R_{H,n}(x) = x$ if $x \in \Lambda_{H,n}$;*
3. *$g(\Lambda_{H,n}) = \Lambda_{gHg^{-1},n}$ and $g(R_{H,n}(x)) = R_{gHg^{-1},n}(gx)$ if $g \in G_0$; in particular, $R_{H,n}$ commute with the action of $\widetilde{\Gamma}_H$;*
4. *if $n \geq n(H)$ and $x \in \widetilde{\Lambda}^H$, then $\mathrm{val}_\Lambda(R_{H,n}(x)) \geq \mathrm{val}_\Lambda(x) - c_2$;*
5. *if $x \in \widetilde{\Lambda}^H$, then $\lim_{n\to+\infty} R_{H,n}(x) = x$.*

(TS3) There exists a constant $c_3 > 0$ and, for any open subgroup G of G_0 an integer $n(G) \geq n_1(H)$, where $H = G \cap H_0$, such that, for every $n \geq n(G)$, if $\gamma \in \widetilde{\Gamma}_H$ verifies $n(\gamma) \leq n$, then $\gamma - 1$ is invertible on $X_{H,n} = (1 - R_{H,n})(\widetilde{\Lambda}^H)$ and we have $\mathrm{val}_\Lambda((\gamma-1)^{-1}(x)) \geq \mathrm{val}_\Lambda(x) - c_3$ if $x \in X_{H,n}$.

Proposition 2. *If $\widetilde{\Lambda}$ is an $\mathbb{Z}_p$-algebra that satisfies the Tate-Sen conditions, and if S is a Banach-algebra, then $\mathcal{O}_S\widehat{\otimes}\widetilde{\Lambda}$ satisfies the Tate-Sen conditions (with the same constants c_1, c_2 and c_3).*

Proof. This follows directly from extending the map $R_{H,n}$ by $\mathcal{O}_S$-linearity. □

Remark 4. Upon initial observation, the conditions appear somewhat complicated. In order to provide some context and clarity, we will offer some explanatory remarks.

(TS1) The sum in this axiom may be considered a sort of trace, and elsewhere this axiom is related to the construction of "normalized traces". In Sen's case, this is a direct consequence of Tate's "almost étale" result.

(TS2) This condition encodes information concerning the "completed normalized traces". We wish to highlight that according to part 2., the map $R_{H,n}$ is a $\Lambda_{H,n}$-linear projector. Consequently, $X_{H,n} = \ker(R_{H,n})$ is a closed $\Lambda_{H,n}$-submodule of $\widetilde{\Lambda}^H$ and there is a topological decomposition $\widetilde{\Lambda}^H = \Lambda_{H,n} \oplus X_{H,n}$. Furthermore, part (3) merely states that an action by $\widetilde{\Gamma}_H = G_0/H$ makes sense on $R_{H,n}$ and that it is trivial.

(TS3) This final condition describes the action of G_0 on the complement to $\Lambda_{H,n}$ in $\widetilde{\Lambda}^H$, as defined by the splitting provided by the section $R_{H,n}$. It is stated that $\gamma - 1$ has a bounded linear inverse on $\ker(R_{H,n})$ (controlled by c_3) provided that n is sufficiently large (depending on G) and that γ is not too close to 1 (depending on n). In order to make any statements about an inverse to $\gamma - 1$, it is necessary to restrict to $\ker(R_{H,n})$. This is because on the complement $\Lambda_{H,n}$ of $\ker(R_{H,n})$ in $\widetilde{\Lambda}^H$ the action of some open subgroup of $\widetilde{\Gamma}_h$ may be trivial. It should be noted that since $n \geq n(G) \geq n_1(G)$, it follows that $1 + p^n\mathbb{Z}_p \subset \chi(G)$. Consequently, there are numerous elements γ in the open subgroup $G/H \subset \widetilde{\Gamma}_H$ for which $n(\gamma) = n$. The sole purpose of requiring $n(G) \geq n_1(G)$ is to guarantee the existence of a multitude of $\gamma \in G/H$ with $n(\gamma) \leq n$.

2.2. **Consequences of Tate-Sen axioms.**

We will work now in the general setting of the Tate-Sen axioms. The firs lemma posits that, provided a 1-cocyle is sufficiently close to the trivial cocycle, it may be considered approximately a coboundary. The strategy of the proof is analogous to that employed in the classical proof of Hilbert's Theorem 90 (see in Hungarian [Záb], Theorem 2.3.8.), which relies on a cocycle construction involving an element with trace 1.

Lemma 5. *Let H be an open normal subgroup of H_0, $a > c_1$ and $k \in \mathbb{N}$. If $\tau \mapsto U_\tau$ is a continuous 1-cocycle on H in $GL_d(\widetilde{\Lambda})$ satisfying $U_\tau - 1 \in p^k M_d(\widetilde{\Lambda})$ and $\mathrm{val}_\Lambda(U_\tau - 1) \geq a$ for every $\tau \in H$, then there exists a matrix $M \in GL_d(\widetilde{\Lambda})$ satisfying $M - 1 \in p^k M_d(\widetilde{\Lambda})$ and $\mathrm{val}_\Lambda(M - 1) \geq a - c_1$ such that the cocycle $\tau \mapsto M^{-1}U_\tau\tau(M)$ satisfies $\mathrm{val}_\Lambda(M^{-1}U_\tau\tau(M) - 1) \geq a + 1$.*

Proof. It should first be noted that for any $h \in H$, we have $\mathrm{val}_\Lambda(U_\tau) \geq \min\{\mathrm{val}_\Lambda(U_\tau - 1), \mathrm{val}_\Lambda(1)\} \geq 0$. By continuity, there exists an an open subgroup H_1 of H (which we may be further reduced to be normal in G_0) such that for any $\tau \in H_1$, $\mathrm{val}_\Lambda(U_\tau - 1) \geq a + 1 + c_1$. By (TS1), there exists an $\alpha \in \widetilde{\Lambda}^{H_1}$ satisfying $\sum_{\tau \in H/H_1} \tau(\alpha) = 1$ and $\mathrm{val}_\Lambda(\alpha) > -c_1$.

Let Q be a system of representatives for H/H_1, let

$$M_Q = \sum_{\sigma \in Q} \sigma(\alpha) U_\sigma.$$

The assumptions on α imply that $M_Q - 1 \in p^k M_d(\widetilde{\Lambda})$ and $\mathrm{val}_\Lambda(M_Q - 1) \geq a - c_1$. In particular, $\mathrm{val}_\Lambda(M_Q - 1) > 0$ and M_Q is invertible and $\mathrm{val}_\Lambda(M_Q^{-1} - 1) > 0$ by Lemma

4. Thus, $\mathrm{val}_\Lambda(M_Q^{-1}) \geq 0$. The 1-cocycle condition implies that

$$U_\tau \tau(M_Q) = \sum_{\sigma \in Q} \tau\sigma(\alpha) U_\tau \tau(U_\sigma) = \sum_{\sigma \in Q} \tau\sigma(\alpha) U_{\tau\sigma} = M_{\tau Q}$$

Let $\tau \in H$ be fixed. For any element $\sigma \in Q$ (as for any element of H), there exists a unique $\sigma' \in Q$ and $h_1 \in H_1$ such that $\tau\sigma = \sigma' h_1$. From the cocycle condition, we have that

$$\mathrm{val}_\Lambda(U_{\sigma h_1} - U_{\sigma'}) = \mathrm{val}_\Lambda(U_{\sigma'} \sigma'(U_{h_1} - 1)) \geq \mathrm{val}_\Lambda(U_{\sigma'}) + \mathrm{val}_\Lambda(U_{h_1} - 1) \geq a + c_1 + 1.$$

Moreover, $\tau\sigma(\alpha) U_{\tau\sigma} = \sigma'(\alpha) U_{\sigma' h_1}$. From these two observation and the fact that $\mathrm{val}_\Lambda(\alpha) > -c_1$, we can conclude that

$$\mathrm{val}_\Lambda(\tau\sigma(\alpha) U_{\tau\sigma} - \sigma'(\alpha) U_{\sigma'})) > a + 1.$$

Consequently,

$$\mathrm{val}_\Lambda(U_\tau \tau(M_Q) - M_Q) > a + 1.$$

Therefore $\mathrm{val}_\Lambda(M_Q^{-1} U_\tau \tau(M_Q) - 1) \geq a + 1$, since $\mathrm{val}_\Lambda(M_Q^{-1}) \geq 0$. □

Corollarry 1. *In accordance with the hypotheses of Lemma 5, it is possible to identify an M such that $\tau \mapsto M^{-1} U_\tau \tau(M)$ is trivial for all $\tau \in H$. In other words, the 1-cocycle $h \mapsto U_\tau$ is a 1-coboundary of the form $U_\tau = M\tau(M)^{-1}$, where $M \equiv 1 \mod p^k$ and $\mathrm{val}_\Lambda(M - 1) \geq a - c_1$.*

Proof. The Lemma 5 is applied repeatedly to construct a sequence $(M_m)_{m \in \mathbb{N}}$ such that for all $m \in \mathbb{N}$, the following holds:

(1) $\mathrm{val}_\Lambda(M_m - 1) \geq a - c_1 + m$ and $M_m - 1 \in p^k M_d(\widetilde{\Lambda})$;
(2) $\mathrm{val}((\prod_{m=0}^n M_m)^{-1} U_\tau \tau(\prod_{m=0}^n M_m)) \geq a + n + 1$ for all $\tau \in H$.

Since $\widetilde{\Lambda}$ is complete with respect to the topology defined by val_Λ, property (1) implies that the product $M = \prod_{m=0}^\infty M_m$ converges in $\mathrm{GL}_d(\widetilde{\Lambda})$. Furthermore, by passing to the limit, we obtain that $M - 1 \in p^k M_d(\widetilde{\Lambda})$ and that $\mathrm{val}_\Lambda(M - 1) \geq a - c_1$. Property (2) guarantees that $M^{-1} U_\tau \tau(M) = 1$ for all $\tau \in H$. □

In addition to considering approximating descent (to $\Lambda_{H,n}$) using approximations relative to val_Λ, it is also necessary to keep track of p-adic approximations when doing descent. A preliminary lemma in this direction (which will be improved by bootstrapping in the subsequent corollary) is as follows:

Lemma 6. *Let $\delta \in (0, +\infty]$, $a \in [c_2 + c_3 + \delta, +\infty]$ and $b \in [\max(a + c_2, 2c_2 + 2c_3 + \delta), +\infty]$. Let H be an open subgroup of H_0, let $n \geq n(H)$, and let $U = 1 + p^k U_1 + p^k U_2$ such that*

$$U_1 \in M_d(\Lambda_{H,n}), \qquad \mathrm{val}_\Lambda(U_1) \geq a - \mathrm{val}_\Lambda(p^k);$$
$$U_2 \in M_d(\widetilde{\Lambda}^H), \qquad \mathrm{val}_\Lambda(U_2) \geq b - \mathrm{val}_\Lambda(p^k).$$

Then for any $\gamma \in \widetilde{\Gamma}_H$ satisfying $n(\gamma) \geq n$, there exists $M \in 1 + p^k M_d(\widetilde{\Lambda}^H)$ such that $\mathrm{val}_\Lambda(M-1) \geq b - c_2 - c_3$ *and $M^{-1}U\gamma(M) = 1 + p^k V_1 + p^k V_2$, where V_1 and V_2 satisfy*

$$V_1 \in M_d(\Lambda_{H,n}), \qquad \mathrm{val}_\Lambda(U_1) \geq a - \mathrm{val}_\Lambda(p^k);$$
$$V_2 \in M_d(\widetilde{\Lambda}^H), \qquad \mathrm{val}_\Lambda(U_2) \geq b - \mathrm{val}_\Lambda(p^k) + \delta.$$

The inclusion of the (generally trivial) case where various estimation parameters (a, b, δ) are infinite is justified by the fact that it allows us to avoid making separate remarks when working with the 1-cocycle $g \mapsto U_g$ at a value U_γ that might equal 1 (i.e., $U_\gamma - 1 = 0$).

Proof. If $p = 0$ in $\widetilde{\Lambda}$, the assertion is self-evident. Therefore, we proceed on the assumption that $p \neq 0$ in $\widetilde{\Lambda}$. The given estimates on the values of $\mathrm{val}_\Lambda(U_i)$ force $\mathrm{val}_\Lambda(U) = 0$. By (TS2) 4., we have that

$$\mathrm{val}_\Lambda(R_{H,n}(U_2)) \geq b - \mathrm{val}_\Lambda(p^k) - c_2 \geq a - \mathrm{val}_\Lambda(p^k).$$

By (TS3), there exists $V \in M_d(\widetilde{\Lambda}^H)$ such that $(\gamma - 1)(V) = (R_{H,n} - 1)(U_2)$ and

$$\mathrm{val}_\Lambda(V) \geq \mathrm{val}_\Lambda(R_{H,n}(U_2) - U_2) - c_3 \geq \min(\mathrm{val}_\Lambda(R_{H,n}(U_2)), \mathrm{val}_\Lambda(U_2)) - c_3,$$

with $\mathrm{val}_\Lambda(U_2) \geq b - \mathrm{val}_\Lambda(p^k) > b - \mathrm{val}_\Lambda(p^k) - c_2$. Consequently, $\mathrm{val}_\Lambda(V) \geq b - \mathrm{val}_\Lambda(p^k) - c_2 - c_3$. Let us now define

$$V_1 := U_1 + R_{H,n}(U_2) \in M_d(\Lambda_{H,n}) \quad \text{and} \quad M := 1 + p^k V \in M_d(\widetilde{\Lambda}^H).$$

In this event, we have that $\mathrm{val}_\Lambda(M-1) = \mathrm{val}_\Lambda(p^k V) \geq b - c_2 - c_3 > 0$ (thus M is invertible and $\mathrm{val}_\Lambda(M) = 0$) and the matrix $V_1 \in M_d(\Lambda_{H,n})$ satisfies

$$\mathrm{val}_\Lambda(V_1) \geq \min\{\mathrm{val}_\Lambda(U_1), \mathrm{val}_\Lambda(R_{H,n}(U_2))\} \geq a - \mathrm{val}_\Lambda(p^k).$$

Since $M^{-1} = 1 - p^k V + p^{2k} V^2 - \ldots$, it is possible to express M^{-1} as $1 - p^k V + p^{2k} V^2 N$, where $N \in M_d(\widetilde{\Lambda}^H)$ satisfying $\mathrm{val}_\Lambda(N) = 0$. We proceed to compute:

$$\begin{aligned} M^{-1}U\gamma(M) - 1 - p^k V_1 &= (1 - p^k V + p^{2k} V^2 N) U (1 + p^k \gamma(V)) - (1 + p^k V_1) \\ &= U + p^k U\gamma(V) - p^k VU - p^{2k}(VU\gamma(V) - V^2 NU\gamma(M)) - 1 - p^k U_1 - p^k R_{H,n}(U_2) \\ &= p^k(U_2 - R_{H,n}(U_2) + U\gamma(V) - VU - p^k(VU\gamma(V) - V^2 NU\gamma(M))), \end{aligned}$$

where the final equality employs the identity $U = 1 + p^k(U_1 + U_2)$. Since $U_2 - R_{H,n}(U_2) = (1 - \gamma)(V)$, we obtain that $M^{-1}U\gamma(M) = 1 + p^k V_1 + p^k V_2$, where

$$V_2 := (U-1)\gamma(V) - V(U-1) - p^k(VU\gamma(V) - V^2 NU\gamma(M)) \in M_d(\widetilde{\Lambda}^H).$$

However, by the definition of U, we have that

$$\mathrm{val}_\Lambda(U-1) \geq \mathrm{val}_\Lambda(p^k) + \min\{\mathrm{val}_\Lambda(U_1), \mathrm{val}_\Lambda(U_2)\},$$

and this is at least a since $b \geq a$. Therefore, we have obtained two lower bounds:

$$\mathrm{val}_\Lambda((U-1)\gamma(V)) \geq b - \mathrm{val}_\Lambda(p^k) - c_2 - c_3 + a \geq b - \mathrm{val}_\Lambda(p^k) + \delta,$$

thus, $\mathrm{val}_\Lambda(V(U-1)) \geq b - \mathrm{val}_\Lambda(p^k) + \delta$. Furthermore,

$$\mathrm{val}_\Lambda(VU\gamma(V) - V^2NU\gamma(M)) \geq 2\mathrm{val}_\Lambda(V),$$

since the other terms vanish. Consequently,

$$\begin{aligned}&\mathrm{val}_\Lambda(p^k(VU\gamma(V) - V^2NU\gamma(M))) \geq \mathrm{val}_\Lambda(p^k) + 2(b - \mathrm{val}_\Lambda(p^k) - c_2 - c_3)\\ &= b - \mathrm{val}_\Lambda(p^k) + b - 2c_2 - 2c_3 \geq b - \mathrm{val}_\Lambda(p^k) + \delta.\end{aligned}$$

Thus, it follows that $\mathrm{val}_\Lambda(V_2) \geq b - \mathrm{val}_\Lambda(p^k) + \delta$, thereby concluding the proof.ˇ □

The application of the lemma will result in a discernible enhancement that is devoid of any of U_1, U_2 and a.

Corollarry 2. *Let $\delta \in (0, +\infty]$ and let $b \in [2c_2 + 2c_3 + \delta, +\infty]$. Let H be an open subgroup of H_0 and pick $n \geq n(H)$. Let $U \in 1 + p^k M_d(\widetilde{\Lambda}^H)$ be such that $\mathrm{val}_\Lambda(U-1) \geq b$, then for any $\gamma \in \widetilde{\Gamma}_H$ satisfying $n(\gamma) \leq n$, there exists an $M \in 1 + p^k M_d(\widetilde{\Lambda}^H)$ such that $\mathrm{val}_\Lambda(M-1) \geq b - c_2 - c_3$ and*

$$M^{-1}U\gamma(M) \in 1 + p^k M_d(\Lambda_{H,n}).$$

It should be noted that, given that $\mathrm{val}_\Lambda(M-1) \geq b - c_2 - c_3 > 0$, it follows that $M \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$. Consequently, the appearance of M^{-1} in the conclusion is justified.

Proof. The case $p = 0$ in $\widetilde{\Lambda}$ is again trivial, so we may and do assume $p \neq 0$ in $\widetilde{\Lambda}$. Let us define $a := b - c_2 \geq c_2 + c_3 + \delta$. Furthermore, let us define $U_{1,1} = 0 \in M_d(\Lambda_{H,n})$ and $U_{2,1} = U \in M_d(\widetilde{\Lambda}^H)$. By repeatedly applying the results of Lemma 6, we obtain matrices M_n ($n \geq 0$) such that $M_n \in 1 + p^k M_d(\widetilde{\Lambda}^H)$ such that

(1) $\mathrm{val}_\Lambda(M_n - 1) \geq b - c_2 - c_3 + n\delta > 0$ and
(2) $(M_0M_1 \ldots M_n)^{-1}U\gamma(M_0 \ldots M_n) = 1 + p^k(U_{1,n} + U_{2,n})$ with $U_{1,n} \in M_d(\Lambda_{H,n})$ and $U_{2,n} \in M_d(\widetilde{\Lambda}^H)$ satisfying $\mathrm{val}_\Lambda(U_{1,n}) \geq a - \mathrm{val}_\Lambda(p^k)$ and $\mathrm{val}_\Lambda(U_{2,n}) \geq b - \mathrm{val}_\Lambda(p^k) + n\delta$ for all $n \geq 0$.

Since $\widetilde{\Lambda}^H$ is complete, (2) implies that the product $M := \prod_{n=0}^{\infty} M_n$ converges in $\mathrm{GL}_d(\widetilde{\Lambda}^H)$. Furthermore, we have that $M \in 1+p^k M_d(\widetilde{\Lambda}^H)$ and $\mathrm{val}_\Lambda(M-1) \geq b-c_2-c_3 > 0$, therefore, $M \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$. From (2), it follows that $M^{-1}U\gamma(M) \in 1 + p^k M_d(\Lambda_{H,n})$ because $\Lambda_{H,n}$ is closed in $\widetilde{\Lambda}^H$ with respect to the topology defined by val_Λ. □

We now establish another result concerning the cohomological triviality of certain 1-cocycles. This incorporates a p-adic estimate on the trivialising 0-cochain.

Proposition 3. *Let $U : G_0 \to \mathrm{GL}_d(\widetilde{\Lambda})$ be a continuous 1-cocycle, with $\sigma \mapsto U_\sigma$. Let us assume that that for some open normal subgroup G of G_0, we have $\mathrm{val}_\Lambda(U_\sigma - 1) > c_1 + 2c_2 + 2c_3$ and $U_\sigma \in 1 + p^k M_d(\widetilde{\Lambda})$ for all $\sigma \in G$ with some $k \in \mathbb{N}$. In this case,* *there exists a matrix* $M \in 1 + p^k M_d(\widetilde{\Lambda})$ *such that* $\mathrm{val}_\Lambda(M-1) > c_2 + c_3$ *(consequently,* $M \in \mathrm{GL}_d(\widetilde{\Lambda}^H)$*) and that the* 1*-cocycle* $\sigma \mapsto V_\sigma = M^{-1}U_\sigma \sigma(M)$ *is trivial on* $H := G \cap H_0$ *in* $\mathrm{GL}_d(\Lambda_{H,n(G)})$.

Proof. Given that $\mathrm{val}_\Lambda(U_\sigma - 1) > 0$, it follows that $\mathrm{val}_\Lambda(U_\sigma) = 0$. By Corollary 1, there exists $M_1 \in 1 + p^k M_d(\widetilde{\Lambda})$ such that $\mathrm{val}_\Lambda(M_1 - 1) > 2c_2 + 2c_3 > 0$ (which implies that $\mathrm{val}_\Lambda(M_1) = 0$) and that the 1-cocycle $\tau \mapsto U'_\tau = M_1^{-1}U_\tau \tau(M_1)$ is trivial on H. Furthermore it is evident that $\mathrm{val}_\Lambda(U'_\sigma) = 0$ for all $\sigma \in G_0$. Since the restriction to H is trivial, the 1-cocycle U' is the inflation of a 1-cocyle $\widetilde{\Gamma}_H = G_0/H \to \mathrm{GL}_d(\widetilde{\Lambda}^H)$, we will continue to denote by U'.

We now select a $\gamma \in G/H \subset \widetilde{\Gamma}_H$ such that $n(\gamma) = n(G)$. It follows that $U'_\gamma - 1 \in p^k M_d(\widetilde{\Lambda}^H)$ and that

$$\begin{aligned} \mathrm{val}_\Lambda(U'_\gamma - 1) &= \mathrm{val}_\Lambda(M_1^{-1}(U_\gamma - 1)\gamma(M_1) + M_1^{-1}\gamma(M_1 - 1) + M_1^{-1} - 1) \\ &\geq \min\{\mathrm{val}_\Lambda(U_\gamma - 1), \mathrm{val}_\Lambda(M_1 - 1)\} > 2c_2 + 2c_3. \end{aligned}$$

It is therefore possible to apply Corollary 2 with $n = n(\gamma) = n(G), U = U'_\gamma, b = \mathrm{val}_\Lambda(U'_\gamma - 1)$ (which is infinite, if $U'_\gamma = 1$) and $\delta = b_2c_2 - 2c_3 > 0$. This yields a matrix $M_2 \in 1 + p^k M_d(\widetilde{\Lambda}^H)$ such that $\mathrm{val}_\Lambda(M_2 - 1) > c_2 + c_3 > 0$ (and thus $\mathrm{val}_\Lambda(M_2) = 0$), and that $M_2^{-1}U'_\gamma\gamma(M_2) \in \mathrm{GL}_d(\Lambda_{H,n(G)})$.

Define $M := M_1M_2 \in 1 + p^k M_d(\widetilde{\Lambda}^H)$. This ensures that the following conditions are met:

(1) $\mathrm{val}_\Lambda(M - 1) \geq \min\{\mathrm{val}_\Lambda(M_1 - 1), \mathrm{val}_\Lambda(M_2 - 1)\} > c_2 + c_3$,
(2) the 1-cocycle $\tau \mapsto V_\tau = M^{-1}U_\tau\tau(M)$ is trivial on H (so it is the inflation of a 1-cocycle $\widetilde{\Gamma}_H \to \mathrm{GL}_d(\widetilde{\Lambda}^H)), V_\gamma \in \mathrm{GL}_d(\Lambda_{H,n(G)})$, and
(3)
$$\begin{aligned} \mathrm{val}_\Lambda(V_\gamma - 1) &= \mathrm{val}_\Lambda((M_2^{-1} - 1)U'_\gamma\gamma(M_2) + (U'_\gamma - 1)\gamma(M_2) + \gamma(M_2 - 1)) \\ &\geq \min\{\mathrm{val}_\Lambda(M_2 - 1), \mathrm{val}_\Lambda(U'_\gamma - 1)\} > c_2 + c_3 > c_3 > 0 \end{aligned}$$

For any $\tau \in \widetilde{\Gamma}_H$, since G/H lies in C_H, it follows that $\tau\gamma = \gamma\tau$. Consequently, $V_\tau\tau(V_\gamma) = V_{\tau\gamma} = V_{\gamma\tau} = V_\gamma\gamma(V_\tau)$, which implies $\gamma(V_\tau) = V_\gamma^{-1}V_\tau\tau(V_\tau)$. We are now in a position to apply Lemma 6 with $n = n(\gamma) = n(G), V_\tau, U_1 = V_\gamma^{-1}$ and $U_2 = \tau(V_\gamma)$ to conclude that $V_\tau \in \mathrm{GL}_d(\Lambda_{H,n(G)})$ (nota bene, $U_2 \in \mathrm{GL}_d(\Lambda_{H,n(G)})$ as a consequence of (TS2) 3.). Since the choice of τ was arbitrary from $\widetilde{\Gamma}_H$, this completes the proof. ☐

2.3. **Application to S-representations.**

Let S be a Banach algebra and let $\widetilde{\Lambda}$ satisfy the Tate-Sen conditions. A $\mathcal{O}_S$-*representation* of G_0 is a free $\mathcal{O}_S$-module of finite rank with a continuous $\mathcal{O}_S$-linear action of G_0. The term *dimension* is employed to denote the rank of the underlying $\mathcal{O}_S$-module.

Let $\widetilde{\Lambda}^+$ (respectively $\Lambda_{H,n}^+$, if $H \subset H_0$ is open and $n \in \mathbb{N}$) be the ring of integers in $\widetilde{\Lambda}$ (respectively $\Lambda_{H,n}$) for the valuation val_Λ (i.e. the set of x satisfying $\mathrm{val}_\Lambda(x) \geq 0$).

With regard to $\mathcal{O}_S$-representations, the following theorem can be stated in order to prove the existence of descent, with a strong form of uniqueness.

Theorem 2. *Let T be a d-dimensional $\mathcal{O}_S$-representation of G_0, $V = S \otimes_{\mathcal{O}_S} T$ and k be an integer such that $\mathrm{val}_\Lambda(p^k) > c_1 + c_2 + c_3$. Let G be a distinguished subgroup of G_0 that acts trivially on T/p^kT, let $H = G \cap H_0$ and let $n \geq n(G)$. Then $\widetilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$ contains a unique free $\Lambda_{H,n}^+$-submodule $D_{H,n}^+(T)$ of rank d satisfying the following properties*

1. *$D_{H,n}^+(T)$ is fixed under H and stable under G_0;*
2. *the natural map $\widetilde{\Lambda}^+ \otimes_{\Lambda_{H,n}^+} D_{H,n}^+(T) \to \widetilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$ is an isomorphism;*
3. *$D_{H,n}^+(T)$ has a basis over $\Lambda_{H,n}^+$ which is c_3-fixed by G/H (i.e if $\gamma \in G/H$, then the matrix W of γ in this basis verifies $\mathrm{val}_\Lambda(W - 1) > c_3$).*

It should be noted that the uniqueness of the proposition ensures that $D_{H,n}^+(T)$ is an actual subset of $\widetilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$, rather than merely an abstract $\Lambda_{H,n}^+$-linear isomorphism.

Proof. Let $v_1, \ldots, v_d$ be a basis of T over $\mathcal{O}_S$ and let $U_\sigma = (u_{i,j}^\sigma)$ be the matrix describing the action of an element $\sigma \in G_0$, that is, $\sigma(v_j) = \sum_{i=1}^d u_{i,j}^\sigma v_i$. Then $\sigma \mapsto U_\sigma$ is a continuous morphism from G_0 to $\mathrm{GL}_d(\mathcal{O}_S)$, which can also be regarded as a continuous 1-cocycle on G_0 with values in $\mathrm{GL}_d(\mathcal{O}_S) \subset \mathrm{GL}_d(\widetilde{\Lambda}^+)$. by virtue of our hypothesis, for any $\sigma \in G$, the corresponding U_σ lies in $1 + p^k M_d(\mathcal{O}_S)$. Consequently, we may apply Proposition 3 to obtain a matrix $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ that satisfies $\mathrm{val}_\Lambda(M - 1) > c_2 + c_3$ (and thus $M \in \mathrm{GL}_d(\widetilde{\Lambda}^+)$) such that the cocycle $\sigma \mapsto V_\sigma = M^{-1} U_\sigma \sigma(M)$ is trivial on H and has values in $\mathrm{GL}_d(\Lambda_{H,n(G)}) \cap \mathrm{GL}_d(\widetilde{\Lambda}^+) = \mathrm{GL}_d(\Lambda_{H,n(G)}^+)$. Let $M = (m_{i,j})$ and $e_k = \sum_{j=1}^d m_{j,k} v_j$. Then we have that for $\sigma \in H$

$$\sigma(e_k) = \sum_{j=1}^d \sigma(m_{i,j})\sigma(v_j) = \sum_{i=1}^d \left(\sum_{j=1}^d u_{i,j}^\sigma \sigma(m_{j,k}) \right) v_i = e_k,$$

that is, the basis $e_1, \ldots, e_d$ of $\widetilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$ over $\widetilde{\Lambda}^+$ is fixed under the action H.

Furthermore, if $\gamma \in G/H$, then the matrix W describing the action of γ in the basis $e_1, \ldots, e_d$ is of the form $M^{-1} U_\sigma \sigma(M)$, where $\sigma \in G$ is a lift of γ. Consequently, we have that $\mathrm{val}_\Lambda(W - 1) > c_2 + c_3 > c_3$. It can therefore be concluded that the $\Lambda_{H,n}^+$-submodule generated by $e_1, \ldots, e_d$ verifies the desired properties and, thus, the existence of such a submodule is established.

In order to demonsrate the uniqueness, let us fix $\gamma \in C_H$ satisfying $n(\gamma) = \gamma$. Let us now consider two bases, of $\widetilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$ over $\widetilde{\Lambda}^+$, namely $e_1, \dots, e_d$ and $e'_1, \dots, e'_d$, which are fixed under the action of H. Let W and W' be the matrices in $\mathrm{GL}_d(\Lambda^+_{H,n})$ (where $n \geq n(G)$), which describe the action of γ in the two bases, respectively. Then $\mathrm{val}_\Lambda(W-1) > c_3$ and $\mathrm{val}_\Lambda(W'-1) > c_3$ hold. Let $B \in GL_d(\Lambda^+_{H,n})$ be the change-of-basis matrix that converts the e_i-coordinates to the e'_i-coordinates. Thus, B is invariant under the action of H and $W' = B^{-1}W_\gamma\gamma(B)$. According to Proposition 3, this implies that the coefficients of B are in $\Lambda_{H,n}$ (and therefore in $\Lambda^+_{H,n}$), which implies that the respective $\Lambda_{H,n}$-spans of $e_1, \dots, e_d$ and $e'_1, \dots, e'_d$ coincide. □

Remark 5. For the sake of simplicity, let us retain the assumptions of Theorem 2. If we define $D_{H,n}(T) = \Lambda_{H,n} \otimes_{\Lambda^+_{H,n}} D^+_{H,n}(T)$, then $D_{H,n}(T)$ is a free $\Lambda_{H,n}$-module of rank d and is the unique $\Lambda_{H,n}$-submodule of $\widetilde{\Lambda}$ satisfying the properties

(1) $D_{H,n}(T)$ is fixed under H and stable under G_0;
(2) the natural map $\widetilde{\Lambda} \otimes_{\Lambda_{H,n}} D_{H,n}(T) \to \widetilde{\Lambda} \otimes_{\mathcal{O}_S} T$ is an isomorphism;
(3) $D_{H,n}(T)$ has a basis over $\Lambda_{H,n}$ which is c_3-fixed by G/H.

3. Ring theoretic constructions

3.1. Brief overview of the theory of Witt vectors.

As will become evident in due course, we will frequently be making use of Witt vectors. Therefore, it seems prudent to provide a brief summary of the basic outline of the theory, but we assume here that the reader has some previous experience with the ring of Witt vectors. For a comprehensive and self-contained development, we direct the reader to [Ser79], Ch. II, §4-§6.

Let A be a (commutative) ring of characteristic p, where p is a prime number. We call A *perfect* if the Frobenius map $\varphi : a \mapsto a^p$ is an isomorphism. The objective of the theory of Witt vectors is to construct a ring $W(A)$, which is a so-called *strict p-ring.*

Definition 7. *A p-ring is a ring B that is Hausdorff and complete for the topology defined by a specified decreasing collection of ideals, namely, $\mathfrak{b}_1 \supset \mathfrak{b}_2 \supset \dots$, such that $\mathfrak{b}_n\mathfrak{b}_m \subseteq \mathfrak{b}_{n+m}$ for all $n, m \geq 1$ and $B/\mathfrak{b}_1$ is a perfect $\mathbb{F}_p$-algebra (i.e. $p \in \mathfrak{m}_1$).*

A ring B is said to be a strict p-ring if it satisfies the additional conditions that $\mathfrak{b}_i = p^iB$ for all $i \geq 1$ (that is, B is p-adically Hausdorff and complete, and B/pB is a perfect $\mathbb{F}_p$-algebra) and p is not a zero-divisor.

The principal result is that if A is a perfect ring of characteristic p, then there exists a unique (up to a unique isomorphism) strict p-ring $W(A)$ with residue ring A. This ring is called the *ring of Witt vectors* over A. Due to the uniqueness of this functor, every map $f : A \to B$ induces a map $\widetilde{f} : W(A) \to W(B)$. In particular, the Frobenius map $\varphi : x \to x^p$ lifts to an automorphism of $W(R)$, given by $(a_i) \mapsto (0, a_0^p, a_1^p, \ldots)$, so the subset $p^n W(A) \subset W(A)$ consists of Witt vectors (a_i), where $a_i = 0$, when $0 \le i \le n-1$. Since A is perfect, it follows by projection to the first n Witt components that $W(A)/p^n W(A) \cong W_n(A)$. Hence, $W(A) \cong \varprojlim_n W_n(A)$.

Example 1. If $A = \mathbb{F}_p$, then $W(A) = \mathbb{Z}_p$. In general, if k is a perfect finite field of characteristic p, then $W(k)$ is the ring of integers of the unique unramified extension of $\mathbb{Q}_p$ whose residue field is k. (See the proposition below.)

As is commonly known, for every p-ring B, there is a unique set-theoretic section $[\cdot]_B : B/\mathfrak{b}_1 \to B$ of the natural reduction map, where $[1]_B = 1$. This is defined by $[x]_B = \lim_{n\to\infty} \widetilde{x^{p^{-n}}}^{p^n} \in B$, where $\widetilde{b} \in B$ is any choice of lifting of $b \in B/\mathfrak{b}_1$. This is referred to as the *Teichmüller map*. Consequently, every element of a strict p-ring B endowed with the p-adic topology (relative to which it is Hausdorff and complete) can be expressed as $x = \sum_{n=0}^{\infty} p^n [x_n]$ with $x_n \in B/pB$.

Now, given two elements x, y of $W(A)$, one can also write from the theory of Witt vectors

$$x + y = \sum_{n=0}^{\infty} p^n [S_n(x_i, y_i)] \quad \text{and} \quad xy = \sum_{n=0}^{\infty} p^n [P_n(x_i, y_i)],$$

where S_n and $P_n \in \mathbb{Z}[X_i^{p^{-n}}, Y_i^{p^{-n}}]_{0 \le i \le n}$ are universal homogeneous polynomials of degree 1 (if $\deg X_i = \deg Y_i = 1$). As an illustration, we have $S_0(X_0, Y_0) = X_0 + Y_0$ and $S_1(X_0, X_1, Y_0, Y_1) = X_1 + Y_1 + \frac{1}{p}((X_0^{\frac{1}{p}} + Y_0^{\frac{1}{p}})^p) - X_0 - Y_0$. It is therefore possible to define $W(A)$ as the set $\prod_{n=0}^{\infty} A$ with the addition and multiplication defined by S_n and P_n.

Proposition 4. *If A is a perfect $\mathbb{F}_p$-algebra and B is a p-ring, then the natural reduction map* $\mathrm{Hom}(W(A), B) \to \mathrm{Hom}(A, B/\mathfrak{m}_1)$ *(which makes sense since $A = W(A)/pW(A)$ and $p \in \mathfrak{b}_1$) is bijective. In general, for a strict p-ring $\mathcal{B}$, the natural map*

$$\mathrm{Hom}(\mathcal{B}, B) \to \mathrm{Hom}(\mathcal{B}/p\mathcal{B}, B/\mathfrak{b}_1)$$

is bijective for every p-ring B.

In particular, since $\mathcal{B}$ and $W(\mathcal{B}/p\mathcal{B})$ satisfy the same universal property in the category of p-rings for any strict p-ring $\mathcal{B}$, strict p-ring are precisely of the form $W(A)$ for perfect $\mathbb{F}_p$-algebras A.

Proof. If $h \in \mathrm{Hom}(\mathcal{B}, B)$, then the associated reduction map $\overline{h} : \mathcal{B}/p\mathcal{B} \to B/\mathfrak{b}_1$ uniquely determines h due to the functorial property of $[\cdot]_{\mathcal{B}}$. Conversely, if $\overline{h} \in \mathrm{Hom}(\mathcal{B}/p\mathcal{B}, B/\mathfrak{b}_1)$, then it is necessary to ascertain whether the $\mathcal{B} \to B$ map defined by the following way is a ring homomorphism:

$$\beta = \sum [\beta_n]_{\mathcal{B}} p^n \mapsto \sum [\overline{h}(\beta_n)]_B p^n$$

It respects multiplicative identity elements. The additivity and multiplicativity of the map can be demonstrated by the universal polynomials $S_n, P_n \in \mathbb{Z}[X_i^{p^{-n}}, Y_i^{p^{-n}}]_{0\leq i\leq n}$, whose existence can be proven in a manner analogous to the case of Witt vectors. These polynomials determine the ring structure for any p-ring. $\square$

As a consequence of this proposition, if K is a p-adic field with a perfect residue field k, then $\mathcal{O}_K$ (the ring of integers of K) endowed with the filtration by powers of its maximal ideal $\mathfrak{m}_K$, namely $\{\mathfrak{m}_K^i\}_{i\geq 1}$, is a p-ring. It follows that there exists a unique and injective $W(k) \to \mathcal{O}_K$ map as the lifting of $W(k)/pW(k) \xrightarrow{\sim} \mathcal{O}_k/\mathfrak{m}_K$. Consequently, $\mathcal{O}_K/p\mathcal{O}_K$ is a vector space over k, with basis $\{1, \pi, \ldots, \pi^{e_K-1}\}$, where π is a uniformiser and $e_K = v_K(p)$ is the ramification index of $\mathfrak{m}_K$. Therefore, by successive approximation and the p-adic completeness and separatedness of $\mathcal{O}_K$, it follows that $\mathcal{O}_K$ is a free $W(k)$-module of rank e_K. Similarly, K is a totally ramified (because the residue fields coincide) finite extension of $K_0 = W(k)[\frac{1}{p}]$ of degree e. We call K_0 the *maximal unramified* subfield of K.

Remark 6. Witt vectors can also be approached from a category-theoretic perspective via differential rings. This is because the Witt vector functor is in fact a right-adjoint functor of the forgetful functor from the category of λ-rings to the category of (commutative) rings. For further insight into this intriguing approach, one may consult the work of Borger ([Bor]).

It has been shown that if k is perfect, then $W(k)$ is the unique complete discrete valuation ring which is absolutely unramified, with residue field k. However, if k is not perfect, the situation becomes considerably more complex. It is sufficient to mention here that one of Matsamura's theorem, namely

Theorem 3 ([Mat87], pp 223-225)**.** *Let $(A, \mathfrak{m}_A, k_A)$ be a complete local ring, and $(B, \mathfrak{m}_B, k_B)$ an absolutely unramified discrete valuation ring of characteristic 0 (i.e., $\mathfrak{m}_B = pB$). Then for every homomorphism $f : k_B \to k_A$, there exists a local (i.e., $g(\mathfrak{m}_B) \subset \mathfrak{m}_A$) homomorphism $g : B \to A$ which induces f on the ground field.*

implies the existence of the following ring.

Definition 8. *Let k be a field of characteristic p. The Cohen ring $\mathcal{C}(k)$ is the unique (up to non-unique [!] isomorphism) absolutely unramified discrete valuation ring of characteristic 0, whose residue field is k.*

3.2. **The ring R.**

Notation 1. From this point onwards, K will be used to denote a p-adic field (for a fixed p prime), $\mathcal{O}_K$ its ring of integers, $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, and k its residue field, which is perfect with characteristic p. We fix an algebraic closure $\overline{K}$. The Galois group of any extension L/K in $\overline{K}$ is denoted by $\mathrm{Gal}(\overline{K}/L) = G_L$, and let $\mathbb{C}_K$ be $\widehat{\overline{K}}$ endowed with its unique absolute value extending the given one on K. For the remainder of this discussion, we will use χ to denote the cyclotomic character on G_{K_0}. $W = W(k)$ is the ring of Witt vectors and $K_0 = \operatorname{Fr} W = W[1/p]$ is its quotient field. From the theory of Witt vectors $\operatorname{rk}_W(\mathcal{O}_K) = [K : K_0] = e_K = v_K(p)$. Furthermore, if π is a generator of $\mathfrak{m}_K$, then $1, \pi, \pi^2, \ldots, \pi^{e_K-1}$ is a basis of $\mathcal{O}_K$ over W and also K over K_0. From now on, v is the normalised (i.e. $v(p) = 1$) valuation of $\mathbb{C}_K$ (or any subfield). Consequently, $v(\pi) = \frac{1}{e_K}$.

Let A be a perfect ring of characteristic p.

Definition 9.

$$R(A) := \varprojlim_{\varphi} A,$$

That is, an element of $R(A)$ is a $(x_n)_{n\in\mathbb{N}}$ sequence in A satisfying $x_{n+1}^p = x_n$.

Remark 7. From $x^p = 0$ follows that $x_n^p = x_{n-1} = 0$ for any $n \geq 1$, thus $R(A)$ is perfect.

The following well-known fact will be referenced frequently in the following sections; therefore, it is beneficial to include it here for the sake of future reference.

Lemma 7. *For $a, b \in \mathcal{O}_K$, if $a \equiv b \mod \mathfrak{m}_K$, then $a^{p^n} \equiv b^{p^n} \mod \mathfrak{m}_K^{n+1}$, where $p = \operatorname{char} k$.*

Now let A be a Hausdorff and p-adically complete ring, i.e. $A \cong \varprojlim A/p^n A$.

Proposition 5. *There is a bijection between $R(A/pA)$ and the set*

$$S = \{(x^{(n)})_{n\in\mathbb{N}} \mid x^{(n)} \in A,\ (x^{(n+1)})^p = x^{(n)}\}.$$

Proof. For an $x = (x_n)_{n\in\mathbb{N}} \in R(A/pA)$ choose an arbitrary lifting $\hat{x}_n \in A$ for every x_n. Now because of $\hat{x}_{n+1}^p \equiv \hat{x}_n \mod pA$, for $n, m \in \mathbb{N}$

$$\hat{x}_{n+1}^{p^{m+1}} \equiv \hat{x}_n^{p^m} \mod p^{m+1}A$$

is also satisfied (here we use Lemma 7). Hence $\hat{x}_n^{p^n}$ is Cauchy, that is, the limit

$$x^{(n)} := \lim_{n\to\infty} \hat{x}_{n+m}^{p^{n+m}}$$

exists and independent of the choice of the liftings. Then the map $x \mapsto (x^{(n)})_{n\in\mathbb{N}}$ defines a $R(A/pA) \to S$ bijection with the inverse map induced by the modulo p reduction $A \to A/pA$. □

As a consequence of the proposition we can identify $R(A/pA)$ with the set S and operations

$$(xy)^{(n)} = (x^{(n)}y^{(n)}), \quad (x+y)^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}.$$

Definition 10. *$R := R(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}) = R(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ and $v_R(x) := v_p(x^{(0)})$. Sometimes we call R the perfectisation of $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$.*

Notice that R is a (non-discrete) valuation ring as $v_R(R) = \mathbb{Q}_{\geq 0} \cup \{\infty\}$. Moreover, $v_R(x) = \infty \Leftrightarrow x = 0$ and $v_R(xy) = v_R(x) + v_R(y)$. It is also straightforward to demonstrate that $v_R(x+y) \geq \min\{v_R(x), v_R(y)\}$, and since

$$v_R(x) \geq p^n \Leftrightarrow v_p(x^{(n)}) \geq 1 \Leftrightarrow x_n = 0,$$

therefore

$$\{x \in R \mid v_R(x) \geq p^n\} = \mathrm{Ker}(\theta_n : R \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}, x \mapsto x_n).$$

Consequently, the topology induced by v_R coincides with the inverse limit topology where the factors are given the discrete topology, hence R is complete. Consider the quotient field $\mathrm{Fr}\, R$ (R is a domain, because it has a valuation)

$$\mathrm{Fr}\, R = \{x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in \mathbb{C}_K, (x^{(n+1)})^p = x^{(n)}\},$$

with the extension of v_R: $v(x) = v_p(x^{(0)})$.

$\mathrm{Fr}\, R$ is a complete nonarchimedean perfect field of characteristic $p > 0$, with $\mathcal{O}_{\mathrm{Fr}\, R} = R$ whose maximal ideal is $\mathfrak{m}_R = \{x \in \mathrm{Fr}\, R; \mid v(x) > 0\}$ and its residue field is isomorphic to $\overline{k}$ (residue field of $\overline{K}$). This is evident from the fact that the map

$$R \stackrel{\theta_0}{\to} \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \cong \overline{k}$$

is surjective and has $\mathfrak{m}_R$ as kernel. $\overline{k}$ is perfect and R is complete, so one can define the section $s : \overline{k} \to R$ given by $a \mapsto ([a^{p^{-n}}])_{n \in \mathbb{N}}$, where $[a^{p^{-n}}] = (a^{p^{-n}}, 0, 0, \ldots)$ is the Teichmüller representative of $a^{p^{-n}}$. What is more, it will also be a homomorphism.

Proposition 6. *$\mathrm{Fr}\, R$ is algebraically closed.*

Proof. It is sufficient to demonstrate that a monic polynomial $f(x) \in R[x]$ of degree d possesses a root in R, given that R is a valuation ring. Let us define the map $\theta_n : R \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ by $(x_i)_{i \in \mathbb{N}} \mapsto x_n$. We may then define $f_m \in (\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})[x]$ as $\theta_m(f)$, where we apply θ_m to coefficients. According to the Hensel lemma, for each f_m there exists a lifting $\widetilde{f}_m$ with coefficients in $\mathcal{O}_{\mathbb{C}_K}$. Since $\mathbb{C}_K$ is known to be algebraically closed, there exists a set of roots of $\widetilde{f}_m$ (with multiplicity) in $\mathcal{O}_{\mathbb{C}_K}$: $\{\alpha_{1,m}, \ldots, \alpha_{d,m}\}$. The $\mod p\mathcal{O}_{\mathbb{C}_K}$ reduced elements of the set, namely $\{\overline{\alpha}_{1,m}, \ldots, \overline{\alpha}_{d,m}\} \subset \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$, are therefore roots of f_m. If we were to arrange a sequence of these roots of f_m in a p-power compatible manner, we would obtain a root of f that is desired. It should be noted that $\overline{\alpha}^p_{i,m+1}$ is a root of f_{m+1}, however, it is unfortunate that $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ is not a domain, which means that f_m always has infinitely many roots.

Note that, due to the fact that $f_m(\overline{\alpha}^p_{i,m+1}) = f_{m+1}(\overline{\alpha}_{i,m+1})^p = 0$ in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$, we have that

$$\prod_{j=1}^{d}(\alpha^p_{i,m+1} - \alpha_{j,m}) \in p\mathcal{O}_{\mathbb{C}_K}.$$

(This is because, in fact, $\widetilde{f}_m(x)$ can be expressed as $\prod_j(x-\alpha_{j,m})$.) Given that, there are d terms in the product, one of them, say $\alpha_{i,m+1}^p - \alpha_{j(i),m}$, must lie in $p^{\frac{1}{d}}\mathcal{O}_{\mathbb{C}_K}$. Consequently, the application of the Lemma 7 demonstrates that $\alpha_{i,m+1}^{p^d} - \alpha_{j(i),m}^{p^{d-1}}$ lies in $p\mathcal{O}_{\mathbb{C}_K}$. In other words, for each integer $1 \le i \le d$, there exist an integer $1 \le j(i) \le d$ such that $\overline{\alpha}_{i,m+1}^{p^d} = \overline{\alpha}_{j(i),m}^{p^{d-1}}$. Therefore, the finite set

$$\{\overline{\alpha}_{1,m+1}^{p^d}, \ldots, \overline{\alpha}_{d,m+1}^{p^d}\}$$

forms an inverse system under the p-power maps. These are the roots of f_{m+1-d}, consequently, if we set $r_{i,m} = \overline{\alpha}_{i,m+1-d}^{p^d} \in \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ for all $m \geq 0$ and $1 \le i \le d$, then the sets $\{r_{1,m}, \ldots, r_{d,m}\}$ constitute an inverse system under the p-power mapping. It is thus possible to select a sequence $r = (r_{i(m),m})$ for all $m \geq 0$ such that $r_{i(m+1),m+1}^p = r_{i(m),m}$ for all m. Consequently, $r \in R$ and $\theta_m(f(r)) = f_m(r_{i(m),m}) = 0$ for all m, which implies that $f(r) = 0$ in R. □

Example 2. An important element of R is

$$\varepsilon = (\varepsilon^{(n)})_{n\in\mathbb{N}} = (1, \zeta_p, \zeta_{p^2}, \ldots),$$

where ζ_p is a p-th root of unity. $v_R(\varepsilon - 1) = v((\varepsilon-1)^{(0)}) = \lim_{n\to\infty}(\varepsilon^{(0)} + (-1)^{(n)})^{p^n}$. Hence if $p > 2$, then

$$\lim_{n\to\infty} p^n v(\zeta_p^n - 1) = \lim_{n\to\infty} \frac{p^n}{p^{n-1}(p-1)} = \frac{p}{p-1},$$

where the second equation follows from the facts that Galois conjugates have the same absolute value and that the product of the Galois conjugates of $\zeta_{p^n} - 1$ is the value obtained by substituting 1 into the p^n-th cyclotomic polynomial, which is p. (The p^n-th cyclotomic polynomial is also irreducible over $\mathbb{Q}_p$ due to the Eisenstein criterion.) If $p = 2$, then

$$\lim_{n\to\infty} 2^n v(\zeta_{2^n} + 1) = \lim_{n\to\infty} 2^n v((\zeta_{2^n} - 1) + 2) = \lim_{n\to\infty} 2^n \min\{v(\zeta_{2^n} - 1), 1\} = 2.$$

In conclusion, $v_R(\varepsilon - 1) = \frac{p}{p-1}$. This implies that $\varepsilon - 1$ lies in the maximal ideal $\mathfrak{m}_R$ of R. This can also be seen by the fact that $\theta_0(\varepsilon) = 1$, so the image of ε is 1 in the residue field $\overline{k}$ of R. Consequently, $\varepsilon - 1$ lies in $\mathfrak{m}_R$.

Notation 2. From now on, let ε be as in Example 2. Furthermore, let us denote $\pi := \varepsilon - 1$.

3.3. **The Galois action on R.**

Let R_L be defined as $R(\mathcal{O}_L/p\mathcal{O}_L) = R(\widehat{\mathcal{O}}_L/p\widehat{\mathcal{O}}_L)$ for any intermediate field $\overline{K} \geqslant L \geqslant K_0$. It is worth noting that this is functorial with respect to inclusions $L \subset L'$ among extensions of $K_0 = W(k)[\frac{1}{p}]$, via the natural injection:

$$R_L = R(\mathcal{O}_L/p\mathcal{O}_L) \hookrightarrow R_{L'} = R(\mathcal{O}_{L'}/p\mathcal{O}_{L'}).$$

As noted, R is the valuation ring of $\operatorname{Fr} R$ with respect to $v_R((x^{(n)})_{n\in\mathbb{N}}) = v(x^{(0)})$. In a similar manner,

Lemma 8. *Let $\overline{K} \geqslant L \geqslant K_0$ be an intermediate field, and ℓ its residue field inside $\overline{k}$. Then R_L is the valuation ring in $\operatorname{Fr} R_L$ with respect to the restriction of the valuation v_R on $\operatorname{Fr} R$, and its residue field is ℓ. In particular, R_L is integrally closed in $\operatorname{Fr} R_L$.*

Notice that v_R may have trivial restriction to R_L, e.g., when L is a finite extension of K_0.

Proof. We have already seen the second part of the theorem, namely that

$$R_L \xrightarrow{\theta_0} \mathcal{O}_L/p\mathcal{O}_L \to \ell$$

gives us that ℓ is the residue field.

In order to demonstrate the first part, it is sufficient to show that if $x, y \in R_L - \{0\}$ and x divides y in R, then x also divides y in R_L. To demonstrate this, we may consider p-power compatible sequences in $\widehat{\mathcal{O}_L}$, as this respects divisibility due to the multiplication defined after Proposition 5. It is evident that if a divides b in $\mathcal{O}_{\mathbb{C}_K}$, then the analogous statement holds in $\widehat{\mathcal{O}}_L$. $\square$

As previously demonstrated, R and $\operatorname{Fr} R$ are perfect rings, Hausdorff and complete with respect to v_R. Consequently, they are also Hausdorff and complete with respect to any $\varpi \in \mathfrak{m} - \{0\}$. Since $\widehat{\mathcal{O}_L}$ is closed in $\mathcal{O}_{\mathbb{C}_K}$, it follows that R_L is also closed in R, hence R_L is ϖ-adically separated and complete for any $\varpi \in \mathfrak{m}_{R_L} - \{0\}$, and similarily $\operatorname{Fr} R_L$ is complete with respect to v_R.

The group $G_{K_0} = \operatorname{Gal}(\overline{K}/K_0)$ acts naturally on R and $\operatorname{Fr} R$.

Proposition 7. *Let $\overline{K} \geqslant L \geqslant K_0$ be an intermediate field. Then*

$$R^{G_L} = R_L, \quad (\operatorname{Fr} R)^{G_L} = \operatorname{Fr}(R_L).$$

The residue field of R^{G_L} is $k_L = \overline{k}^{G_L}$, the residue field of L.

Proof. This is a straightforward consequence of Theorem 1, which states that $\mathbb{C}_K^{G_L} = \widehat{L}$ and $(\mathcal{O}_{\mathbb{C}_K})^{G_L} = \mathcal{O}_{\widehat{L}} = \varprojlim_n \mathcal{O}_L/p\mathcal{O}_L$. Furthermore, $\overline{k} \hookrightarrow R \twoheadrightarrow \overline{k}$ induces the map $k_L \hookrightarrow R \twoheadrightarrow k_L$, with the identity map as composition. Consequently, the residue field of R^H is k_L. The proof is analogous for the case of $\operatorname{Fr} R$. $\square$

Remark 8. In the event that $v(L^\times)$ is discrete (which occurs when L is a finite extension of K_0), then $R^{G_L} = k_L$. Indeed, it remains to be shown that if $x = (x^{(n)})_{n\in\mathbb{N}} \in R^{G_L}$ with $v(x) = v(x^{(0)}) > 0$, then $x = 0$. However, since $v(\widehat{L}^\times) = v(L^\times)$ is discrete, it follows from $v(x^n) = p^{-n}v(x^{(0)})$ that $v(x) = v(x^{(n)}) = \infty$.

The action of G_{K_0} on $\operatorname{Fr} R$ leaves the valuation v_R invariant, and thus the action is continuous with respect to the v_R-adic topology. It should be noted, however, that the action of G_{K_0} on R is not continuous with respect to the discrete topology. This is due to the fact that it is possible to provide an explicit element, namely ε, which has a non-open stabiliser. Indeed, as observed, $\varepsilon \in 1+\mathfrak{m}_R$ and clearly $g(\varepsilon) = \varepsilon^{\chi(g)}$ holds for any $g \in G_{K_0}$, where $\chi : G_{K_0} \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character. As $\operatorname{char} R = p$, $1+\mathfrak{m}_R$ with its natural $\mathbb{Z}_p$-module structure is torsion-free. Consequently, for $x, y \in \mathbb{Z}_p$ we have that $\varepsilon^x = \varepsilon^y$ if and only if $x = y$. Hence $g(\varepsilon) = \varepsilon$ if and only if $\chi(g) = 1$. Nevertheless, χ does not possess an open kernel, and thus the stabiliser of ε is not open.

Since R is complete, we obtain a unique local k-algebra map $k[[\pi]] \to R$ satisfying $\pi \mapsto \varepsilon - 1$, which depends on the choice of ε. However, its image does not:

Lemma 9. *The image of $k[[\pi]]$ in R is independent of the choice of ε.*

Proof. Consider a second choice of ε, denoted by ε'. In this case, $\varepsilon' = \varepsilon^a$ for some $a \in \mathbb{Z}_p^\times$ (this is sensible, given that ε belongs to the multiplicative group of $1+\mathfrak{m}_R$, which has p-adically separated and complete $1+c\mathfrak{m}_R$ neighbourhoods of 1). The unique local k-algebra automorphism of $k[[\pi]]$ given by $u \mapsto (1+\pi)^a - 1$ composed with the map $k[[\pi]] \to R$ resting on ε gives us the map which rests on ε'. □

In view of the lemma, we may define the canonical subfield

Definition 11.

$$E_0 := k((\pi))$$

in $\operatorname{Fr} R$, to be the fraction field of the canonical image of $k[[\pi]]$ in R for any choice of ε. From the definition of π we can calculate at ease the action of G_{K_0} and φ:

$$g(\pi) = (\pi+1)^{\chi(g)} - 1, \quad \text{if} \quad g \in G_{K_0} \quad \text{and} \quad \varphi(\pi) = (\pi+1)^p - 1.$$

Set

$$K_0^{\mathrm{cyc}} = \varinjlim_{n \in \mathbb{N}} K_0(\varepsilon^{(n)}).$$

Then, by the above calculation, $\varepsilon = (\varepsilon^{(n)})_{n\in\mathbb{N}}$ is a unit of $R_{K_0^{\mathrm{cyc}}}$ (since $v_R(\varepsilon - 1) > 1$).

For $G_{K_0^{\mathrm{cyc}}} = \operatorname{Gal}(\overline{K}/K_0^{\mathrm{cyc}})$, $R^{G_{K_0^{\mathrm{cyc}}}} = R_{K_0^{\mathrm{cyc}}}$ by Proposition 7. Since $v(\pi) = \frac{p}{p-1}$, $\pi \in R^{G_{K_0^{\mathrm{cyc}}}}$, $k \subset R^{G_{K_0^{\mathrm{cyc}}}}$, and $R^{G_{K_0^{\mathrm{cyc}}}}$ is complete,

$$k[[\pi]] \subset R^{G_{K_0^{\mathrm{cyc}}}} \quad \text{and} \quad E_0 \subset (\operatorname{Fr} R)^{G_{K_0^{\mathrm{cyc}}}}.$$

The following notion exists uniquely for every field K (up to a unique isomorphism).

Definition 12. *Let K^{perf} be the smallest perfect field containing K. This is called the perfect closure of K.*

Since $R^{G_{K_0^{\mathrm{cyc}}}}$ and $(\operatorname{Fr} R)^{G_{K_0^{\mathrm{cyc}}}}$ are perfect and complete, we have

$$\widehat{k[[\pi]]^{\mathrm{perf}}} \subset R^{G_{K_0^{\mathrm{cyc}}}} \quad \text{and} \quad \widehat{E_0^{\mathrm{perf}}} \subset (\operatorname{Fr} R)^{G_{K_0^{\mathrm{cyc}}}}.$$

Moreover, equality is established in place of mere containment.

Proposition 8.

$$\widehat{k[[\pi]]^{\mathrm{perf}}} = R^{G_{K_0^{\mathrm{cyc}}}} = R_{K_0^{\mathrm{cyc}}}, \quad \widehat{E_0^{\mathrm{perf}}} = (\operatorname{Fr} R)^{G_{K_0^{\mathrm{cyc}}}}.$$

Proof. It suffices to prove that $\mathcal{O}_{\widehat{E_0^{\mathrm{perf}}}}$ is dense in $R_{K_0^{\mathrm{cyc}}}$, for which it would even be sufficient to demonstrate that $\mathcal{O}_{E_0^{\mathrm{perf}}}$ is dense in $R_{K_0^{\mathrm{cyc}}}$. Since $R_{K_0^{\mathrm{cyc}}} = \varprojlim \mathcal{O}_{K_0^{\mathrm{cyc}}}/p\mathcal{O}_{K_0^{\mathrm{cyc}}}$, it would follow from

$$\theta_m(\mathcal{O}_{E_0^{\mathrm{perf}}}) = \mathcal{O}_{K_0^{\mathrm{cyc}}}/p\mathcal{O}_{K_0^{\mathrm{cyc}}} \quad \forall m \in \mathbb{N}.$$

To demonstrate this, it suffices to show that $\mathcal{O}_{K_0^{\mathrm{cyc}}}/p\mathcal{O}_{K_0^{\mathrm{cyc}}} \subset \theta_m(\mathcal{O}_{E_0^{\mathrm{perf}}})$. Set $\pi_n = \varepsilon^{(n)} - 1$, then

$$\mathcal{O}_{K_0}[\varepsilon^{(n)}] = W[\pi_n], \quad \mathcal{O}_{K_0^{\mathrm{cyc}}} = \cup_n W[\pi_n].$$

In light of the discussion in Section 3.1, $\mathcal{O}_{K_0^{\mathrm{cyc}}}/p\mathcal{O}_{K_0^{\mathrm{cyc}}}$, viewed as a k-algebra, is generated by $\overline{\pi}_n$-s, the reduction of $\pi_n \mod p\mathcal{O}_{K_0^{\mathrm{cyc}}}$. In conclusion, it suffices to show that $\overline{\pi}_n$ is contained in $\theta_m(k[[\pi]]^{\mathrm{perf}})$ for all $m, n \in \mathbb{N}$. This can be achieved through a straightforward calculation. □

Since $\operatorname{Fr} R$ is algebraically closed, there exists a unique unique separable closure of E_0 inside $\operatorname{Fr} R$, which we denote by E_0^s. In order to provide a more focused thesis, we will only present the following theorem without providing a proof. The proof, however, can be found in [FO] (§ 4.2.3.).

Theorem 4. *E_0^s is dense in $\operatorname{Fr} R$ and stable under G_{K_0}. Moreover, for any $g \in G_{K_0^{\mathrm{cyc}}}$, the restriction $g|_{E_0^s}$ is in $\operatorname{Gal}(E_0^s/E_0)$, and the map*

$$\operatorname{Gal}(\overline{K}/K_0^{\mathrm{cyc}}) \to \operatorname{Gal}(E_0^s/E_0)$$

is an isomorphism.

In fact, this theorem is a special case of a much deeper theorem in the theory of norm fields. There is a functorial equivalence between the categories of finite extensions of K_0^{cyc} in $\overline{K}$ and finite separable extensions of E_0 in $\operatorname{Fr} R$. (see [BC], Theorem 13.4.3.)

4. Fontaine's rings

4.1. Étale φ-modules.

In this section, we fix a field E with $\operatorname{char}(E) = p$ and a separable closure E^s. Let G_E denote $\operatorname{Gal}(E^s/E)$. It must be emphasised that we do not assume E to be perfect;

thus, in general, the p-power endomorphism φ is not surjective. It follows from Fermat's little theorem and the fact that there are at most p roots of $x^p = x$ that $(E^s)^{\varphi=1} = \mathbb{F}_p$.

The Cohen ring $\mathcal{C}(E)$ of E will be denoted here by $\mathcal{O}_{\mathcal{E}}$. This is the unique (up to isomorphism) absolutely unramified discrete valuation ring of characteristic 0, with residue field equal to E. Furthermore, let $\mathcal{E}$ be the field of fractions of $\mathcal{O}_{\mathcal{E}}$. In summary, the setup is as follows: $\mathcal{O}_{\mathcal{E}} = \varprojlim_n \mathcal{O}_{\mathcal{E}}/p^n\mathcal{O}_{\mathcal{E}}$, $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}} = E$, $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[\frac{1}{p}]$, and $\mathcal{O}_{\mathcal{E}}$'s maximal ideal is generated by p. Moreover, if $\mathcal{E}'$ is another field with the same property as before, then there exists a continuous isomorphism $\mathcal{E} \to \mathcal{E}'$ that induces the identity map on E. (If E is perfect then this isomorphism is unique and $\mathcal{O}_{\mathcal{E}}$ may be identified with $W(E)$.) In addition, a continuos Frobenius endomorphism φ can be provided on $\mathcal{E}$. It satisfies $\varphi(\mathcal{O}_{\mathcal{E}}) \subset \mathcal{O}_{\mathcal{E}}$. (Once again, in the case of a perfect field, there is a unique such endomorphism.)

For the remainder of this subsection, we shall fix a choice of $\mathcal{E}$ and φ.

Let $\mathcal{F}$ be a finite extension of $\mathcal{E}$ with ring of integers $\mathcal{O}_{\mathcal{F}}$. It should be recalled that an extension $\mathcal{F}/\mathcal{E}$ is said to be *unramified* if it satisfies conditions

(1) p generates the maximal ideal of $\mathcal{O}_{\mathcal{F}}$;
(2) $F = \mathcal{O}_{\mathcal{F}}/p\mathcal{O}_{\mathcal{F}}$ is a separable extension of E.

The functoriality of Cohen rings (see Theorem 3) implies that for any homomorphism $f : E \to F$, where F is a field of characteristic p, there is a unique local homomorphism $\mathcal{C}(E) \to \mathcal{C}(F)$, which induces f on the residue fields. Consequently, for each finite separable extension F of E, there exist a unique (up to a unique isomorphism) unramified extension of $\mathcal{F} = \mathrm{Fr}\,\mathcal{C}(F)$ of $\mathcal{E}$ whose residue field is F. Furthermore, there exists a unique $\varphi' : \mathcal{F} \to \mathcal{F}$ Frobenius endomorphism such that $\varphi'_{\mathcal{E}} = \varphi$. We may write $\mathcal{F} = \mathcal{E}_F$ and continue to denote φ' as φ. By the same argument if $\sigma : F \to F'$ is a homomorphism such that $\sigma|_E = \mathrm{Id}$, we obtain an induced map $\mathcal{E}_F \to \mathcal{E}_{F'}$. Moreover, this commutes with φ. In paricular, if F/E is Galois, then $\mathcal{E}_F/\mathcal{E}$ is also Galois, and

$$\mathrm{Gal}(F/E) = \mathrm{Gal}(\mathcal{E}_F/\mathcal{E}).$$

Since $E^s = \bigcup_{F \leqslant E^s \text{ finite}} F$, we set

$$\mathcal{E}^{\mathrm{ur}} := \bigcup_{F \leqslant E^s \text{ finite}} \mathcal{E}_F.$$

It follows that $\mathcal{E}^{\mathrm{ur}}/\mathcal{E}$ is Galois with $\mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}/\mathcal{E}) = G_E$. Let $\widehat{\mathcal{E}^{\mathrm{ur}}}$ be the completion of $\mathcal{E}^{\mathrm{ur}}$. Then we have

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} = \varprojlim \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}/p\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$$

as its ring of integers. Furthermore, there persists the Frobenius φ on $\mathcal{E}^{\mathrm{ur}}$ satisfying $\varphi(\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}) \subset \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$, which extends by continuity to $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ and $\widehat{\mathcal{E}^{\mathrm{ur}}}$. Similarly, the group G_E acts on both $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ and $\widehat{\mathcal{E}^{\mathrm{ur}}}$. In addition, this action commutes with the aforementioned φ.

Firstly, we shall formulate the statement corresponding to the basic identities $(E^s)^{G_E} = E$ and $(E^s)^{\varphi_{E^s}=1} = \mathbb{F}_p$ in this context.

Lemma 10.

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}^{G_E} = \mathcal{O}_{\mathcal{E}}; \qquad (\widehat{\mathcal{E}^{\mathrm{ur}}})^{G_E} = \mathcal{E};$$
$$(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\varphi=1} = \mathbb{Z}_p; \qquad (\widehat{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1} = \mathbb{Q}_p.$$

Proof. Since both G_E and φ fix p, and $\widehat{\mathcal{E}^{\mathrm{ur}}} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}[\frac{1}{p}]$, it is sufficient to prove the integral claim. Moreover, the inclusions $\mathcal{O}_{\mathcal{E}} \hookrightarrow \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}^{G_E}$ and $\mathbb{Z}_p \hookrightarrow (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\varphi=1}$ are local homomorphisms between p-adically separated and complete rings, consequently, it suffices to show surjectivity $\mod p^n$ $(n \geq 1)$, given that surjectivity (exactness) is a local property. This is to be demonstrated by induction on n.

The formation of G_E-invariants is a left-exact functor, which implies that the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \overset{\mathrm{mod}\ p}{\longrightarrow} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \longrightarrow E^s \longrightarrow 0$$

of $\mathcal{O}_{\mathcal{E}}$-modules gives a linear injection $(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{G_E}/p(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{G_E} \hookrightarrow (E^s)^{G_E} = E$ of modules over $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}} = E$. However, the ranks of these modules are identical, thus establishing a bijection. Since $(E^s)^{\varphi=1} = \mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$, the surjectiveness of $\mathbb{Z}_p \hookrightarrow (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\varphi=1}$ follows in an analogous manner.

We now consider the case where $n > 1$. Let us assume that the map $\mathcal{O}_{\mathcal{E}}/p^{n-1}\mathcal{O}_{\mathcal{E}} \hookrightarrow (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{G_E}/p^{n-1}(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{G_E}$ is surjective and let $\xi \in (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{G_E}$. We must identify an element $x \in \mathcal{O}_{\mathcal{E}}$ such that $\xi \equiv x \mod p^n\mathcal{O}_{\mathcal{E}}$. We can choose $y \in \mathcal{O}_{\mathcal{E}}$ such that $\xi \equiv y \mod p^{n-1}\mathcal{O}_{\mathcal{E}}$, so now $\xi - y = p^{n-1}\xi'$ with $\xi' \in (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{G_E}$. By virtue of the preceding paragraph, there exists a $z \in \mathcal{O}_{\mathcal{E}}$ such that $\xi' \equiv z \mod p\mathcal{O}_{\mathcal{E}}$. Consequently, we have that $\xi \equiv y + p^{n-1}z \mod p^n\mathcal{O}_{\mathcal{E}}$ with $y + p^{n-1}z \in \mathcal{O}_{\mathcal{E}}$. Analogous arguments can be employed in the case of Frobenius invariants. □

Before proceeding, it is essential to define a concept that is of paramount importance to us.

Definition 13.

(1) *A module M over $\mathcal{O}_{\mathcal{E}}$ is said to be a φ-module over $\mathcal{O}_{\mathcal{E}}$, if it is equipped with a semi-linear map $\varphi : M \to M$, that is:*

$$\varphi(x+y) = \varphi(x) + \varphi(y), \qquad \varphi(\lambda x) = \varphi(\lambda)\varphi(x), \quad \forall\, x, y \in M,\ \lambda \in \mathcal{O}_{\mathcal{E}}.$$

(2) *A module D over $\mathcal{E}$ is said to be a φ-module over $\mathcal{E}$, if it is equipped with a semi-linear map $\varphi : D \to D$.*

Set

$$M_\varphi := \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} M, \qquad D_\varphi := \mathcal{E} \otimes_{\mathcal{E}} D.$$

Consequently, the assignment of a semi-linear map $\varphi : M \to M$ is equivalent to the assignment of an $\mathcal{O}_{\mathcal{E}}$-linear map $\Phi : M_\varphi \to M$. A similar argument can be made in the case of D.

Definition 14.

(1) *A φ-module M over $\mathcal{O}_{\mathcal{E}}$ is said to be étale if it is of finite type and $\Phi : M_\varphi \to M$ is an isomorphism.*
(2) *A φ-module D over $\mathcal{E}$ is said to be étale if $\dim_{\mathcal{E}} D < \infty$ and if there exists an $\mathcal{O}_{\mathcal{E}}$-lattice M of D, which is stable under φ such that M is an étale φ-module over $\mathcal{O}_{\mathcal{E}}$.*

Remark 9. It should be noted that an $\mathcal{O}_{\mathcal{E}}$-lattice M is a $\mathcal{O}_{\mathcal{E}}$-submodule of finite type that contains a basis.

Furthermore, it is a simple matter to verify that if M is an $\mathcal{O}_{\mathcal{E}}$-module of finite type with an action of φ, then M is étale if and only if M/pM is étale as an E-module.

While not discussed in detail here, it should be noted that the category of p-adic representations of G_E is equivalent to the category of étale φ-modules over $\mathcal{E}$. The equivalence of categories is established by the functor $\mathbf{M}(V) = (\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Q}_p} V)^G$, while the quasi-inverse is given by $\mathbf{V}(D) = (\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{E}} D)^{\varphi=1}$.

4.2. The rings $\mathbf{A}, \mathbf{B}, \mathbf{A}_K$ and $\mathbf{B}_K$.

Let us consider the case where $E = E_0 = k((\pi))$. We denote $\mathcal{E}_0$ the corresponding field. Thus $E_0 = \mathcal{O}_{\mathcal{E}_0}/p\mathcal{O}_{\mathcal{E}_0}$. Let $K_\infty = \varinjlim_n K(\mu_n)$. We shall now fix the following notational conventions:

Definition 15.

$$\mathbf{A} = \mathcal{O}_{\widehat{\mathcal{E}_0^{\mathrm{ur}}}}; \qquad \mathbf{B} = \widehat{\mathcal{E}_0^{\mathrm{ur}}};$$
$$\mathbf{A}_K = \mathbf{A}^{G_{K_\infty}}; \qquad \mathbf{B}_K = \mathbf{B}^{G_{K_\infty}};$$
$$\mathbf{E}_K = \mathbf{A}_K/p\mathbf{A}_K.$$

In particular,

$$\mathbf{A}_{K_0} = \mathbf{A}^{G_{E_0}} = \mathcal{O}_{\mathcal{E}_0}, \ \ \mathbf{B}_{K_0} = \mathbf{B}^{G_{E_0}} = \mathcal{E}_0, \quad \mathbf{A}^{\varphi=1} = \mathbb{Z}_p, \quad \mathbf{B}^{\varphi=1} = \mathbb{Q}_p \quad \text{and} \quad \mathbf{E}_{K_0} = E_0.$$

In conclusion, $\mathbf{B}_{K_0} = \mathbf{A}_{K_0}[\frac{1}{p}]$ and $\mathbf{B}$ is the p-adic completion of the maximal unramified extension of $\mathbf{B}_{K_0}$ inside $\widetilde{\mathbf{B}} = W(\mathrm{Fr}\, R)[\frac{1}{p}]$ (see the next section), and $\mathbf{A} \subset \mathbf{B}$ is the ring of integers.

From Theorem 4, we can infer that the completion of E_0^s with respect to the valuation v_R is $\mathrm{Fr}\, R$ and $\mathrm{Gal}(E_0^s/E_0) = G_{K_0^{\mathrm{cyc}}}$. Moreover, $\mathbf{E}_K = (E_0^s)^{G_{K_\infty}}$ and $G_{K_\infty} \cong \mathrm{Gal}(\mathbf{E}_K^s/\mathbf{E}_K) \cong \mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}_{\mathbf{E}_K}/\mathcal{E}_{\mathbf{E}_K})$. The following lemma is a corollary of the structure theorem for local fields of equal characteristic (see [Ser79], §4.) and the observation that the residue field of $\mathbf{E}_K$ is $\overline{k}^{G_{K_\infty}} = k_{K_\infty}$.

Lemma 11. *If π_K is a uniformiser of $\mathbf{E}_K$, the $\mathbf{E}_K = k_{K_\infty}((\pi))$.*

In fact, the ring $\mathbf{A}_{K_0} = \mathcal{O}_{\mathcal{E}_0}$ can be explicitly described as follows.

$$\mathbf{A}_{K_0} = \left\{ \sum_{n\in\mathbb{Z}} \lambda_n \pi^n \mid \lambda_n \in W(k),\ \lambda_n \to 0 \text{ as } n \to -\infty \right\}.$$

Indeed,

Proposition 9. *If $\pi_K \in \mathbf{A}_K$ is such that $\overline{\pi}_K \in \mathbf{E}_K$ is a uniformiser, then A_K is the p-adic completion of $W(k)[[\pi_K]][\frac{1}{\pi_K}]$.*

Proof. The notation $\mathcal{S}_K$ will be used to denote the p-adic completion of $W(k)[[\pi_K]][\frac{1}{\pi_K}]$. It follows from the preceding lemma that $\mathcal{S}_K/p\mathcal{S}_K = \mathbf{E}_K$, which implies that the inclusion $\mathcal{S}_K \hookrightarrow \mathbf{A}_K$ is surjective $\mod p$. Since $\mathcal{S}_K$ is p-adically complete, this map is in fact an isomorphism. □

Proposition 10. *Every element of $\mathbf{A}_K$ can be uniquely expressed as $\sum_{n\in\mathbb{Z}} a_n \pi_K^n$, where $(a_n)_{n\in\mathbb{Z}}$ is a sequence in $W(k)$ such that $\lim_{n\to-\infty} a_n = 0$.*

Proof. Let $s : \mathbf{E}_K \to \mathbf{A}_K$ denote the section of the reduction $x \mapsto \overline{x} \mod p$, given by the formula

$$s(\sum_{k\in\mathbb{Z}} b_n \overline{\pi}_K^n) = \sum_{n\in\mathbb{Z}} [b_n] \pi_K^n.$$

If $x \in \mathbf{A}_K$, then let us define recursively the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{A}_K$ by putting $x_0 := x$ and $x_{n+1} := \frac{x_s(\overline{x}_n)}{p}$. Then we have

$$x = \sum_{n\geq 0} p^n s(\overline{x}_n).$$

□

In conclusion, we shall once more define a category. To this end, we shall employ the following notation:

$$\Gamma_K := \mathrm{Gal}(K_\infty/K).$$

Definition 16. *An étale φ-module D over $\mathcal{O}_{\mathcal{E}_{\mathbf{E}_K}}$ (or $\mathcal{E}_{\mathbf{E}_K}$) is said to be an étale (φ,Γ)-module over $\mathcal{O}_{\mathcal{E}}$ (respectively $\mathcal{E}$) if it is endowed with a continuous semi-linear action of Γ_K, which commutes with φ.*

As was the case in the preceding section, we can also state an analogous equivalence of categories.

Proposition 11. *The category of p-adic representations of G_K is equivalent to the category of étale (φ,Γ)-modules over $\mathcal{E}_{\mathbf{E}_K}$.*

4.3. The field $\widetilde{\mathrm{B}}$ and some of its subrings.

Definition 17. *Let $\overline{K} \geqslant L \geqslant K_0$ be an intermediate field (possibly of infinite degree over K_0). Let us define*

$$\widetilde{\mathbf{A}}_L^+ := W(R_L), \quad \text{and} \quad \widetilde{\mathbf{A}}_L := W(\mathrm{Fr}\, R_L).$$

In the event that $L = \overline{K}$, we will shortly denote $\widetilde{\mathbf{A}}^+ = W(R)$ and $\widetilde{\mathbf{A}} = W(\mathrm{Fr}\, R)$.

The functoriality of Witt vectors allows for the unique endowment of these rings with the Frobenius φ and the Galois action of $G_{K_0} = \mathrm{Gal}(\overline{K}/K_0)$, as demonstrated in section 3.1. In explicit terms,

$$\varphi(\sum_{n=0}^{\infty} p^n[x_n]) = \sum_{n=0}^{\infty} p^n[x_n^p] \quad \sigma(\sum_{n=0}^{\infty} p^n[x_n]) = \sum_{n=0}^{\infty} p^n[\sigma(x_n)], \text{ if } \sigma \in G_{K_0}.$$

Then φ commutes with the action of G_{K_0}. We may now apply Proposition 7 to the Witt coordinates to derive the following proposition.

Proposition 12. *Let $\overline{K} \geqslant L \geqslant K_0$ be an intermediate field. Then*

$$(\widetilde{\mathbf{A}}^+)^{G_L} = \widetilde{\mathbf{A}}_L^+, \quad (\widetilde{\mathbf{A}})^{G_L} = \widetilde{\mathbf{A}}_L.$$

For $k \geq 0$, let the map $w_k : \widetilde{\mathbf{A}} \to \mathbb{R} \cup \{\infty\}$ (resp. for $\widetilde{\mathbf{A}}_L, \widetilde{\mathbf{A}}_L^+$) be defined by

$$w_k(x) = \min_{i \leq k} v_R(x_i), \quad \text{where} \quad x = \sum_{i \geq 0} p^i[x_i].$$

It is evident that for any elements $x, y \in \widetilde{\mathbf{A}}$ (resp. in $\widetilde{\mathbf{A}}_L, \widetilde{\mathbf{A}}_L^+$),

(1) $w_k(x) = \infty$ if and only if $x \in p^{k+1}\widetilde{\mathbf{A}}$ (resp. in $p^{k+1}\widetilde{\mathbf{A}}_L, p^{k+1}\widetilde{\mathbf{A}}_L^+$);
(2) $w_k(x+y) \geq \min\{w_k(x), w_k(y)\}$;
(3) $w_k(xy) \geq \min_{i+j \leq k}\{w_i(x) + w_j(y)\}$,

that is, w_k is a semi-valuation on $\widetilde{\mathbf{A}}$ (resp. on $\widetilde{\mathbf{A}}_L, \widetilde{\mathbf{A}}_L^+$). Furthermore, it can be demonstrated with ease that

(4) $w_k(\varphi(x)) = p w_k(x)$;
(5) $w_k([\lambda]x) = w_k(x) + v_R(\lambda)$ $\forall \lambda \in \mathrm{Fr}\, R$ (resp. in $\mathrm{Fr}\, R_L$);
(6) $w_k(\sigma(x)) = w_k(x)$ $\forall \sigma \in G_{K_0}$.

Definition 18. *The weak topology on $\widetilde{\mathbf{A}}_L$ and $\widetilde{\mathbf{A}}_L^+$ is the one defined by the semi-valuations w_k.*

This means that for a sequence $\{a_n\}_{n\in\mathbb{N}}$ and element a in $\widetilde{\mathbf{A}}_L$ or $\widetilde{\mathbf{A}}_L^+$, $a_n \to a$ for the weak topology if and only if $\forall k \geq 0$ $\ w_k(a_n - a) \to 0$ as $n \to \infty$. In fact, the weak topology on $\widetilde{\mathbf{A}}$ is the product topology of the v_R-adic topology under the identification of $W(\mathrm{Fr}\, R) = \prod_{n\geq 0} \mathrm{Fr}\, R$. In other words, it is the inverse limit topology of the product topologies on each $W_n(\mathrm{Fr}\, R) = (\mathrm{Fr}\, R)^n$. It can be therefore stated that $\widetilde{\mathbf{A}}$ for the explicit topology described above is complete and Hausdorff. Moreover, the map $(x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} p^n[x_n]$ is a homeomorphism between $(\mathrm{Fr}\, R_L)^{\mathbb{N}}$ (resp. $R_L^{\mathbb{N}}$) and $\widetilde{\mathbf{A}}_L$ (resp. $\widetilde{\mathbf{A}}_L^+$).

Remark 10. The ring $\widetilde{\mathbf{A}}$ (resp. $\widetilde{\mathbf{A}}^+$) is naturally endowed with another topology, which is the finest topology, for which the projection $\widetilde{\mathbf{A}} \to \mathrm{Fr}\, R$ (resp. $\widetilde{\mathbf{A}}^+ \to R$) is continuous, where $\mathrm{Fr}\, R$ (resp. R) is endowed with the discrete topology. This is referred to as the *strong topology*, which is in fact the p-adic topology on $\widetilde{\mathbf{A}}$ or $\widetilde{\mathbf{A}}^+$. However, the Galois action will not be continuous with respect to this topology, as it is not even for R.

Proposition 13. *φ acts continuously on $\widetilde{\mathbf{A}}_L$ and $\widetilde{\mathbf{A}}_L^+$, endowed with the weak topology. Furthermore, we have that*

$$\widetilde{\mathbf{A}}_L^{\varphi=1} = (\widetilde{\mathbf{A}}_L^+)^{\varphi=1} = \mathbb{Z}_p.$$

Proof. The continuity of φ is evident from the preceding property (4) of w_k. If $x = \sum_{n=0}^{\infty} p^n[x_n]$ is invariant under φ, then $x_n^p = x_n$, so $x_n \in \mathbb{F}_p$ for all $n \in \mathbb{N}$. The claim thus follows from the fact that $W(\mathbb{F}_p) = \mathbb{Z}_p$. □

The following definition is now to be added.

Definition 19.

$$\widetilde{\mathbf{B}}_L^+ := \mathrm{Fr}\, \widetilde{\mathbf{A}}_L^+ = \widetilde{\mathbf{A}}_L^+[\frac{1}{p}] \qquad \widetilde{\mathbf{B}}_L := \mathrm{Fr}\, \widetilde{\mathbf{A}}_L = \widetilde{\mathbf{A}}_L[\frac{1}{p}],$$

and if $L = \overline{K}$, then the subscripts are omitted, in accordance with previous notation.

Every element $\widetilde{\mathbf{B}}_L$ (resp. $\widetilde{\mathbf{B}}_L^+$) is of the form $\sum_{n=n_0}^{\infty} p^n[x_n]$, where $n_0 \in \mathbb{Z}$ and $(x_n)_{n\geq n_0}$ is a sequence of elements of $\mathrm{Fr}\, R_L$ (reps. R_L). The actions of φ and G_{K_0} extends by $\mathbb{Q}_p$-linearity to $\widetilde{\mathbf{B}}_L$ (resp. $\widetilde{\mathbf{B}}_L^+$). Furthermore,

$$\widetilde{\mathbf{B}}_L^{\varphi=1} = (\widetilde{\mathbf{B}}_L^+)^{\varphi=1} = \mathbb{Q}_p, \quad \widetilde{\mathbf{B}}^{G_L} = \widetilde{\mathbf{B}}_L \quad \text{and} \quad (\widetilde{\mathbf{B}}^+)^{G_L} = \widetilde{\mathbf{B}}_L^+.$$

Furthermore, we define the element π as $[\varepsilon] - 1$ in $\widetilde{\mathbf{A}}$. Consequently, the image of π in $\mathrm{Fr}\, R$ is given by $\overline{\pi} = \varepsilon - 1$. Hence, we have

$$\varphi(\pi) = (1+\pi)^p - 1 \qquad \text{and} \qquad g(\pi) = (1+\pi)^{\chi(g)} - 1, \text{ if } g \in G_{K_0}.$$

4.4. **The field of overconvergent elements and some of its subrings.**

Let $r > 0$ and $x = \sum_{n\geq 0} p^n[x_n] \in \widetilde{\mathbf{A}}$.

Definition 20.

$$\begin{aligned}\widetilde{\mathbf{A}}^{\dagger,r} &:= \left\{x \in \widetilde{\mathbf{A}} \mid \inf_{n\geq 0}\left(w_n(x) + \frac{nrp}{p-1}\right) \geq 0 \textit{ and } \lim_{n\to\infty}\left(w_n(x) + \frac{nrp}{p-1}\right) = \infty\right\} \\ &= \left\{x \in \widetilde{\mathbf{A}} \mid \inf_{n\geq 0}\left(v_R(x_n) + \frac{nrp}{p-1}\right) \geq 0 \textit{ and } \lim_{n\to\infty}\left(v_R(x_n) + \frac{nrp}{p-1}\right) = \infty\right\}\end{aligned}$$

The equality in question is upheld by the following lemma:

Lemma 12. $\lim_{n\to\infty}(v_R(x_n) + nr) = \infty$ *if and only if* $\lim_{n\to\infty}(w_n(x) + nr) = \infty$. *Moreover, if the aforementioned condition is satisfied, then* $\inf_{n\geq 0}(v_R(x_n) + nr) = \inf_{n\geq 0}(w_n(x) + nr)$.

Proof. Since $w_n(x) \leq v_R(x_n)$ by definition, it is sufficient to prove the non-trivial direction (and inequality). To this end, define $K_n = \sup\{k \mid w_n(x) = v_R(x_k),\ k \leq n\}$, which is evidently at most n. If $\lim_{n\to\infty} K_n = \infty$, then

$$\lim_{n\to\infty}(w_n(x) + nr) = \lim_{n\to\infty}(v_R(x_{K_n}) + nr) \geq \lim_{n\to\infty}(v_R(x_{K_n}) + K_n r) = \infty.$$

In the event that $\lim_{n\to\infty} K_n = N < \infty$, then for sufficiently large values of n

$$\lim_{n\to\infty}(w_n(x) + nr) = \lim_{n\to\infty}(v_R(x_N) + nr) = \infty.$$

With regard to the remaining part of the lemma, it can be observed that

$$\inf_{n\geq 0}(w_n(x) + nr) = \inf_{n\geq 0}(v_R(x_{K_n}) + nr) \geq \inf_{n\geq 0}(v_R(x_{K_n}) + K_n r) \geq \inf_{n\geq 0}(v_R(x_n) + nr).$$

□

Remark 11. It should be noted that $\widetilde{\mathbf{A}}^{+} \subset \widetilde{\mathbf{A}}^{\dagger,r}$ for all $r > 0$.

For the sake of simplicity, we set $s(r) = \frac{nrp}{p-1}$.

Definition 21.

$$v_r : \widetilde{\mathbf{A}}^{\dagger,r} \to \mathbb{R}_{\geq 0} \cup \{\infty\}, \qquad v_r(x) := \inf_{n\in\mathbb{N}}(v_R(x_n) + s(r)) = \inf_{n\geq 0}(w_n(x) + s(r))$$

If $r_1 \geq r_2$, then

1. $\widetilde{\mathbf{A}}^{\dagger,r_1} \subset \widetilde{\mathbf{A}}^{\dagger,r_2}$ and
2. $v_{r_1}(x) \leq v_{r_2}(x)$ for all $x \in \widetilde{\mathbf{A}}^{\dagger,r}$.

Hence we can define a function $f_x : \mathbb{R}_{\geq r} \to \mathbb{R}$ by $f_x(t) = v_t(x)$.

Proposition 14 (Newton polygon of x)**.** *Assume* $r > 0$ *and* $x = \sum_{n\in\mathbb{N}} p^n[x_n] \in \widetilde{\mathbf{A}}^{\dagger,r}$.

(1) The function f_x *is an increasing, piecewise linear, concave, and continuous function. All slopes of* f_x *belong to* $\frac{p-1}{p}\mathbb{Z}_{\geq 0}$*. Furthermore,* f_x *has finitely many slopes and cusps.*

(2) *Let us denote with $\partial_l f_x$ (resp. $\partial_r f_x$) the left (resp. right) derivation of f_x. Then $\frac{p-1}{p}\partial_l f_x(t)$ (resp. $\frac{p-1}{p}\partial_r f_x(t)$) is the maximal (resp. minimal) integer N satisfying $v_t(x) = v_R(x_N) + \frac{tpN}{p-1}$. Consequently, $f_x(t)$ is derivable at $t_0 > r$ if and only if there exists exactly one $n \geq 0$ such that $v_{t_0}(x) = v_R(x_n) + ns(t_0)$, and $n = \frac{p-1}{p} f_x'(t_0)$.*

(3) *If $x_0 \neq 0$, then $\exists r_0 \geq r$ such that for any $t \geq r_0 :\ f_x(t) = v_R(x_0)$. In particular, the slope of the last segment of f_x is 0.*

Proof. The aforementioned observation regarding v_r leads to that f_x is increasing. If $r_0 \geq r$ and $x \in \widetilde{\mathbf{A}}^{\dagger,r_0}$, then there are only finite number of $n \in \mathbb{N}$ for which $f_x(r_0) = v_R(x_n) + ns(r_0)$. These integers may be denoted by $n_1 < n_2 < \ldots < n_k$. Consequently, for any $t \geq r$, we have that

$$f_x(t) = \min_{1\leq i\leq k}\{v_R(x_{n_i}) + n_i s(t)\} = f_x(r_0) + \min_{1\leq i\leq k}\left\{\frac{pn_i}{p-1}(t - r_0)\right\}.$$

Therefore, it can be stated that

$$\frac{f_x(t) - f_x(r_0)}{t - r_0} = \begin{cases} \frac{pn_1}{p-1}, & \text{if} \quad t \geq r_0 \\ \frac{pn_k}{p-1}, & \text{if} \quad t \leq r_0 \end{cases},$$

which implies (1) and (2).

For (3), it can be observed that for $t \geq r$, we have that

$$v_t(x) = \inf(v_R(x_0), \inf_{n\geq 1} v_R(x_n) + ns(t)).$$

Here $\inf_{n\geq 1}(v_R(x_n) + ns(t)) \geq v_r(x) + s(t) - s(r)$ holds. It follows that, for $t \geq r$ sufficiently large, the second term is invariably greater, thus establishing $f_x(t) = v_R(x_0)$. □

Lemma 13. *$\widetilde{\mathbf{A}}^{\dagger,r}$ is a subring of $\widetilde{\mathbf{A}}$, which is stable under the action of G_{K_0}, and $\varphi : \widetilde{\mathbf{A}}^{\dagger,r} \to \widetilde{\mathbf{A}}^{\dagger,pr}$ is a bijection.*

Proof. If $x, y \in \widetilde{\mathbf{A}}^{\dagger,r}$, then both $w_n(x + y) + sn \geq \min\{w_n(x) + s(r)n, w_n(y) + s(r)n\}$ and $w_n(xy) + s(r)n \geq \min i + j \leq n\{(w_i(x) + is(r)) + (w_j(y) + js(r))\}$. In particular, both $x + y$ and xy belong to $\widetilde{\mathbf{A}}^{\dagger,r}$. w_n is invariant under the Galois action, hence $\widetilde{\mathbf{A}}^{\dagger,r}$ is indeed stable under G_{K_0}. Moreover, the last part of the lemma is implied by

$$w_n(\varphi(x)) + ns(pr) = pw_n(x) + ns(pr) = p(w_n(x) + ns(r)).$$

□

Furthermore, the following assertion is also true.

Lemma 14. *Let $x, y \in \widetilde{\mathbf{A}}^{\dagger,r}$ and $\alpha \in \operatorname{Fr} R$.*

(1) *$v_r(x) = \infty$ if and only if $x = 0$;*

(2) $v_r(x+y) \geq \min\{v_r(x), v_r(y)\}$*;*
(3) $v_r(xy) = v_r(x) + v_r(y)$*;*
(4) $v_{pr}(\varphi(x)) = pv_r(x)$*;*
(5) $v_r(px) = v_r(x) + s(r)$*;*
(6) $v_r([\alpha]x) = v_R(\alpha) + v_r(x)$*;*
(7) $v_r(\sigma(x)) = v_r(x)$ *for all* $\sigma \in G_{K_0}$*.*

Proof. (1), (2), (4), (5), (6) and (7) is evident from the definition of v_r and partly a by-product of the proof of the preceding lemma. The sole noteworthy item is (3). Furthermore, it was demonstrated that $v_r(xy) \geq v_r(x) + v_r(y)$. The objective is to demonstrate $f_xy(t) = f_x(t) + f_y(t)$. By Proposition 14 we know that for any $r' \geq r$ (except for the finitely many cusps), there exists a unique n and m, respectively, such that $v_{r'}(x) = v_R(x_n) + ns(r')$ and $v_{r'}(y) = v_R(y_m) + ms(r')$. Observe that

$$xy = [x_n y_n]p^{m+n} + (x_[x_n]p^n)y + [x_n]p^n(x_[y_m]p^m).$$

By virtue of the fact that $v_r(xy) \geq v_r(x) + v_r(y)$ and that $v_{r'}(x) < v_{r'}(x - [x_n]p^n)$ and $v_{r'}(y) < v_{r'}(y - [y_m]p^m)$, it follows that the last two terms of the left-hand side of the aforementioned expression is strictly greater then $v_{r'}(x) + v_{r'}(y)$. This implies, in accordance with (2) and (5), that

$$v_{r'}(x) = v_R(x_n y_n) + (n+m)s(r') = v_{r'}(x) + v_{r'}(y).$$

In other words, $f_xy(t) = f_x(t) + f_y(t)$ for every $t \geq r$,with the exception of the cusps. By virtue of the continuity, the desired equality holds for all $t \geq r$. □

The preceding lemma demonstrates that v_r is a valuation on $\widetilde{\mathbf{A}}^{\dagger,r}$. By virtue of Lemma 14, (5), the valuation can be extended to the following subring of $\widetilde{\mathbf{B}}$, such that the results of Proposition 14 and Lemma 14 will also hold on this ring.

Definition 22.

$$\widetilde{\mathbf{B}}^{\dagger,r} = \widetilde{\mathbf{A}}^{\dagger,r}[\frac{1}{p}],$$

Remark 12. If $x = \sum_{n=n_0}^{\infty} p^n[x_n]\widetilde{\mathbf{B}}^+$, where $n_0 \in \mathbb{Z}$, then we can also define $v_0(x)$ as well, as $\inf_n v_R(x_n)$. Then the usual properties are still hold, with the exception that f_x has infinitely many slopes and cusps in the neighbourhood of 0.

Remark 13. Note that for every $0 \neq a \in \operatorname{Fr} R$, there is an integer $N \geq 0$ such that $p^N[a]$ belongs to $\widetilde{\mathbf{A}}^{\dagger,r}$, hence $[a] \in \widetilde{\mathbf{B}}^{\dagger,r}$ and it is a unit.

It should be noted that $\widetilde{\mathbf{A}}^{\dagger,r}$ is not the ring of integers of $\widetilde{\mathbf{B}}^{\dagger,r}$ for v_r. This is illustrated by the case, where $r = \frac{p-1}{p}$. In this instance, $v_r(\frac{[\tilde{p}]}{p}) = v_R(p) + (-1)s(r) = 0)$. However, $x \in \widetilde{\mathbf{A}}^{\dagger,r}$ if and only if it belongs to the ring of integers of $x \in \widetilde{\mathbf{B}}^{\dagger,r} \cap \widetilde{\mathbf{A}}$ (that is, $v_r(x) \geq 0$). Furhtermore, we have the following lemma:

Lemma 15. *For* $x = \sum_{n=n_0}^{\infty} p^n[x_n] \in \widetilde{\mathbf{B}}^{\dagger,r}$*, define* $x^- = \sum_{n\geq 0} p^n[x_n]$ *and* $x^+ = \sum_{n\leq -1} p^n[x_n]$*. If* $x \in \widetilde{\mathbf{B}}^{\dagger,r}$ *satisfies* $v_r(x) \geq 0$*, then* $x^- \in \widetilde{\mathbf{A}}^{\dagger,r}$*,* $x^+ \in \widetilde{\mathbf{B}}^+$*, and* $v_t(x^{\pm}) \geq v_t(x)$ *for all* $t \geq r$*.*

Proof. The condition $v_R(x_n) + ns(r) \geq 0$ for all $n \in \mathbb{Z}$ implies that $x^- \in \widetilde{\mathbf{A}}^{\dagger,r}$ and that if $n \geq -1$, then $x_n \in R$, thus $x^+ \in \widetilde{\mathbf{B}}^+$. It is evident that the other part of the proposition also holds. □

Remark 14. It can be observed that, in accordance with property (6) of Lemma 14, if $x \in \widetilde{\mathbf{A}}^{\dagger,r}$ and $a \in R$ (with $v_R(a) \geq 0$), then $v_r(x) \geq v_R(a)$ if and only if $x \in [a]\widetilde{\mathbf{A}}^{\dagger,r}$.

Proposition 15. $\widetilde{\mathbf{A}}^{\dagger,r}$ *is Hausdorff and complete with respect to the* v_r*-adic topology.*

Proof. From (1) in the Lemma 14, it is clear that $\widetilde{\mathbf{A}}^{\dagger,r}$ is Hausdorff.

Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in $\widetilde{\mathbf{A}}^{\dagger,r}$ that converges to 0. Then a_i also tends to 0 in $\widetilde{\mathbf{A}}$ as well with respect to the weak topology. Consequently, since $\widetilde{\mathbf{A}}$ is complete with respect to the weak topology, the series $\sum_{i=0}^{\infty} a_i$ converges to an element $a \in \widetilde{\mathbf{A}}$. The objective is now to show that a is even in $\widetilde{\mathbf{A}}^{\dagger,r}$. This element satisfies $w_n(a) \geq \inf_{i\in\mathbb{N}} w_n(a_i)$ for all n. Since $\lim_{n\to\infty} w_n(a_i) + ns(r) = \infty$ and $\lim_{i\to\infty}(\inf_{n\in\mathbb{N}} w_n(a_i) + ns(r)) = \infty$, it follows that $\lim_{n\to\infty} w_n(a) + ns(r) = \infty$. Therefore, $a \in \widetilde{\mathbf{A}}^{\dagger,r}$. □

Remark 15. It is additionally beneficial to highlight another straightforward fact. If $x \in \widetilde{\mathbf{A}}^{\dagger,r}$ and $\sum_{n=0}^{\infty} p^n[x_n]$ converges to x in $\widetilde{\mathbf{A}}$, then the sum converges to x in $\widetilde{\mathbf{A}}^{\dagger,r}$ as well.

Lemma 16. *Let* $r < 0$*.*

(1) The action of G_{K_0} *is continuous on* $\widetilde{\mathbf{A}}^{\dagger,r}$*.*
(2) The Frobenius map $\varphi : \widetilde{\mathbf{A}}^{\dagger,r} \to \widetilde{\mathbf{A}}^{\dagger,pr}$ *is a homeomorphism.*

Proof. (2) is an immediate consequence of Lemma 14, (5).

In order to prove (1), since $v_r(\sigma(x)) = v_r(x)$, it is sufficient to show that for a fixed $x = \sum_{n=0}^{\infty} p^n[x_n] \in \widetilde{\mathbf{A}}^{\dagger,r}$, the function $\sigma \mapsto [\sigma(x)]$ is continuous. In the event that $x = [\alpha]$, with $\alpha \in \operatorname{Fr} R$, then since $\sigma \mapsto \sigma(\alpha)$ is also continuous, it follows that $\sigma \mapsto [\sigma(\alpha)]$ is also, provided that the weak topology is considered on $\widetilde{\mathbf{A}}$. Consequently, for every $n \in \mathbb{N}$ $w_n([\sigma(\alpha)] - [\alpha]) \to \infty$ as $\sigma \to 1$. Moreover, we have from basic properties of w_n that $w_n([\sigma(\alpha)] - [\alpha]) \geq v_R(\alpha)$, which implies that $v_r([\sigma(\alpha)] - [\alpha]) \to \infty$ as $\sigma \to 1$. The general case can be derived from the observation, that if $x = \sum_{n=0}^{\infty} p^n[x_n]$, then the function $\sigma \mapsto \sigma(x)$ is the sum of the functions $\sum_{n=0}^{\infty}(\sigma \mapsto p^n[\sigma(x_n)])$, which converges uniformly in $\widetilde{\mathbf{A}}^{\dagger,r}$. □

Definition 23.

$$\widetilde{\mathbf{A}}^{\dagger} := \varinjlim_{r>0} \widetilde{\mathbf{A}}^{\dagger,r},$$

which is endowed with topology of direct limit.

Lemma 17. $x = \sum_{n\geq 0} p^n[x_n]$ *is a unit in* $\widetilde{\mathbf{A}}^{\dagger,r}$ *if and only if* $x_0 \neq 0$ *and* $v_R(x_n)+ns(r) > v_R(x_0)$ *for all* $n \geq 1$.

Proof. We shall first assume that $x0 \neq 0$ and $v_R(x_n) + ns(r) > v_R(x_0)$ is satisfied for all $n \geq 1$. Since $x_0 \in R$, $[x_0]$ is a unit in $\widetilde{\mathbf{A}}^{\dagger,r}$. Hence we may assume that $x_0 = 1$ by multiplying with $[x_0]^{-1}$ if necessary. Consequently, we may express x as $1 - x'$, where $x' \in \widetilde{\mathbf{A}}^{\dagger,r}$ and $v_r(x') > 0$. Then $\sum_{n\geq 0}(x')^n$ converges in $\widetilde{\mathbf{A}}^{\dagger,r}$ to the inverse of x.

Now, for the converse, if $x = \sum_{n\geq 0} p^n[x_n]$ is a unit in $\widetilde{\mathbf{A}}^{\dagger,r}$ with inverse $y = \sum_{n\geq 0} p^n[y_n]$, then $x_0y_0 = 1 \mod p$, which implies that $x_0 \neq 0$. As $\lim_n v_R(x_n) + ns(r) \lim_n v_R(y_n) + ns(r) = +\infty$, there are only a finite number of x_m and y_n for which $v_r(x) = v_R(x_m) + ms(r)$ and $v_r(y) = v_R(y_n) + ns(r)$. Let m_0 and n_0 be the largest such elements, respectively. In the event that $m_0 + n_0 > 1$, the coefficient of $p^{m_0+n_0}$ in $xy = 1$ is

$$[x_{m_0+n_0}] + \ldots + [x_{m_0}y_{n_0}] + \ldots + [y_{m_0+n_0}] = 0.$$

Consequently,

$$v_R(x_{m_0}y_{n_0}) + (m_0 + n_0)s(r) \geq \min_{\substack{i+j=m_0+n_0,\\ i\neq m_0}} \{v_R(x_iy_j) + (m_0 + n_0)s(r)\}$$

. In contrast, due to the choice of m_0 and n_0, we have that $v_R(x_i)+v_R(x_j)+(m_0+n_0)r(s)$ is strictly greater than $v_R(x_{m_0}) + v_R(x_{n_0}) + (m_0 + n_0)r(s)$, which is a contradiction. It can be concluded, therefore, that $m_0 = n_0 = 0$ and that for all $n \geq 1$ $v_R(x_n) + ns(r) > v_R(x_0)$. □

Example 3. For $r > \frac{p-1}{p}$, $\frac{\pi}{[\overline{\pi}]}$ is a unit in $\widetilde{\mathbf{A}}^{\dagger,r}$. Indeed. $\pi = [\varepsilon] - 1 = \sum_{n\geq} p^n[x_n]$ and $x_0 = \overline{\pi} = \varepsilon - 1$. Furthermore $n \geq 1$, x_n is a polynomial in $\varepsilon^{\frac{1}{p^n}} - 1$ of degree p^n with no constant term. Consequently, $v_R(x_n) \geq v_R(\varepsilon^{\frac{1}{p^n}} - 1) = \frac{1}{p^{n-1}(p-1)}$. Therefore, if $r > \frac{p-1}{p}$, it follows that $v_r(\frac{\pi}{[\overline{\pi}]}) = v_R(x_n) + ns(r) - v_R(\overline{\pi}) \geq \frac{1}{p^{n-1}(p-1)} + \frac{p}{p-1}(nr - 1) > 0$, and thus $\frac{\pi}{[\overline{\pi}]}$ is a unit in $\widetilde{\mathbf{A}}^{\dagger,r}$.

Proposition 16. *Every* $x \in \widetilde{\mathbf{A}}_K^{\dagger,r}$ *can be expressed as* $x = \sum_{n\geq 0} x_n(p/\pi^r)^k$, *where* $x_n \in \widetilde{\mathbf{A}}_K^{+}$ *and* $x_n \to 0$ *with respect to the weak topology.*

Proof. Let $x = \sum_{n\geq 0} p^n[y_n] \in \widetilde{\mathbf{A}}_K^{\dagger,r}$, where $y_n \in \operatorname{Fr} R_K$, and such that $v_R(y_n)+ns(r) \geq 0$ and $v_R(y_n) + ns(r) \to \infty$ as $n \to \infty$. By the preceding example, it follows that $x = \sum_{n\geq 0}(p/[\overline{\pi}^r])[\pi^r y_n]$, where $[\pi^r y_n] \in \widetilde{\mathbf{A}}_K^{+}$ and tends to 0 with respect to the weak topology. It is therefore sufficient to demonstrate that $p/[\overline{\pi}^r]$ can be written as $\sum_{n\geq 1} x_n(p/\pi^r)^n$, where $x_n \in \widetilde{\mathbf{A}}_K^+$ and $x_n \to 0$. However,

$$\frac{p}{[\overline{\pi}^r]} = \frac{p}{\pi^r}\frac{\pi^r}{[\overline{\pi}^r]} = \frac{p}{\pi^r}\left(1+\frac{p}{[\overline{\pi}^r]}\beta\right)^r = \frac{p}{\pi^r}\left(1+\frac{p}{[\overline{\pi}^r]}y\right),$$

where $\beta = \sum_{k\geq 1} p^{k-1}[\overline{\pi}^{r-1}\beta_k] \in (p,[\overline{\pi}])\widetilde{\mathbf{A}}_K^+$. Consequently, $p/[\overline{\pi}^r] = \sum_{n\geq 1} y^{n-1}(p/\pi^r)^n$. □

Remark 16. As a direct consequence of the preceding lemma, we may conclude that x is a unit in $\widetilde{\mathbf{A}}^{\dagger,r}$ if and only if the only slope of f_x is 0, and only for $n = 0$ does $0 = v_r(x) = v_R(x_n) + ns(r)$ hold.

Corollarry 3. *If $x = \sum_{n\geq 0} p^n[x_n] \in \widetilde{\mathbf{A}}^{\dagger,r}$ such that $[x_0] \neq 0$, then there is an $r_0 > r$ such that $\frac{x}{[x_0]}$ is a unit in $\widetilde{\mathbf{A}}^{\dagger,r_0}$.*

Proof. For the sake of simplicity, we shall denote $y = \frac{x}{[x_0]}$. The preceding remark will be used to prove the statement. By Proposition 14 (3), we may select an $r_1 \geq r$ such that for any $t \geq r_1$: $f_x(t) = v_R(x_0)$. Since $f_y(t) = f_x(t) + v_R(x_0)$, it follows that $v_t(y) = 0$ and that y lies in $\widetilde{\mathbf{A}}^{\dagger,t}$ for all $t \geq r_1$. Proposition 14 (2) indicates that, by choosing a number $r_0 > r_1$, it also holds that $n = \frac{p-1}{p} f_y'(r_0)$ is the only integer for which $0 = v_{r_0}(y) = v_R(y_n) + ns(r_0)$ holds. Consequently, since $f_y'(r_0) = 0$, it follows that $\frac{x}{[x_0]}$ is indeed a unit in $\widetilde{\mathbf{A}}^{\dagger,r_0}$. □

Definition 24.

$$\widetilde{\mathbf{B}}^{\dagger} := \varinjlim_{r>0} \widetilde{\mathbf{B}}^{\dagger,r} = \widetilde{\mathbf{A}}^{\dagger}[\frac{1}{p}]$$

is called the field of overconvergent elements. It is endowed with the direct limit topology.

Proposition 17. *$\widetilde{\mathbf{B}}^{\dagger}$ is a field.*

Proof. Since p is invertible, it is sufficient to show that for a given $x \in \widetilde{\mathbf{A}}^{\dagger,r}$ (with $r > 0$) ,it is invertible in $\widetilde{\mathbf{A}}^{\dagger,r_0}$ for a suitable $r_0 \geq r$.

Claim: If $y \in \widetilde{\mathbf{A}}^{\dagger,r} \cap p\widetilde{\mathbf{A}}$, then there exists a $0 \neq a \in \operatorname{Fr} R$ such that $\frac{[a]y}{p} \in \widetilde{\mathbf{A}}^{\dagger,r}$.

Proof of the Claim. Given that $w_n(\frac{y}{p}) = w_{n+1}(y)$, if we choose an $0 \neq a$ with $v_R(a) > s(r)$, then

$$w_n([a]\frac{y}{p}) + ns(r) \geq w_{n+1}(y) + (n+1)s(r).$$

□

If $0 \neq a \in \operatorname{Fr} R$, then $[a]$ is invertible in $\widetilde{\mathbf{B}}^{\dagger,r}$ (Remark 13). Consequently, in light of the claim, we may assume that for $x = \sum_{n\geq 0} p^n[x_n]$, $x_0 \neq 0$. Accordingly, by Corollary 3, there exists an $r_0 > r$ such that $\frac{x}{[x_0]}$ is invertible in $\widetilde{\mathbf{A}}^{\dagger,r_0}$ and thus in $\widetilde{\mathbf{B}}^{\dagger}$ as well. □

Remark 17. These rings also have a geometrical interpretation.. Let $\mathcal{A}^r$ be the ring of Laurent series $f(T) = \sum_{n\in\mathbb{Z}} a_n T^n$, where $a_n \in W(k) = \mathcal{O}_{K_0}$ such that $v(a_n) + nr \geq 0$ and that $\lim_{n\to -\infty} v(a_n) + nr = +\infty$. If $f \in \mathcal{A}^r$, we define $\omega_r(f) := \inf_n\{v_p(a_n) + nr\}$. Then it can be demonstrated that ω_r is a valuation on $\mathcal{A}^r$ and this ring can be interpreted as the ring of analytic functions on the annulus $\{0 < v(T) \leq r\}$, which are bounded by 1 with coefficients in $W(k)$. Similarly, if we define $\mathcal{B}^r$ as $\mathcal{A}^r[\frac{1}{p}]$, then $\mathcal{B}^r$ is the ring of analytic functions on the annulus $\{0 < v(T) \leq r\}$ whose coefficents belong to K_0. It can be demonstrated that $(\mathcal{A}^{1/re_K}, s(r)\omega_{1/re_K})$ is homeomorphic to $(\mathbf{A}_K^{\dagger,r}, v_r)$. In a similar fashion, $\mathcal{B}^{1/re_K}$ is isomorphic to $\mathbf{B}_K^{\dagger,r}$. This is the origin of the term "overconvergence".

Let K be a finite extension of $\mathbb{Q}_p$. In a similar manner to that previously outlined in this thesis, the following rings are now to be defined:

Definition 25.

$$\widetilde{\mathbf{B}}_K^{\dagger,r} := (\widetilde{\mathbf{B}}^{\dagger,r})^{G_{K_\infty}}; \quad \mathbf{B}^{\dagger,r} = \widetilde{\mathbf{B}}^{\dagger,r} \cap \mathbf{B}; \quad \mathbf{B}_K^{\dagger,r} = (\mathbf{B}^{\dagger,r})^{G_{K_\infty}};$$
$$\widetilde{\mathbf{A}}_K^{\dagger,r} := (\widetilde{\mathbf{A}}^{\dagger,r})^{G_{K_\infty}}; \quad \mathbf{A}^{\dagger,r} = \widetilde{\mathbf{A}}^{\dagger,r} \cap \mathbf{A}; \quad \mathbf{A}_K^{\dagger,r} = (\mathbf{A}^{\dagger,r})^{G_{K_\infty}};$$
$$\mathbf{B}^{\dagger} = \varinjlim_{r>0} \mathbf{B}^{\dagger,r}; \quad \mathbf{B}_K^{\dagger} = \varinjlim_{r>0} \mathbf{B}_K^{\dagger,r};$$
$$\mathbf{A}^{\dagger} = \varinjlim_{r>0} \mathbf{A}^{\dagger,r}; \quad \mathbf{A}_K^{\dagger} = \varinjlim_{r>0} \mathbf{A}_K^{\dagger,r}.$$

Let V be a p-adic representation of G_K. Then

Definition 26.

$$D^{\dagger}(V) := (V \otimes \mathbf{B}^{\dagger})^{G_{K_\infty}}$$

is a vector space of dimension $\leq \dim(V)$ over $\mathbf{B}^{\dagger}{}_K$.

Definition 27. *V is an overconvergent p-adic representation, if*

$$D^{\dagger}(V) \otimes_{\mathbf{B}^{\dagger}{}_K} \mathbf{B}^{\dagger} \cong V \otimes \mathbf{B}^{\dagger}.$$

Equivalently, if $\dim_{\mathbf{B}^{\dagger}{}_K}(D^{\dagger}(V)) = \dim(V)$.

5. Applications of Sen's method

In this chapter, we provide two applications of Sen's method, as previously outlined. The first is the classical example of Sen's theory, while the second concerns the construction of (ϕ, Γ)-modules.

5.1. **The field $\mathbb{C}_p$ and the Sen operator.**

This section will present the classic example of Sen theory. This topic is not a central focus of the thesis, and thus will be discussed from a purely historical perspective. The results will be presented without detailed proof.

The first non-trivial example of a ring that satisfies the Tate-Sen conditions is the field $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ with the p-adic valuation, val_p, and the action of G_K. This example was first proposed by Tate in [Tat67]. Let L be a finite extension of K. For $n \geq 1$, let $L_n = L(\mu_{p^n})$ and set $L_\infty = \varinjlim_n L_n$. Let $H_L = \mathrm{Gal}(\overline{K}/L_\infty)$ and $\Gamma_L = \mathrm{Gal}(L_\infty/L)$. Theorem 1 then implies that $\mathbb{C}_p^{H_L} = \widehat{L}_\infty$, the completion of L_∞ for val_p. Moreover, if $n \in \mathbb{N}$ and $x \in L_\infty$, it can be demonstrated that $[L_{n+k} : L_n]^{-1} \operatorname{Tr}_{L_{n+k}/L_n}(x)$ is independent of the choice of integer k, where $x \in L_{n+k}$, and that the map from L_∞ to L_n thus defined extends uniform continuously to a map $R_{L,n} : \widehat{L}_\infty \to L_n$. (see in Hungarian, [Hev], §5.)

Proposition 18. *$\widetilde{\Lambda} = \mathbb{C}_p$ satisfies the conditions (TS1), (TS2) and (TS3), with $\widetilde{\Lambda}^{H_L} = \widehat{L}_\infty$, $\Lambda_{H_L,n} = L_n$, $R_{H_L,n} = R_{L,n}$ and $\mathrm{val}_\Lambda = \mathrm{val}_p$, the constants $c_1, c_2 > 0$ and $c_3 > 1/(p-1)$ can be chosen arbitrarily.*

In the remainder of this section, we will assume that the constants $c_1, c_2 > 0$ and $c_3 > 1/(p-1)$ satisfy the inequality $c_1 + 2c_2 + 2c_3 < \mathrm{val}_p(12p)$. We will denote $n(L) = n(G_L)$.

Proposition 19. *Let S be a Banach algebra, T a d-dimensional $\mathcal{O}_S$-representation of G_K and $V = S \otimes_{\mathcal{O}_S} T$. Let L be a finite Galois extension of K such that G_L acts trivially on $T/12pT$ and let $n \geq n(L)$. Then $(S\widehat{\otimes}\mathbb{C}_p) \otimes_S V$ contains a unique free $S \otimes L_n$-submodule $D_{\mathrm{Sen}}^{L_n}(V)$ of rank d satisfying the following properties*

1. *$D_{\mathrm{Sen}}^{L_n}(V)$ is fixed under H_L and stable under G_K;*
2. *the natural map $(S\widehat{\otimes}\mathbb{C}_p) \otimes_{S\otimes L_n} D_{\mathrm{Sen}}^{L_n}(V) \to (S\widehat{\otimes}\mathbb{C}_p) \otimes_S V$ is an isomorphism;*
3. *$D_{\mathrm{Sen}}^{L_n}(V)$ has a basis over $S \otimes L_n$ which is c_3-fixed by Γ_L.*

It follows that $S/\mathfrak{m}_x \otimes_S D_{\mathrm{Sen}}^{L_n}(V) \simeq D_{\mathrm{Sen}}^{L_n}(V_x)$.

Proof. This follows from Sen's method (more precisely from Theorem 2 and Remark 5 and from the fact that $\mathbb{C}_p$ verifies the Tate-Sen conditions. The final part follows from the proposition being applied to $S/\mathfrak{m}_x$ and from the fact that the image of $S/\mathfrak{m}_x \otimes_S D_{\mathrm{Sen}}^{L_n}(V)$ in $(E_x \otimes \mathbb{C}_p) \otimes_{E_x} V_x$ satisfies conditions (1), (2) and (3). □

Let $\gamma \in \Gamma_L$ such that $n(\gamma) \geq n$, then γ acts trivially on L_n and linearly on $D_{\mathrm{Sen}}^{L_n}(V)$. If M_γ is the matrix describing the action of γ in a $D_{\mathrm{Sen}}^{L_n}(V)$-basis such that $\mathrm{val}_p(M_\gamma - 1) > 0$, then the logarithm $\log \gamma$ of γ can be defined as the series $-\sum_{m=1}^{+\infty}(1-\gamma)^m/m$. It is straightforward to show that $\log(\gamma^k) = k \log \gamma$. Since Γ_L being a p-adic Lie group of dimension 1, it follows that operator $(\log_p \chi(\gamma))^{-1}$ is independent of the choice of γ. We will denote it by Θ_{Sen}. Since the choice of γ is unrestricted, we may select γ from the centre of $\widetilde{\Gamma}_L$. Consequently, the operator Θ_{Sen} commutes with the action of $\widetilde{\Gamma}_L$. This in turn implies that the coefficients of its characteristic polynomial and minimal polynomial are in $S \otimes K$. The operator Θ_{Sen} is referred to as the *Sen operator*. If $x \in \mathcal{X}$, the eigenvalues of $\Theta_{\mathrm{Sen}}(x)$ are designated as the *generalised Hodge-Tate weights* of V_x.

Remark 18. Let $n = n(L)$ and $\gamma \in \Gamma_L$ be such that $n(\gamma) = n(L)$, and using the fact that the matrix M_γ of γ in a c_3-fixed basis of $D_{\mathrm{Sen}}^{L_n}(V)$ satisfies $\mathrm{val}_p(M_\gamma - 1) > c_3 > 1/(p-1)$, it can be concluded that the eigenvalues of $\log M_\gamma$ valued $> c_3$ and thus the generalised Hodge-Tate weights of V possess p-adic value $> 1/(p-1) - n(L)$.

5.2. Tate-Sen condition for $\widetilde{\mathbf{B}}^{\dagger,r}$.

In order to facilitate comprehension, we are now recalling some of the notations that have been employed thus far and introducing some new ones: K is finite extension of $\mathbb{Q}_p$, with residue field k_K. $K_0 = W(k)[\frac{1}{p}]$ is the maximal unramified subfield of K with ring of integers $\mathcal{O}_{K_0} = W(k)$ and with residue field k. $\mathbf{E}_K = k_K((\overline{\pi}_K))$ with ring of integers denoted by $\mathbf{E}_K^+ := k_K[[\overline{\pi}_K]]$. $\mathbf{A}_K = (\mathcal{O}_K[[\pi]][\pi^{-1}])^\wedge$ and let us denote $\mathbf{A}_K^+ := \mathcal{O}_K[[\pi]]$. $\mathbf{B}_K = \mathbf{A}_K[\frac{1}{p}]$. $\mathbf{E}_K/\mathbf{E}_{K_0}$ is totally ramified with index $e_K = [K : K_0]$ and $\mathbf{E}_{K_0}/\mathbf{E}_{\mathbb{Q}_p}$ is unramified of degree f_K. Therefore, $d_K = e_K f_K = [K : \mathbb{Q}_p] = [\mathbf{E}_K : \mathbf{E}_{\mathbb{Q}_p}] = [\mathbf{B}_K : \mathbf{B}_{\mathbb{Q}_p}]$.[4] Moreover, $\widetilde{\mathbf{A}}_K^+ = W(R_K)$ and $\widetilde{\mathbf{A}}_K = W(\mathrm{Fr}\, R_K)$.

In this section, we demonstrate that if $r > 0$, then $\widetilde{\Lambda} = \widetilde{\mathbf{B}}^{\dagger,r}$ satisfies the Tate-Sen conditions with $\Lambda_{H_K,n} = \varphi^{-n}(\widetilde{\mathbf{B}}_K^{\dagger,p^n r})$ for $\mathrm{val}_\Lambda = v_r$ and for some maps $R_{K,n} : \widetilde{\mathbf{B}}_K^{\dagger,r} \to \varphi^{-n}(\widetilde{\mathbf{B}}_K^{\dagger,p^n r})$, defined below. For the sake of simplicity, we assume that r is an integer ≥ 1.

Proposition 20 (TS1)**.** *Let L/K be finite extensions of $\mathbb{Q}_p$. Fix an $r > 0$. Then, for any $\delta > 0$, there exists an $\alpha \in \widetilde{\mathbf{B}}_L^{\dagger,r}$ such that $v_r(\alpha) > -\delta$ and $\mathrm{Tr}_{L_\infty/K_\infty}(\alpha) = 1$.*

It should be noted that $\delta > 0$ was chosen here arbitrarily.

Proof. Proposition 7 implies that the map $\mathrm{Tr}_{L_\infty/K_\infty}$ on the field $\mathrm{Fr}\, R_L$ coincides with $\mathrm{Tr}_{\mathrm{Fr}\, R_L/\mathrm{Fr}\, R_K}$. Since the extension $\mathrm{Fr}\, R_L/\mathrm{Fr}\, R_K$ is separable, there exists a $\beta \in \mathrm{Fr}\, R_L$ such that $\mathrm{Tr}_{L_\infty/K_\infty}(\beta) = 1$. Given that $v_R(\varphi^{-n}(\beta)) = p^{-n} v_R(\beta)$, it is possible to assume that $v_R(\beta)$ is as small as desired; in particular, that $v_R(\beta) > \max\{-s(r), -\delta\}$. Consequently, we have $\mathrm{Tr}_{L_\infty/K_\infty}([\beta]) = 1 + \sum_{k \geq 1} p^k [x_k]$ in $\widetilde{\mathbf{A}}$ with $v_R(x_k) \geq v_R(\beta) > -ks(r)$. Therefore, $\mathrm{Tr}_{L_\infty/K_\infty}([\beta]) \in \widetilde{\mathbf{A}}^{\dagger,r}$ and it is a unit, since $v_r(\sum_{k\geq 1} p^k [x_k]) > 0$ (Lemma 17). Hence, the choice $\alpha = [\beta]/\mathrm{Tr}_{L_\infty/K_\infty}([\beta])$ is appropriate. □

[4]The latter two equalities are demonstrated by the Artin's lemma.

Let us define $I = \mathbb{Z}[\frac{1}{p}] \cap [0,1)$ and let $I_n = \{x \in I \mid v_p(x) \geq -n\}$ for $n \geq 0$. Consequently, I_n is a system of representatives for $p^{-n}\mathbb{Z}_p/\mathbb{Z}_p$ and I is a system of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$. Note that if x lies in a perfect ring and $i \in I$, then x^i is well defined. If $\{x_i\}_{i\in I}$ is a sequence, we say that $x_i \to 0$, if for any U open neighbourhood of 0, $|\{i \in I \mid x_i \notin U\}| < \infty$.

Lemma 18. *Every $x \in \operatorname{Fr} R_{\mathbb{Q}_p}$ can be written in a unique way as $x = \sum_{i\in I} \varepsilon^i a_i(x)$, where $a_i(x) \in \mathbf{E}_{Q_p}$ and $a_i(x) \to 0$. Moreover, $x \in R_{\mathbb{Q}_p}$ if and only if $a_i(x) \in \mathbf{E}^+_{\mathbb{Q}_p}$.*

Proof. Since $\mathbb{F}_p$ is perfect, an element $x \in \varphi^{-n}(\mathbf{E}^+_{\mathbb{Q}_p})$ can be expressed as $\sum_{i\in I_n} \varepsilon^i a_i(x)$, where $a_i(x) \in \mathbf{E}^+_{\mathbb{Q}_p}$. The functions $a_i(\cdot)$ are extended linearly to $\varphi^{-n}(\mathbf{E}_{Q_p})$. Let $k \in \mathbb{Z}$, then $x \in \pi^k R_{\mathbb{Q}_p}$ if and only if $a_i(x) \in \pi^k \mathbf{E}^+_{Q_p}$ for all $i \in I_n$. Consequently,

$$v_R(x) - v_R(\pi) < \min_{i\in I_n}\{v_R(a_i(x))\} \leq v_R(x).$$

Thus, the function $x \mapsto a_i(x)$ is uniformly continuous on $(\mathbf{E}_{Q_p})^{\text{perf}} = \cup_{n\geq 0}\varphi^{-n}(\mathbf{E}_{Q_p})$. By Proposition 8, $(\mathbf{E}_{Q_p})^{\text{perf}}$ is dense in $\operatorname{Fr} R_{\mathbb{Q}_p}$, therefore the functions a_i extend to $\operatorname{Fr} R_{\mathbb{Q}_p}$, and every element $x \in \operatorname{Fr} R_{\mathbb{Q}_p}$ may be written as $\sum_{i\in I} \varepsilon^i a_i(x)$, where $a_i(x) \in \mathbf{E}_{\mathbb{Q}_p}$ and $a_i(x) \to 0$.

Finally, if $\sum_{i\in I} \varepsilon^i a_i(x) = 0$ and $x_n := \sum_{i\in I_n} \varepsilon^i a_i(x)$, then $x_n \in \varphi^{-n}(\mathbf{E}_{Q_p})$. Since $x_n \to 0$, it follows that $a_i(x) = 0$ for all $i \in I$. □

Corollarry 4. *Every $x \in \widetilde{\mathbf{A}}_{\mathbb{Q}_p}$ can be expressed in a unique way as $x = \sum_{i\in I}[\varepsilon^i]a_i(x)$, where $a_i(x) \in \mathbf{A}_{\mathbb{Q}_p}$ and $a_i(x) \to 0$ with respect to the weak topology. Moreover, $x \in \widetilde{\mathbf{A}}^+_{\mathbb{Q}_p}$ if and only if $a_i(x) \in \mathbf{A}^+_{\mathbb{Q}_p}$ for any $i \in I$.*

Proof. Let $M := \{x = \sum_{i\in I}[\varepsilon^i]a_i(x) \mid a_i(x) \in \mathbf{A}_{\mathbb{Q}_p},\ a_i(x) \to 0 \text{ w. r. t. } v_r\}$. Then, by the preceding lemma, the map $M \to \widetilde{\mathbf{A}}_{\mathbb{Q}_p}$ is bijective $\mod p$, and hence in general. A similar argument can be employed to demonstrate the second part of the proposition. □

Let

$$R_{\mathbb{Q}_p,n} : \widetilde{\mathbf{A}}_{\mathbb{Q}_p} \to \varphi^{-n}(\mathbf{A}_{\mathbb{Q}_p}), \quad \forall n \in \mathbb{N},$$

such that $R_{\mathbb{Q}_p,n}(x) = \sum_{i\in I_n}[\varepsilon^i]a_i(x)$. These maps extends to $\widetilde{\mathbf{B}}_{\mathbb{Q}_p}$ by $\mathbb{Q}_p$-linearity.

Proposition 21. *The map $R_{\mathbb{Q}_p,n} : \widetilde{\mathbf{B}}_{\mathbb{Q}_p} \to \varphi^{-n}(\mathbf{B}_{\mathbb{Q}_p})$ is $\varphi^{-n}(\mathbf{B}_{\mathbb{Q}_p})$-linear and is the identity on $\varphi^{-n}(\mathbf{B}_{\mathbb{Q}_p})$. Let r be an integer, greater than 1. Then,*

(1) $R_{\mathbb{Q}_p,n}(\widetilde{\mathbf{A}}^{\dagger,r}_{\mathbb{Q}_p}) \subset \widetilde{\mathbf{A}}^{\dagger,r}_{\mathbb{Q}_p}$;
(2) if $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathbb{Q}_p}$, then $v_r(R_{\mathbb{Q}_p,n}(x)) \geq v_r(x) - p/(p-1)$;
(3) if $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathbb{Q}_p}$, then $v_r(R_{\mathbb{Q}_p,n}(x) - x) \to +\infty$ as $n \to +\infty$.

Proof. From the preceding corollary, it can be deduced that the map is $R_{\mathbb{Q}_p,n} : \widetilde{\mathbf{B}}_{\mathbb{Q}_p} \to \varphi^{-n}(\mathbf{B}_{\mathbb{Q}_p})$ is indeed $\varphi^{-n}(\mathbf{B}_{\mathbb{Q}_p})$-linear and is the identity on $\varphi^{-n}(\mathbf{B}_{\mathbb{Q}_p})$.

If $x \in \widetilde{\mathbf{A}}^{\dagger,r}_{\mathbb{Q}_p}$, then Proposition 16 implies that it can be expressed as $\sum_{k\geq 0} x_k(p/\pi^r)^k$, where $x_k \in \widetilde{\mathbf{A}}^+_{\mathbb{Q}_p}$ and $x_k \to 0$ with respect to the weak topology. The fact that $R_{\mathbb{Q}_p,n}(x) = \sum_{k\geq 0} R_{\mathbb{Q}_p,n}(x_k)(p/\pi^r)^k$ and that $R_{\mathbb{Q}_p,n}(\widetilde{\mathbf{A}}^+_{\mathbb{Q}_p}) \subset \widetilde{\mathbf{A}}^+_{\mathbb{Q}_p}$ (by Corollary 4) demonstrates item (1).

Let $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathbb{Q}_p}$. Since $v_r(\pi) = p/(p-1)$ there exist $k, \ell \in \mathbb{Z}$ such that $p^k\pi^\ell x \in \widetilde{\mathbf{A}}^{\dagger,r}_{\mathbb{Q}_p}$ and $0 \leq v_r(p^k\pi^\ell x) < p/(p-1)$. Consequently, by (1), $v_r(R_{\mathbb{Q}_p,n}(p^k\pi^\ell x)) \geq 0$, which implies (2).

Finally, (3) follows from the fact that $R_{\mathbb{Q}_p,n}(x) \to x$ as $n \to \infty$. $\square$

Lemma 19. *Let $e_1, \dots, e_{d_K}$ be a basis of ${\mathbf{B}^\dagger}_K$ over ${\mathbf{B}^\dagger}_{\mathbb{Q}_p}$. Them, for any $m \geq 0$, $\varphi^{-m}(e_1), \dots, \varphi^{-m}(e_{d_K})$ is a basis of $\widetilde{\mathbf{B}}_K$ over $\widetilde{\mathbf{B}}_{\mathbb{Q}_p}$.*

Proof. It can be demonstrated with ease that the map ${\mathbf{B}^\dagger}_K \otimes_{{\mathbf{B}^\dagger}_{\mathbb{Q}_p}} \widetilde{\mathbf{B}}_{\mathbb{Q}_p} \to \widetilde{\mathbf{B}}_K$ is injective. Furthermore, by comparing dimensions, it can be shown that this map is surjective as well. This proves the case where $m = 0$. For $m \geq 0$, it is sufficient to observe that $\varphi : \widetilde{\mathbf{B}}_K \to \widetilde{\mathbf{B}}_K$ is a bijection. $\square$

The dual basis with respect to the perfect pairing $(x, y) \mapsto \mathrm{Tr}_{K_\infty/K_{0,\infty}}(xy)$ is denoted by $e_1^*, \dots, e_{d_K}^*$. Thus, an element $x \in \widetilde{\mathbf{B}}_K$ may be expressed as $\sum_{i=1}^{d_K} x_i\varphi^{-m}(e_i^*)$, where $x_i = \mathrm{Tr}_{K_\infty/K_{0,\infty}}(x\varphi^{-m}(e_i))$.(For further details, please refer to [FO], A. 3. 4.)

Definition 28. *Define*

$$R_{K,n}(x) = \sum_{i=1}^{d_K} R_{\mathbb{Q}_p,n}(x_i)\varphi^{-m}(e_i^*),$$

where $x_i = \mathrm{Tr}_{K_\infty/K_{0,\infty}}(x\varphi^{-m}(e_i))$.

A straightforward calculation reveals that the map $R_{K,n}$ defined in this manner is independent of the choice of basis and the choice of $m \geq 0$. Moreover, it can be shown that $R_{K,n}(\widetilde{\mathbf{B}}_K) \subset \varphi^{-n}(\mathbf{B}_K)$.

We will now prove that $\widetilde{\mathbf{B}}^{\dagger,r}$ satisfies condition (TS2).

Proposition 22 (TS2)**.** *Let $c_2 \in (\frac{p}{p-1}, +\infty)$ and $r \in \mathbb{Z}_{\geq 1}$. The map $R_{K,n} : \widetilde{\mathbf{B}}_K \to \varphi^{-n}(\mathbf{B}_K)$ is $\varphi^{-n}(\mathbf{B}_K)$-linear and is the identity on $\varphi^{-n}(\mathbf{B}_K)$. Furthermore, there exists $n(K)$ such that if $n \geq n(K)$, then*

(1) $R_{K,n}(\widetilde{\mathbf{B}}^{\dagger,r}_K) \subset \widetilde{\mathbf{B}}^{\dagger,r}_K \cap \varphi^{-n}(\mathbf{B}_K) = \varphi^n(\mathbf{B}^{\dagger,p^n r}_K)$;

(2) *if* $x \in \widetilde{\mathbf{B}}_K^{\dagger,r}$, *then* $v_r(R_{K,n}(x)) \geq v_r(x) - c_2$;
(3) *if* $x \in \widetilde{\mathbf{B}}_K^{\dagger,r}$, *then* $v_r(R_{K,n}(x) - x) \to +\infty$ *as* $n \to \infty$.

Proof. The initial portion of the proposition is derived from Proposition 21.

Let $e_1, \dots, e_{d_K}$ be a basis of $\mathbf{B}_K^\dagger$ over $\mathbf{B}_{\mathbb{Q}_p}^\dagger$, then there exists an $s > 0$ such that e_i and e_i^* belong to $\mathbf{B}_K^{\dagger,s}$. Furthermore, if $p^m r \geq s$, then $\varphi^{-m}(e_i) \in \varphi^{-m}(\mathbf{B}_K^{\dagger,p^m r}) \subset \widetilde{\mathbf{B}}_K^{\dagger,r}$. This yields (1) with $n(K) = m$. It is possible to multiply the e_i's by $p^k\pi^\ell$ such that $v_r(\varphi^{-m}(e_i)) \geq 0$ and $e_i^* \in \widetilde{\mathbf{A}}_k$ for $1 \leq i \leq d_K$. Consequently, $v_r(\varphi^{-n}(e_i^*)) \geq 0$ for all $n \geq m$. Moreover, if $\delta = c_2 - p/(p-1) > 0$, then $v_r(\varphi^{-n}(e_i^*)) \geq -\delta$ for sufficiently large n. In conclusion, for $n(K) = \max(n(\delta), m)$, item (2) and (3) follow from Proposition 21 (2) and (3), respectively. □

Remark 19. It should be noted that with a modicum of ingenuity, it is possible to make c_2 arbitrary.

In order to prove the third condition (TS3), it is necessary to invoke a technical tool concerning the action of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ on $\widetilde{\mathbf{B}}_K^{\dagger,r}$. The proof can be found in [Col08] (§9.) or in [Ber] (§23.).

Proposition 23. *There exists an* $m(K)$ *such that if* $\gamma \in \Gamma_K$ *with* $n(\gamma) \geq m(K)$ *and* $s \geq p^{n(\gamma)+1}$ *and* $1 \leq i \leq p-1$, *then*

(1) *the map* $\gamma - 1 : [\varepsilon]^i\varphi(\mathbf{B}_K^{\dagger,s/p}) \to [\varepsilon]^i\varphi(\mathbf{B}_K^{\dagger,s/p})$ *is invertible;*
(2) *For* $x \in [\varepsilon]^i\varphi(\mathbf{B}_K^{\dagger,s/p})$, $v_s((1-\gamma)^{-1}x) \geq v_s(x) - p^{n(\gamma)}v_R(\overline{\pi})$.

Proposition 24 (TS3)**.** *There exists* $c_3 > 0$ *and* $m(K) \geq n(K)$ *such that if* $\gamma \in \Gamma_K$ *and* $n \geq \max(n(\gamma), m(K))$, *then*

(1) $(1-\gamma)$ *is invertible on* $(1 - R_{K,n})(\widetilde{\mathbf{B}}_K^{\dagger,r})$;
(2) $v_r((1-\gamma)^{-1}x) \geq v_r(x) - c_3$ *if* $x \in (1 - R_{K,n})(\widetilde{\mathbf{B}}_K^{\dagger,r})$.

Proof. Let $m(K) \geq n(K)$ be as in Proposition 23. By Proposition 22, item (3) and by the fact that for any $m \geq 1$, $I_m = I_{m-1} \sqcup (\bigsqcup_{i=1}^{p-1} ip^{-m} + I_{m-1})$, we may express an element x of $\widetilde{\mathbf{B}}_K^{\dagger,r}$ in the following form:

$$x = R_{K,n}(x) + \sum_{m \geq n+1} \sum_{i=1}^{p-1} x_{m,i}, \text{ where}$$
$$x_{m,i} = [\varepsilon^{ip^{-m}}]R_{K,m-1}([\varepsilon^{-ip^{-m}}]x) \in \varphi^{-m}([\varepsilon]^i\varphi(\mathbf{B}_K^{\dagger,rp^{m-1}})),$$

where $v_r(x_{m,i}) \geq v_r(x) - c_2$.

It remains to select a $\gamma \in \Gamma_K$ such that $n(\gamma) \geq m(K)$ and let $n \geq n(\gamma)$. Once more, by Proposition 23, $1-\gamma$ is invertible on $[\varepsilon]^i \varphi(\mathbf{B}_K^{\dagger, rp^{m-1}})$ and

$$v_{rp^m}((1-\gamma)^{-1}\varphi^m(x_{m,i})) \geq v_{rp^m}(\varphi^m(x_{m,i})) - p^{n(\gamma)} v_R(\overline{\pi}).$$

This implies that

$$v_r((1-\gamma)^{-1}x_{m,i}) \geq v_r(x_{m,i}) - p^{n(\gamma)-m} v_R(\overline{\pi}),$$

moreover,

$$(1-R_{K,n})(x) = (1-\gamma)\sum_{m\geq n+1}\sum_{i=1}^{p-1}(1-\gamma)^{-1}x_{m,i}.$$

This demonstrates the proposition with the constant $c_3 = c_2 + p/(p-1) + p^{-m(K)}C$. $\square$

5.3. **Overconvergent representations.**

Proposition 25. *The ring $\widetilde{\Lambda} = \widetilde{\mathbf{B}}^{\dagger,r}$ satisfies the conditions (TS1), (TS2) and (TS3) for $\widetilde{\Lambda}^{H_L} = \varphi^{-n}(\widetilde{\mathbf{B}}^{\dagger,p^n r})$, $\mathrm{val}_\Lambda = v_r$ and $R_{H_L,n} = R_{L,n}$, with arbitrary constants $c_1 > 0, c_2 > 0$ and $c_3 > 1/(p-1)$.*

Proof. The veracity of conditions (TS1), (TS2), and (TS3), respectively, follows from Proposition 20, 22 and 24. $\square$

Lemma 20. *Let $M \in 1 + pM_d(\widetilde{\mathbf{A}}_K)$ be such that there exist $U, V \in 1 + pM_d(\mathbf{A}_K)$ satisfying $U\gamma(M) = MV$, for any Γ_H. It follows that $M \in 1 + pM_d(\mathbf{A}_K)$.*

Proof. Let $R_{K,0} : \widetilde{\mathbf{A}}_K \to \mathbf{A}_K$ be the mapping defined in the previous section and $N := (1 - R_{K,0})(M)$. It must be verified that $N = 0$. For the sake of contradiction, let $k \in \mathbb{N}$ be the largest integer such that $N \in p^k M_d(\mathbf{A}_K)$ and let $\overline{N}$ be the image of $p^{-k}N$ in $M_d(\mathbf{E}_K)$. Since $R_{K,0}$ commutes with the action of γ and it is $\mathbf{A}_K$-linear, it follows that $U\gamma(N) = NV$, and therefore also $U\gamma(p^{-k}N) = (p^{-k}N)V$. Upon reduction of this identity $\mod p\mathbf{A}_K$, we obtain that $\gamma(\overline{N}) = \overline{N}$. On the other hand, since $R_{K,0}(N) = 0$, it follows that $R_{K,0}(\overline{N}) = 0$. This implies that

$$0 = R_{K,0}(\gamma(\overline{N})) = \gamma(R_{K,0}(\overline{N})) = \gamma(\overline{N}) = \overline{N}.$$

Therefore, $N \in p^{k+1}M_d(\mathbf{A}_K)$, which is contrary to the initial assumption. $\square$

In the remainder of this section, we will fix some constants $c_1 > 0, c_2 > 0$ and $c_3 > 1/(p-1)$, such that $c_1 + c_2 + 2c_3 < v_p(12p)$. Let T be a $\mathbb{Z}_p$-representation of G_K, $s > 0$, and let L be a finite Galois extension of K.

Definition 29.

$$D_L^{\dagger,s}(T) := (\mathbf{A}^{\dagger,s} \otimes_{\mathbb{Z}_p} T)^{H_L}, \quad D_{L,n}^{\dagger,s}(T) := \varphi^{-n}(D_L^{\dagger,p^n s}(T)).$$

$D_L^{\dagger,s}(T)$ is an $\mathbf{A}_L^{\dagger,s}$-module with Galois action of $\widetilde{\Gamma}_L$. $D_{L,n}^{\dagger,s}(T)$ is an $\mathbf{A}_{L,n}^{\dagger,s} = \varphi^{-n}(\mathbf{A}_L^{\dagger,p^n s})$-module.

Theorem 5. *Let K be a finite extension of $\mathbb{Q}_p$ and let T be a $\mathbb{Z}_p$-representation of G_K. Let L be a finite Galois extension of K such that G_L acts trivially on $T/12pT$. If $n \geq n(L)$, then $D_{L,n}^{\dagger,(p-1)/p}(T)$ is the unique $\mathbf{A}_{L,n}^{\dagger,(p-1)/p}$-submodule of rank d of $\widetilde{\mathbf{A}}^{\dagger,(p-1)/p} \otimes_{\mathbb{Z}_p} T$ satisfying the following properties:*

(1) $D_{L,n}^{\dagger,(p-1)/p}(T)$ is fixed under H_L and stable under G_0;
(2) the natural map $\widetilde{\mathbf{A}}^{\dagger,(p-1)/p} \otimes_{\mathbf{A}_{L,n}^{\dagger,(p-1)/p}} D_{L,n}^{\dagger,(p-1)/p}(T) \to \widetilde{\mathbf{A}}^{\dagger,(p-1)/p} \otimes_{\mathbb{Z}_p} T$ is an isomorphism;
(3) the $\mathbf{A}_{L,n}^{\dagger,(p-1)/p}$-module $D_{L,n}^{\dagger,(p-1)/p}(T)$ has a basis such that the corresponding matrix W_γ of γ in this basis satisfies $v_{(p-1)/p}(W_\gamma - 1) > c_3$, for any $\gamma \in \Gamma_L$.

Proof. Given that $v_p(12p) > c_1 + 2c_2 + 2c_3$, it follows immediately from Proposition 25 and Theorem 2 that the uniqueness of the module satisfying the above conditions is guaranteed. It remains to verify that the module defined by Theorem 2 coincides with $(\varphi^{-n}(\mathbf{A}^{\dagger,p^{n-1}(p-1)} \otimes_{\mathbb{Z}_p} T))^{H_L}$. Nevertheless, the proof of Theorem 2 provides a concrete construction of the module. If U_τ denotes the matrix corresponding to the action of $\tau \in G_K$ in a basis of T, then from Proposition 3 provides us with a matrix $M \in 1 + 12pM_d(\widetilde{\mathbf{A}}^{\dagger,(p-1)/p})$ such that $v_{(p-1)/p}(M-1) > c_2 + c_3$ and that the cocycle $\tau \mapsto M^{-1}U_\tau\tau(M)$ is trivial on H_L and has values in $\mathrm{GL}_d(\mathbf{A}_{L,n}^{\dagger,(p-1)/p})$. Consequently, the cocycle $\tau \mapsto C_\tau = \varphi^n(M^{-1}U_\tau\tau(M)) = \varphi^n(M)U_\tau\varphi^n(\tau(M))$ is also trivial on H_L and has values in $\mathrm{GL}_d(\mathbf{A}_L^{\dagger,p^{n-1}(p-1)})$.

On the other hand, in the theory of (φ, Γ)-modules as presented in [FO], it is shown that there exists a matrix $P \in 1 + 12pM_d(\mathbf{A})$ such that the cocycle $\tau \mapsto D_\tau = P^{-1}U_\tau\tau(P)$ is trivial on H_L and takes values in $\mathrm{GL}_d(\mathbf{A}_L)$.

The elimination of U_τ between C_τ and D_τ, accompanied by the assumption that $N = \varphi^n(M)^{-1}P$, results in the relation $ND_\tau = C_\tau\tau(N)$. In particular, since $C_\tau = D_\tau = 1$ if $\tau \in H_L$, it follows that N is stable under the action of $\tau \in H_L$, that is, $N \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L)$. Moreover, since $U_\tau - 1$ is divisible by $12p$ if $\tau \in G_L$ and since the same is true for M and P, the matrices N and, C_τ and D_τ belong to $1 + 12pM_d(\widetilde{\mathbf{A}}_L)$, if $\tau \in G_L$. However, since the coefficients of C_τ and D_τ are drawn from $\mathbf{A}_L$, it follows from Lemma 20 that the coefficients of N are also drawn from $\mathbf{A}_L$ and that M has coefficients from $\varphi^{-n}(\mathbf{A}_L)$.

This leads to the conclusion that the basis $e_1, \ldots, e_d$ in $\widetilde{\mathbf{A}}^{\dagger,(p-1)/p} \otimes_{\mathbb{Z}_p} T$, which was defined by the matrix M as in the proof of Theorem 2, consists of elements of $D_{L,n}^{\dagger,(p-1)/p}(T)$. Furthermore, the matrix M belongs to $\mathrm{GL}_d(\widetilde{\mathbf{A}}^{\dagger,(p-1)/p})$ and its coefficients are elements of $\varphi^{-n}(\mathbf{A}^{\dagger,p^{n-1}(p-1)})$. This implies that $e_1, \ldots, e_d$ form a basis of

$\varphi^{-n}(\mathbf{A}^{\dagger,p^{n-1}(p-1)})\otimes_{\mathbb{Z}_p}T$ over $\varphi^{-n}(\mathbf{A}^{\dagger,p^{n-1}(p-1)})$. From this, we can infer that $D_{L,n}^{\dagger,(p-1)/p}(T)$ is a $\mathbf{A}_{L,n}^{\dagger,(p-1)/p}$-submodule generated by $e_1,\dots,e_d$. This completes the proof. □

Corollarry 5. *Let K be a finite extension of $\mathbb{Q}_p$ and let T be a $\mathbb{Z}_p$-representation of G_K. Let L be a finite Galois extension of K such that G_L acts trivially on $T/12pT$. If $s \geq (p-1)p^{n(L)-1}$, then $D_L^{\dagger,s}(T)$ is a free $\mathbf{A}^{\dagger,s}$-module of rank d, and the natural map $\mathbf{A}^{\dagger,s}\otimes_{\mathbf{A}_L^{\dagger,s}} D_L^{\dagger,s}(T)\to \mathbf{A}^{\dagger,s}\otimes_{\mathbb{Z}_p} T$ is an isomorphism.*

Proof. Theorem 5 and the fact that $D_L^{\dagger,(p-1)p^{n(L)-1}}(T) = \varphi^{n(L)}(D_{L,n(L)}^{\dagger,(p-1)/p}(T))$ demonstrate that if $s=(p-1)p^{n(L)-1}$, then $D_L^{\dagger,s}(T)$ is a free module of rank d and the map $\widetilde{\mathbf{A}}^{\dagger,s}\otimes_{\widetilde{\mathbf{A}}_L^{\dagger,s}} D_L^{\dagger,s}(T)\to \mathbf{A}^{\dagger,s}\otimes_{\mathbb{Z}_p} T$ an isomorphism. By writing a basis of T in terms of a basis of $D_L^{\dagger,s}(T)$, we obtain a matrix in $M_d(\mathbf{A}^{\dagger,s})$. This matrix is also due to the previous isomorphism in $\mathrm{GL}_d(\widetilde{\mathbf{A}}^{\dagger,s})$, and thus in conclusion in $\mathrm{GL}_d(\mathbf{A}^{\dagger,s})$. This implies the corollary if $s=(p-1)p^{n(L)-1}$. In general, the statement can be demonstrated by extension of scalars. □

We now proceed to descent from L to K.

Lemma 21. *Let L be a finite Galois extension of K, then there exists a number $s(L/K)$ such that*

(1) if $s\geq s(L/K)$, then there exists an $\alpha\in\mathbf{B}_L^{\dagger,s}$ which generates a normal basis[5] of $\mathbf{B}_L^{\dagger,s}$ over $\mathbf{B}_K^{\dagger,s}$ and the discriminant of the minimal poynomial of α is invertible in $\mathbf{B}_K^{\dagger,s}$;
(2) if $s\geq s(L/K)$ and $G=\mathrm{Gal}(L/K)$, then

$$(\mathbf{B}_L^{\dagger,s})^{\natural}\otimes_{\mathbf{B}_K^{\dagger,s}}\mathbf{B}_L^{\dagger,s}\cong\oplus_{g\in G}(\mathbf{B}_L^{\dagger,s})^{\natural}e_g.$$

Proof. Select an element α from $\mathbf{B}^{\dagger}{}_L$ that forms a normal basis over $\mathbf{B}^{\dagger}{}_K$. Then, the discriminant of the minimal polynomial of α is in $\mathbf{B}^{\dagger}{}_K$, that is, for sufficiently large s, it is invertible in $\mathbf{B}_K^{\dagger,s}$, which proves the first part.

Let $s\geq s(L/K)$ and let $f(x)=\prod(x-\alpha_i)\in\mathbf{B}_K^{\dagger,s}[x]$ be the minimal polynomial of α. That is, $\mathbf{B}_L^{\dagger,s}=\mathbf{B}_K^{\dagger,s}[x]/f(x)$ and thus we have that

$$\begin{aligned}(\mathbf{B}_L^{\dagger,s})^{\natural}\otimes_{\mathbf{B}_K^{\dagger,s}}\mathbf{B}_L^{\dagger,s} &\cong (\mathbf{B}_L^{\dagger,s})^{\natural}\otimes_{\mathbf{B}_K^{\dagger,s}}\mathbf{B}_K^{\dagger,s}[x]/f(x)\\ &\cong (\mathbf{B}_L^{\dagger,s})^{\natural}[x]/f(x)\cong\oplus_i(\mathbf{B}_L^{\dagger,s})^{\natural}[x]/(x-\alpha_i)\cong\oplus_{g\in G}(\mathbf{B}_L^{\dagger,s})^{\natural}e_g,\end{aligned}$$

where the third isomorphism is due to the fact that the discriminant of $f(x)$ is invertible, which implies that the ideals $(x-\alpha_i)$ are relatively prime in pairs. □

[5]That is, the Galois conjugates form a basis.

Theorem 6. *Let K be a finite extension of $\mathbb{Q}_p$ and let T be a $\mathbb{Z}_p$-representation of G_K. Let L be a finite Galois extension of K such that G_L acts trivially on $T/12pT$. Let $n \geq n(L)$ and let $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$. If $s \geq \max\{(p-1)p^{n(L)-1}, s(L/K)\}$, then*

(1) the $\mathbf{A}_K^{\dagger,s}$-module $D_K^{\dagger,s}(T)$ and the $\mathbf{B}_K^{\dagger,s}$-module $D_K^{\dagger,s}(V)$ are of rank d;
(2) the natural map $\mathbf{B}^{\dagger,s} \otimes_{\mathbf{B}_K^{\dagger,s}} D_K^{\dagger,s}(V) \to \mathbf{B}^{\dagger,s} \otimes_{\mathbb{Q}_p} V$ is an isomorphism.

Proof. Lemma 21 implies that $M = D_L^{\dagger,s}(V)$ and $B = \mathbf{B}_L^{\dagger,s}$ satisfy the conditions of Proposition 1. Consequently,

$$\mathbf{B}_L^{\dagger,s} \otimes_{\mathbf{B}_K^{\dagger,s}} D_K^{\dagger,s}(V) \cong D_L^{\dagger,s}(V).$$

This, in conjunction with Corollary 5 (tensoring additionally with $\mathbb{Q}_p$), implies item (2) and that the $\mathbf{B}_K^{\dagger,s}$-module $D_K^{\dagger,s}(V)$ is of rank d. (It should be noted that since $\mathbf{B}_K^{\dagger,s}$ is principal, $D_K^{\dagger,s}(V)$ is necessarily free.)

It remains to be shown that the $\mathbf{A}_K^{\dagger,s}$-module $D_K^{\dagger,s}(T)$ is free. A sufficiently large integer n should be chosen, and $D_K^{\dagger,s}(V)/Q_n$ should be regarded, where $Q_n = ((1+x)^{p^n}-1)/((1+x)^{p^{n-1}}-1)$. As demonstrated in [Ber02] (cf. Lemma 4.9.), this is a d-dimensional vector space over $K(\mu_{p^n})$. Furthermore, the image of $D_K^{\dagger,s}(T)$ in $D_K^{\dagger,s}(V)/Q_n$ is an $\mathcal{O}_{K(\mu_{p^n})}$-lattice. Select d elements of $D_K^{\dagger,s}(T)$, whose images generate this lattice. Since $\mathbf{A}_K^{\dagger,s}$ is complete with respect to the Q_n-adic topology and the kernel of the map $D_K^{\dagger,s}(T) \to D_K^{\dagger,s}(V)/Q_n$ is $Q_n D_K^{\dagger,s}(T)$, it follows that the chosen d elements also generate $D_K^{\dagger,s}(T)$ over $\mathbf{A}_K^{\dagger,s}$. □

The subsequent corollary is a previously established classical result by Cherbonnier and Colmez [CC98].

Corollarry 6. *If $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$, then $D^\dagger(V) = (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}$ is a d dimensional vector space over ${\mathbf{B}^\dagger}_K$ that is stable under both Γ_K and φ. Moreover,*

$$D(V) = \mathbf{B}_K \otimes_{{\mathbf{B}^\dagger}_K} D^\dagger(V) \qquad \text{and} \qquad \mathbf{B}^\dagger \otimes_{{\mathbf{B}^\dagger}_K} D^\dagger(V) = \mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V.$$

In particular, the functor $V \mapsto D^\dagger(V)$ is an equivalence of categories between the category of $\mathbb{Q}_p$-representations of G_K and the category of étale (φ, Γ)-modules over ${\mathbf{B}^\dagger}_K$.

Let S be a Banach $\mathbb{Q}_p$-algebra, K a finite extension of $\mathbb{Q}_p$, V an S-representation of G_K, T an $\mathcal{O}_S$-lattice of V that is table under G_K, and let L be a finite Galois extension of K such that G_L acts trivially on $T/12pT$. Let $s(V)$ be defined as $\max\{(p-1)p^{n(L)-1}, s(L/K)\}$. If necessary, increase $s(V)$ slightly to ensure the existence of an integer $n(V)$ such that $p^{n(V)-1}(p-1) = s(V)$.

Proposition 26. *Let V be a d dimensional S-representation of G_K and $n \geq n(L)$. Then, $(\mathcal{O}_S \widehat{\otimes} \widetilde{\mathbf{A}}^{\dagger,(p-1)/p}) \otimes_{\mathcal{O}_S} T$ has a unique free $\mathcal{O}_S \widehat{\otimes} \mathbf{A}_{L,n}^{\dagger,(p-1)/p}$-submodule $D_{L,n}^{\dagger,(p-1)/p}(T)$*

of rank d, that is fixed by H_L, stable under G_K, and possesses an invariant basis under Γ_L. Furthermore,

$$(\mathcal{O}_S\widehat{\otimes}\widetilde{\mathbf{A}}^{\dagger,(p-1)/p})\otimes_{\mathcal{O}_S\widehat{\otimes}\mathbf{A}_{L,n}^{\dagger,(p-1)/p}} D_{L,n}^{\dagger,(p-1)/p}(T)\cong(\mathcal{O}_S\widehat{\otimes}\widetilde{\mathbf{A}}^{\dagger,(p-1)/p})\otimes_{\mathcal{O}_S} T.$$

Proof. This is another immediate consequence of the application of Theorem 2, using Proposition 2 and 25. □

Let V be an d dimensional S-representation of G_K, let $s\geq s(V)$, and let $n(V)$ be defined as above.

Definition 30.

$$D_K^{\dagger,s}(V)=(S\widehat{\otimes}\mathbf{B}_L^{\dagger,s}\otimes_{S\widehat{\otimes}\mathbf{B}_L^{\dagger,s(V)}}\varphi^{n(V)}(D_{L,n(V)}^{\dagger,(p-1)/p}(V)))^{H_K}$$

In conclusion, the main theorem, which gave the title to the thesis, can now be stated. It should be noted that in this instance, the functor $V\mapsto D^\dagger(V)$ that arises from the theorem will cease to be an equivalence of categories.

Theorem 7. *Let S be a d dimensional S-representation of G_K and $s\geq s(V)$. Then,*

(1) $D_K^{\dagger,s}(V)$ is an $S\widehat{\otimes}\mathbf{B}_K^{\dagger,s}$-module of rank d;
(2) the map

$$(S\widehat{\otimes}\mathbf{B}^{\dagger,s})\otimes_{S\widehat{\otimes}\mathbf{B}_K^{\dagger,s}} D_K^{\dagger,s}(V)\to(S\widehat{\otimes}\mathbf{B}^{\dagger,s})\otimes_S V$$

is an isomorphism;
(3) if $x\in\mathcal{X}$, then the map $S/\mathfrak{m}_x\otimes_S D_K^{\dagger,s}(V)\to D_K^{\dagger,s}(V_x)$ is an isomorphism.

Proof. Proposition 26 implies that $D_L^{\dagger,s}(V)$ is an $S\widehat{\otimes}\mathbf{B}_L^{\dagger,s}$-module of rank d and that

$$(S\widehat{\otimes}\mathbf{B}^{\dagger,s})\otimes_{S\widehat{\otimes}\mathbf{B}_L^{\dagger,s}} D_L^{\dagger,s}(V)\to(S\widehat{\otimes}\mathbf{B}^{\dagger,s})\otimes_S V$$

is an isomorphism. In a similar manner to the previous argument, items (1) and (2) follow from Proposition 1. Finally, in the event that $K=L$ and $s=p^{n(V)-1}(p-1)$, item (3) follows from the uniqueness property in Proposition 26. The general case follows from extension by scalars and taking the invariants under H_K. □

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